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RÜCKERT NULLSTELLENSATZ FOR NORMED, NON-DISCRETE FIELDS USING NON-STANDARD ANALYSIS

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

Rückert Nullstellensatz is the analogous for convergent series over K^n (K-normed, non-discrete, complete field) of the well-known Hilbert Nullstellensatz for polynomials over an algebraically closed field. For K an arbitrary algebraically closed (normed, non-discrete, complete) field, the Rückert Nullstellensatz is proved in [A] using algebraic methods. The particular case $K = \mathbf{C}$ (= the field of complex numbers) is proved, for instance, in [Tg] using Puiseux series and in [Ro2] using generic points in a non-standard context. In this note we prove a new version of the Rückert Nullstellensatz for the extension $K \subset \tilde{K}$, where K is a normed, non-discrete, complete field and $\tilde{K}$ is the completion of the algebraic closure of K (see Theorem 2.2). When K is algebraically closed, we obtain, as a Corollary (Corollary 2.3), the Rückert Nullstellensatz fom [A]. The proof consists in clarifications and adaptations of the proof from [Ro2] to the present context. We also use [I].

1 Germs on K^n

For the non-standard context we use the notations, terminology and Principles from [Ro1], [Dv] and for the standard context the notations, terminology and Theorems from [ACJ], part. I.

1.1. Let K be a normed (the norm will be denoted by $|\cdot|$), non-discrete, complete field; then K^n becomes, naturally, a (complete) metric space. We define on $\mathcal{P}(K^n)$ the following (equivalence) relation: if $p \in K^n$ and $A_1, A_2 \subset$

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K^n, $p \in A_1 \cap A_2$, then $A_1 \sim A_2$ if and only if there is an open neighborhood U of p in K^n such that $A_1 \cap U = A_2 \cap U$. The classes of equivalence of the previous relation are called the *germs of sets in p*.

We consider the following (well-defined) relations:

1) If α, β are germs of sets in p, then $\alpha \leq \beta$ if and only if $(\exists)A_1 \in \alpha$, $(\exists)A_2 \in \beta$, $(\exists)U =$ open neighborhood of p in K^n such that $A_1 \cap U \subseteq A_2 \cap U$. It follows that $\alpha = \beta$ if and only if $\alpha \leq \beta$ and $\beta \leq \alpha$.

2) If α, β, γ are germs of sets in p, then $\gamma = \alpha \wedge \beta$ if and only if $(\exists)A_1 \in \alpha$, $(\exists)A_2 \in \beta$, $(\exists)A_3 \in \gamma$, such that $A_1 \cap A_2 = A_3$.

3) If α, β, γ are germs of sets in p, then $\gamma = \alpha \vee \beta$ if and only if $(\exists)A_1 \in \alpha$, $(\exists)A_2 \in \beta$, $(\exists)A_3 \in \gamma$, such that $A_1 \cup A_2 = A_3$.

Let there be (cf. [ACJ])

$$\mathcal{A}_{n,p} : \{f : A \subset K^n \to K | f \text{ analytic on } A \text{ and } (\exists)\cup \subset K^n,$$
$$\text{open neighborhood of } p,\ U \subset A\} = \mathcal{A}_{n,p,K}.$$

We consider on $\mathcal{A}_{n,p}$ the following (equivalence) relation: $f_1 \sim f_2$ if and only if $(\exists)U \subset K^n$, open neighborhood of p, such that $f_1|_U = f_2|_U$. A class of functions as before is called a *germ of analytic function in p*. Let's denote by $\mathcal{O}_{n,p}$ the set of germs of analytic functions in p. Since K is a (commutative) ring, it is easy to see that $\mathcal{O}_{n,p} = (\mathcal{O}_{n,p}, +, \cdot)$ can be naturally organized as a (commutative) ring with identity. If necessary, we also write $\mathcal{O}_{n,p} = \mathcal{O}_{n,p,K}$.

Let $K \subset L$ be an extension of normed, non-discrete, complete fields.

If $p \in K^n$ and $S \subset \mathcal{O}_{n,p}$ is a finite set, *the associated variety* of S in L is

$$\mathcal{V}_L(S) := \wedge\{\varphi^{-1}(0) | \varphi \in S\};$$

here $\varphi^{-1}(0)$ is the class of $f^{-1}(0)$ for some $f \in \varphi$.

If $p \in K^n$ and α is a germ of sets from L^n in p, then *the ideal of* α is the set

$$\mathcal{I}(\alpha) := \{\varphi \in \mathcal{O}_{n,p,K} | \alpha \leq \varphi^{-1}(0)\} \in Id(\mathcal{O}_{n,p,K}).$$

If I is an ideal in some ring R, then the radical of I is

$$\sqrt{I} := \{x \in R | (\exists) n \in \mathbf{N}^*,\ x^n \in I\}.$$

The germs of sets are not usual sets and the germs of analytic functions are not usual analytic functions. We'll show how, by using non-standard methods, we can replace the germs of sets by sets and the germs of analytic functions by analytic functions (following [Ro2]).

1.2. We recall here some results from [Ro1] and [Dv].

We consider that all objects we need belong to a standard universe $\mathcal{U}$, endowed with a language $\mathcal{L} = (\equiv, \in)$. Let's denote by ${}^*\mathcal{U}$ the corresponding non-standard universe (an enlargement of $\mathcal{U}$) and by ${}^*\mathcal{L} = ({}^*\equiv, {}^*\in)$ the corresponding language. If T is a standard object in $\mathcal{U}$, we denote by ${}^*T$ its enlargement in ${}^*\mathcal{U}$; if s is a sentence of $\mathcal{L}$, we denote by ${}^*s$ its extension to ${}^*\mathcal{L}$ (i.e. we keep all the logic connectors and the bounded quantifiers and their order; we replace the constants and objects T from s with the corresponding ${}^*T$). In the non-standard universe some Principles hold. We recall here two of them, useful in the sequel.

(T.P.) **Transfer Principle**: *Let s be a sentence of $\mathcal{L}$. Then*

$$* \models {}^*s \quad \textit{if and only if} \quad \models s.$$

(We write $\models s$ if and only if s holds in $\mathcal{U}$ and $* \models {}^*s$ if and only if $*\, s$ holds in ${}^*\mathcal{U}$.)

Let r be a binary relation $r \in \mathcal{U}$. We denote by $dom\,(r) := \{x|(\exists)y$ such that $(x, y) \in r\}$. The relation r is called *concurrent* if for any finite set $\{a_1, \dots, a_m\} \subset dom\,(r)$, there is b such that $(a_i, b) \in r$, $i = \overline{1, m}$.

(C.P.) **Concurrence Principle**: *Let r be a concurrent relation in $\mathcal{U}$. Then there is an element $b \in {}^*\mathcal{U}$ such that $({}^*a, b) \in {}^*r$, for all $a \in dom\,(r)$.*

1.3. Let's consider now an extension ${}^*K^n$ of K^n; then ${}^*K^n$ is a normed, non-discrete, non-complete space, $K^n \subset {}^*K^n$. If $p \in K^n$, the *halo* of p is

$$hal_n(p) := \{q \in {}^*K^n | {}^*d(p, q) \simeq 0\}.$$

Here, for $x \in {}^*\mathbf{R}$ (= the field of hyperreal numbers), we write $x \simeq 0$ if ${}^*|x| < \varepsilon$, $(\forall)\varepsilon \in \mathbf{R}$, $\varepsilon > 0$; ${}^*d$ is the extension with hyperreal values to ${}^*K^n$ of the usual metric d on K^n. If $p = (p_1, \dots, p_n) \in K^n$, $p_i \in K$, $i = \overline{1, r}$, then

$$hal_n(p) = hal_1(p_1) \times hal_1(p_2) \times \dots \times hal_1(p_n).$$

We can see that

$$hal_n(p) = \cap\{{}^*U | U \text{ is an open neighborhood of } p \text{ in } K^n\}.$$

Let τ be the (metric) topology on K^n. Then the elements of ${}^*\tau$ are the *-open sets from ${}^*K^n$. It can be seen that there is a *-open set $\nu \subseteq hal_n(p)$ (indeed, apply the Concurrence Principle (C.P.) to the concurrent relation: UrV if and only if $U, V \in \tau$ and $p \in V \subseteq U$).

Definition 1.3.1. A set $\alpha \in {}^*K^n$ is called *a germ of non-standard sets in* p if $(\exists)A \subseteq K^n$ such that $\alpha = {}^*A \cap hal_n(p)$. A function $\varphi : hal_n(p) \to {}^*K$ is called *a germ of non-standard analytic functions in* p if $(\exists)\psi \in \mathcal{O}_{n,p}$, $(\exists)f \in \psi$ such that $\varphi = {}^*f|hal_n(p)$ (it is easy to see that $A \sim B \Rightarrow {}^*A \cap hal_n(p) = {}^*B \cap hal_n(p)$ and $f \sim g \Rightarrow {}^*f|hal_n(p) = {}^*g|hal_n(p)$).

Let's denote by $\mathcal{G}_{n,p}$ the lattice of germs of sets in p, by $\mathcal{N}_{n,p}$ the lattice of germs of non-standard sets in p (with the usual operations on sets) and by

$\Gamma_{n,p}$ the ring of germs of non-standard analytic functions in p (with the usual operations on functions); we recall that $\mathcal{O}_{n,p}$ is the set of germs of analytic functions in p. If necessary, we write $\mathcal{N}_{n,p,K}$, $\mathcal{G}_{n,p,K}$, $\Gamma_{n,p,K}$, $\mathcal{O}_{n,p,K}$. We define the following functions:

$$\sigma : \mathcal{N}_{n,p} \to \mathcal{G}_{n,p}, \quad \delta : \Gamma_{n,p} \to \mathcal{O}_{n,p} \quad \text{by}$$

$$\sigma({}^*A \cap hal_n(p)) = [A] \text{ (= the germ of sets in } p \text{ with the representative } A)$$

$$\delta({}^*f|hal_n(p)) = [f] \text{ (= the germ of analytic functions in } p \text{ with representative } f).$$

Let's prove that σ is well-defined. If ${}^*A \cap hal_n(p) = {}^*B \cap hal_n(p)$, we consider the sentence

$$s = (\exists x \in \tau)(p \in x \wedge A \cap x = B \cap x).$$

Then

$${}^*s = (\exists x \in {}^*\tau)(p \in x \wedge {}^*A \cap x = {}^*B \cap x).$$

But ${}^*s$ is true, since any *-open set $S \subset hal_n(p)$ satisfies ${}^*A \cap \nu = {}^*B \cap \nu$, and we proved that such a ν exists (before the Definition 1.3.1). By the Transfer Principle (T.P.) we deduce that s is true, so $A \sim B$, hence $[A] = [B]$.

As for δ, if ${}^*f|hal_n(p) = {}^*g|hal_n(p)$, then for any *-open set $\nu \subset hal_n(p)$ we have ${}^*f|\nu = {}^*g|\nu$. Again by the Transfer Principle (T.P.) we deduce that the sentence

$$(\exists x \in \tau)(p \in x \wedge f|x = g|x)$$

is true, so $f \sim g$, hence $[f] = [g]$.

Further, it is straightforward to prove that σ and δ are isomorphisms (of lattices and rings, respectively).

1.4. In 1.1 we defined, for $K \subset L$ an extension of normed, non-discrete, complete fields, the variety $\mathcal{V}_L(S)$ associated to a finite set $S \subset \mathcal{O}_{n,p,L}$ of germs of analytic functions in $p \in K^n$. Now we define *the non-standard variety* $\mathcal{V}_L(S)$ *associated to a finite set* $S \subset \Gamma_{n,p,L}$ *of germs of non-standard analytic functions.* So, let $S \subset \Gamma_{n,p,L}$ be a set as before. Then, in this context

$$\mathcal{V}_L(S) := \cap\{\varphi^{-1}(0)|\varphi \in S\}.$$

For $p \in K^n \subset L^n$, we denote by $hal_n(p) = hal_n^K(p)$ the halo of p in K^n and by $hal_n^L(p)$ the halo of p in L^n (clearly, $hal_n(p) \subset hal_n^L(p)$).

Now,

$$\varphi^{-1}(0) = ({}^*f|hal_n(p))^{-1}(0) = \{x \in hal_n^L(p)|{}^*f(x) = 0\} \subset hal_n^L(p)$$

is a set of points, so an usual set (here $\varphi = {}^*f|hal_n(p)$).

It is easy to see, since σ and δ are isomorphisms, that, if $\varphi_i = {}^*f_i|hal_n(p)$, $i = \overline{1,m}$, we have

$$\sigma(\mathcal{V}_L(\varphi_1, \ldots, \varphi_m)) = \mathcal{V}_L(\delta(\varphi_1), \ldots, \delta(\varphi_m))$$

(use the definitions of $\mathcal{V}(S)$ from 1.1 and 1.4).

1.5. Let $f : V \subset K^n \to K$ be an analytic function on an open neighborhood V of the origin and put

$$f(t_1, \ldots, t_n) = \sum_{j \geq 0} f_j(t_1, \ldots, t_n),$$

where f_j is a homogeneous polynomial of degree j, for any $j \geq 0$. We say that f is *regular in t_n of order $k > 0$* if $f_j \equiv 0$, $(\forall) j < k$ and t_n^k has a non-zero coefficient in f_k. If $f = \sum_{j \geq k} f_j$, $f_k \not\equiv 0$, it is easy to find a non-singular linear transformation $t_j \to t'_j$, transforming f into a regular function of order k in t'_n.

Let $\mathcal{O}_n := \mathcal{O}_{n,0}$. A *Weierstrass polynomial of degree $k > 0$ in t_n* is a function $h \in \mathcal{O}_n$ of the form

$$h(t_1, \ldots, t_n) = t_n^k + a_1(t_1, \ldots, t_{n-1}) t_n^{k-1} + \ldots + a_k(t_1, \ldots, t_{n-1}),$$

where $a_j \in \mathcal{O}_{n-1}$ and $a_j(0, \ldots, 0) = 0$, $j = \overline{1,k}$.

A *germ* of non-standard analytic functions regular in t_n of order $k > 0$ (resp. of non-standard polynomials in t_n, resp. of non-standard Weierstrass polynomials of degree $k > 0$ in t_n) is $f^*|hal_n(0)$, where f is an analytic function regular in t_n of degree $k > 0$ (resp. a polynomial in t_n, resp. a Weierstrass polynomial of degree $k > 0$ in t_n). By the Transfer Principle (P.T.) we have the following non-standard versions of the well-known (see [ACJ]) Weierstrass Preparation and Division Theorems:

Theorem 1.5.1. (non-standard Weierstrass Preparation): *Let φ be a germ of non-standard analytic functions in the origin, regular of order $k > 0$ in t_n. Then there is a germ of non-standard Weierstrass polynomials of degree k in t_n, denoted by ω, and a germ of non-standard analytic functions in the origin, denoted by ψ, such that $\psi(0) \neq 0$ and $\varphi = \omega.\psi$.*

Theorem 1.5.2. (non-standard Weierstrass Division): *Let ω be a germ of non-standard Weierstrass polynomials of degree k in t_n and φ a germ of non-standard analytic functions in the origin. Then, there is a germ of non-standard analytic functions, denoted by Δ, and a germ of non-standard polynomials of degree $< k$ in t_n, denoted by ρ, such that $\varphi = \omega.\Delta + \rho$.*

2 Rückert Nullstellensatz

Let K be a non-discrete, complete normed field $K = (K, |\cdot|_K)$. Let $\bar{K}$ be an algebraic closure of K. One knows that $|\cdot|_K$ extends uniquely to a non-discrete norm $|\cdot|_{\bar{K}}$ on $\bar{K}$ ([La], page 291), not necessarily complete. Let's denote by $\tilde{K} = \hat{\bar{K}}$ (the completion of $\bar{K}$) (see [La], page 286), $\tilde{K} = (\tilde{K}, |\cdot|_{\tilde{K}})$.

Lemma 2.1. *$K \subset \tilde{K}$ and $\tilde{K}$ is an algebraically closed, non-discrete, complete normed field.*

Proof. If $K = (K, |\cdot|_K)$ is archimedean, then the characteristic of K is zero (if not, $|\cdot|_K|_P$, $P =$ the prime field of K is bounded, so $|\cdot|_K$ is non-archimedean, by [IM], page 12), so $\mathbf{Q} \subset K$. We denote by $|\cdot|_{\mathbf{Q}} := |\cdot|_K|_{\mathbf{Q}}$, so $|\cdot|_{\mathbf{Q}} = |\cdot|^\alpha$, $0 < \alpha \leq 1$, where $|\cdot|$ is the usual module on $\mathbf{Q}$, by Ostrovschi Theorem ([IM], page 15). But K is complete, so $K \supset \hat{\mathbf{Q}} = \mathbf{R}$ (the completion of $\mathbf{Q}$, any two norms on $\mathbf{R}$ being equivalent, by [La], Prop. 3, page 288). By Ghelfand-Mazur Theorem ([La], page 290), we deduce that $K = \mathbf{R}$ or $K = \mathbf{C}$, so $\bar{K} = \mathbf{C}$, so $\tilde{K} = \mathbf{C}$, i.e. an algebraically closed, non-discrete, complete normed field.

If K is non-archimedean, take $\bar{K} =$ the algebraic closure of K, which is again a non-archimedean field. But then $\tilde{K} = \hat{\bar{K}}$ (the completion of $\bar{K}$) remains algebraically closed by [R], page 146.

□

Let $\mathcal{O}_n := \mathcal{O}_{n,0,K}$ and $\tilde{\mathcal{O}}_n := \mathcal{O}_{n,0,\tilde{K}}$ (see 1.1). Clearly, $\mathcal{O}_n \subset \tilde{\mathcal{O}}_n$. Considering the extension $K \subset \tilde{K}$ of normed, non-discrete, complete fields, we recall that we defined in 1.1 $\mathcal{V}_{\tilde{K}}(S)$, for $S \subset \tilde{\mathcal{O}}_n$ a finite set and $I(\alpha)$ if $\alpha \in \mathcal{G}_{n,0,\tilde{K}}$ (notation from 1.3). If $I \in Id(\mathcal{O}_n)$ is an ideal, it is finitely generated (use noetherianity) by, say, $\varphi_1, \ldots, \varphi_m$. Then $\mathcal{V}_{\tilde{K}}(I) := \mathcal{V}_{\tilde{K}}(\varphi_1, \ldots, \varphi_m)$. In this paragraph we prove the following extension of the classical Rückert Nullstellensatz:

Theorem 2.2. *Let K be a normed, non-discrete, complete field and $I \in Id(\mathcal{O}_n)$ be an ideal. Then*

$$\mathcal{I}(\mathcal{V}_{\tilde{K}}(I)) = \sqrt{I}.$$

Corollary 2.3. (Rückert Nullstellensatz from [A]) *If K is a normed, non-discrete, complete, algebraically closed field, then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$, for any ideal $I \in Id(\mathcal{O}_n)$.*

Proof. We have $K = \tilde{K}$ in this case.

□

We shall prove an analogous of the previous Theorem 2.2 for germs of non-standard analytic functions (Theorem 2.16); then the Theorem 2.2 will follow using the isomorphisms σ and δ from Section 1.3, using the last equality from Section 1.4 (see §1).

We put $\Gamma_n := \Gamma_{n,0,K}$, $\tilde{\Gamma}_n := \Gamma_{n,0,\tilde{K}}$ (see 1.3). We have $0 \in K^n \subset \tilde{K}^n$. We put $hal_n(0) := hal_n^K(0)$ and $\widehat{hal_n}(0) := hal_n^{\tilde{K}}(0)$ (see 1.4). We have $hal_n(0) \subseteq \widehat{hal_n}(0)$.

Definition 2.4. If $I \subset \Gamma_n$ is an ideal, then $p \in \widehat{hal_n}(0)$ is a *generic point for I* if $I = \mathcal{I}(p)$ $(= \{\varphi \in \Gamma_n | \varphi(p) = 0\})$.

Lemma 2.5. *The ideal (0) from Γ_n has a generic point in $hal_n(0)$ (so it has a generic point in $\widehat{hal_n}(0)$).*

Proof. We define the binary relation $(f, U)rW$ if and only if f is analytic and non-zero on the open neighborhood U of the origin and $W \subseteq U$ is a non-empty open subset such that $f(x) \neq 0$, $(\forall)x \in W$. It is easy to see that r is a concurrent relation. By the Concurrence Principle (C.P.), we find a *-open set $\nu \neq \emptyset$ such that for any $f \in \Gamma_n \setminus \{0\}$, f analytic on U implies $\nu \subseteq {}^*U$ and $f(x) \neq 0$, $(\forall)x \in \nu$. It follows that $\nu \subseteq \cap\{{}^*U | U$ open neighborhood of $0\} = hal_n(0)$. Then $(0) = \mathcal{I}(\xi)$, for any $\xi \in \nu$.

□

If $P \subseteq \Gamma_n$ is an ideal, $P \neq 0$, it is easy to see that $P \subseteq \mathcal{I}(0)$, which is the only maximal ideal of Γ_n, so $\Gamma_n \setminus \mathcal{I}(0)$ is the set of invertible elements of the local ring Γ_n.

Lemma 2.6: *If $P \in Spec(\Gamma_1)$, $P \neq 0$, then $P = \mathcal{I}(0)$.*

Proof. Let $\varphi \neq 0$, $\varphi \in P$. Then $\varphi(0) = 0$, so $\varphi(z) = z^k \psi(z)$, with $\psi(0) \neq 0$. Hence $\varphi \in P$, $\psi \notin P$, P prime, so $z^k \in P$, so $z \in P$.

□

Theorem 2.7: *If $P \in Spec(\Gamma_n)$, $P \neq \Gamma_n$, then P has a generic point in $\widehat{hal_n}(0)$.*

Proof. If $P = 0$, use Lemma 2.5. If $P \neq 0$, we use induction on n. If $n = 1$, use Lemma 2.6. Suppose now that we know the Theorem 2.7 for Γ_n and we want to prove it for Γ_{n+1}.

Let $P \in Spec(\Gamma_{n+1})$, $P \neq 0$. Put $P' := P \cap \Gamma_n$ and $P'' := P \cap \Gamma_n[t_{n+1}]$. It is easy to see that if P is proper then P' and P'' are also proper ideals and $P' \in Spec(\Gamma_n)$. By the induction hypothesis, we know that P' has a generic point $(\xi_1, \ldots, \xi_n) \in \widehat{hal_n}(0)$. Let's define $\varepsilon : \Gamma_n \to {}^*\tilde{K}$ by $\varepsilon(\varphi) := \varphi(\xi_1, \ldots, \xi_n)$ the evaluation morphism. We have $P' = Ker\ \varepsilon$. We get the embedding $G : \Gamma_n/P' \to {}^*L'$, induced by ε.

Let $l : \Gamma_{n+1} \to \Gamma_{n+1}/P$ be the canonical projection and $i : \Gamma_n \hookrightarrow \Gamma_{n+1}$ be the natural inclusion. Then $\Gamma_n \stackrel{l \circ i}{\to} \Gamma_{n+1}/P$. We have $Ker(l \circ i) = P \cap \Gamma_n = P'$, so we get the extension of rings $\Gamma_n/P' \hookrightarrow \Gamma_{n+1}/P$.

Lemma 2.8: *Let $P \in Spec(\Gamma_n)$, $P \neq 0$. Then there is a non-standard Weierstrass polynomial $\omega \in P$.*

Proof: Let $\varphi \in P$, $\varphi \neq 0$. We may suppose that φ is regular of order $k > 0$ in t_n. By Theorem 1.5.1, we have $\varphi = \omega\pi$, where $\pi \in \Gamma_n$ is invertible and ω is a non-standard Weierstrass polynomial. $\varphi = \omega\pi \in P$, $\pi \notin P$, hence $\omega \in P$.

□

Lemma 2.9: Γ_{n+1}/P *is an integral extension of* Γ_n/P'.

Proof. Take $\omega \in P$ a non-standard Weierstrass polynomial of degree m in t_{n+1} (cf. Lemma 2.8). Then

$$0 = l(\omega) = l(t_{n+1})^m + \sum_{j=0}^{m-1} l(a_j) l(t_{n+1})^j,$$

where $l(a_j) \in \Gamma_n/P'$, for $0 \leq j \leq m-1$. So $l(t_{n+1})$ is integer over Γ_n/P'.

Let now $\nu \in \Gamma_{n+1}$. By Theorem 1.5.2 there is $\Delta \in \Gamma_{n+1}$ and $\rho \in \Gamma_n[t_{n+1}]$ such that $\nu = \omega\Delta + \rho$. So $l(\nu) = l(\omega)l(\Delta) + l(\rho) = l(\rho)$ ($\omega \in P$, so $l(\omega) = 0$). Since $\rho \in \Gamma_n[t_{n+1}]$, $l(\rho)$ (hence $l(\nu)$) is a polynomial in $l(t_{n+1})$ with coefficients in Γ_n/P'. Since we already proved that $l(t_{n+1})$ is integral over Γ_n/P' we have the lemma.

□

Lemma 2.10: *Let* p *be a polynomial over the ring of continuous functions on an open neighborhood* V *of the origin of* $\tilde{K}^n$ *with values in* $\tilde{K}$. *Suppose that*

$$p(z_1, \ldots, z_{n+1}) = z_{n+1}^k + \sum_{j=0}^{k-1} a_j(z_1, \ldots, z_n) z_{n+1}^j,$$

where $a_0, \ldots, a_{k-1}$ *are continuous functions on* V *and* $a_j(0, \ldots, 0) = 0$, $0 \leq j \leq k-1$. *Let* $(\xi_1, \ldots, \xi_n) \in \widehat{hal_n}(0)$. *Then any root of the polynomial* $q(z) := p(\xi_1, \ldots, \xi_n, z)$ *from* ${}^*\tilde{K}$ *is infinitesimal.*

Proof. Let $\xi \in {}^*\tilde{K}$ be a root of q. If $\xi = 0$, O.K. If $\xi \neq 0$, we have

$$0 = \xi^k + \sum_{j=0}^{k-1} b_j(\xi_1, \ldots, \xi_n)\xi^j | : \xi^k \neq 0 \quad \text{so} \quad -1 = \sum_{j=0}^{k-1} b_j(\xi_1, \ldots, \xi_n)\xi^{j-k}.$$

If ξ *is not* infinitesimal, then ξ^{j-k} is finite for any j, $0 \leq j \leq k-1$. Because the functions b_j are all continuous that $b_j(\xi_1, \ldots, \xi_n)$ are all infinitesimal. So, 1 is infinitesimal, a contradiction.

□

Construction 2.11: Let Λ be the field of fractions of $\Gamma_n/P'(\Lambda = (\Gamma_n/P')_0)$-see the beginning of the proof of Theorem 2.7, and put

$$p(X) := X^n + \sum_{j=0}^{m-1} l(a_j) X^j, \quad p \in \Lambda[X],$$

where $\omega \in P$, $\omega = t_{n+1}^m + \sum_{j=0}^{m-1} a_j t_{n+1}^j$ is a Weierstrass polynomial, cf. Lemma 2.8. Then $p(l(t_{n+1})) = 0$ and let $q \in \Lambda[X]$ be an irreducible factor of p such that $q(l(t_{n+1})) = 0$. We extend G (induced by the evaluation morphism ε), defined before Lemma 2.8 to an injective morphism, denoted also by G, $G : \Lambda \hookrightarrow {}^*\tilde{K}$, in a natural way. Then $\Lambda[X] \overset{G}{\hookrightarrow} {}^*\tilde{K}[X]$ and let ξ_{n+1} be a zero of $G(q)$ in ${}^*\tilde{K}$, since $\tilde{K}$ (so ${}^*\tilde{K}$) is algebraically closed. Then $G(p)(\xi_{n+1}) = 0$ (because $G(q)|G(p)$) and

$$G(p)(X) = X^m + \sum_{j=0}^{m-1} G(l(a_j))X^j = X^m + \sum_{j=0}^{m-1} \hat{a}_j(\xi_1, \ldots, \xi_n)X^j.$$

From Lemma 2.10 we deduce

Lemma 2.12. *Let $P \in Spec(\Gamma_{n+1})$, $0 \neq P \neq \Gamma_{n+1}$ and suppose that $P' = P \cap \Gamma_n$ has a generic point $(\xi_1, \ldots, \xi_n) \in \widehat{hal}_n(0)$. Then, if ξ_{n+1} is from Construction 2.11, we have $(\xi_1, \ldots, \xi_{n+1}) \in \widehat{hal}_{n+1}(0)$.*

Lemma 2.13: *If $P, P', (\xi_1, \ldots, \xi_n)$ are as in the previous lemma and ξ_{n+1} is from Construction 2.11, then $(\forall)\varphi \in \Gamma_{n+1}$, $(\exists)\rho \in \Gamma_n[t_{n+1}]$ such that $\varphi(\xi_1, \ldots, \xi_{n+1}) = \rho(xi_1, \ldots, \xi_{n+1})$.*

Proof. $\xi_{n+1} \in {}^*\tilde{K}$ is a zero of $G(q)$. From Lemma 2.12, $(\xi_1, \ldots, \xi_{n+1}) \in \widehat{hal}_{n+1}(0)$ and we have for $p \in \Lambda[X]$ (see Construction 2.11) with q as a factor: $0 = G(p)(\xi_{n+1}) = \xi_{n+1}^m + \sum_{j=0}^{m-1} a_j(\xi_1, \ldots, \xi_n)\xi_{n+1}^j = \omega(\xi_1, \ldots, \xi_{n+1})$, where ω is the non-standard Weierstrass polynomial with coefficients $a_1, \ldots, a_{m-1} \in \Gamma_n$. From Theorem 1.5.2 we find $\Delta \in \Gamma_{n+1}$ and $\rho \in \Gamma_n[t_{n+1}]$, $deg\rho < m$, such that $\varphi = \omega\Delta + rho$. But then $\varphi(\xi_1, \ldots, \xi_{n+1}) = \rho(\xi_1, \ldots, \xi_{n+1})$. $\square$

Construction 2.14: We recall that the evaluation morphism ε induces the embedding $G : \Gamma_n/P' \hookrightarrow {}^*\tilde{K}$ (before Lemma 2.8). Let's consider the following fields: $\Lambda := (\Gamma_n/P')_0$, $\Omega := (\Gamma_n[t_{n+1}]/P'')_0$, $\Phi := (\Gamma_{n+1}/P)_0$. We have the natural inclusions: $\Lambda \hookrightarrow \Omega \hookrightarrow \Phi$. From Lemma 2.9 we deduce that the extension $\Lambda \hookrightarrow \Phi$ is algebraic, so the extensions $\Lambda \hookrightarrow \Omega$ and $\Omega \hookrightarrow \Phi$ are also algebraic. So, if we consider the polynomial q from Construction 2.11, supposing that its dominant coefficient is 1, then $q = Irr(l(t_{n+1}), \Omega)$. Firstly, the canonical projection $l : \Gamma_n[t_{n+1}] \to \Gamma_n[t_{n+1}]/P''$ factorizes to $l_1 : (\Gamma_n/P')[t_{n+1}] \to \Gamma_n[t_{n+1}]/P''$ and extends to fractions $l_2 : \Lambda[t_{n+1}] \to \Omega$, $l_2(t_{n+1}) = l(t_{n+1})$ (i.e. $t_{n+1} \pmod{P''}$). Secondly, the evaluation map ε extends to $\varepsilon' : \Gamma_n[t_{n+1}] \to {}^*\tilde{K}$ by $\pi \mapsto \pi(\xi_1, \ldots, \xi_{n+1})$. Because $(\xi_1, \ldots, \xi_n)$ is a generic point for P', we deduce that ε' factorizes to $\varepsilon_1 : (\Gamma_n/P')[t_{n+1}] \to {}^*\tilde{K}$

and extends to fractions $\bar{\varepsilon}' : \Lambda[t_{n+1}] \to {}^*\tilde{K}$. We ex tend the embedding G to $G' : \Gamma_n[t_{n+1}]/P'' \to {}^*\tilde{K}$ such that $G' \circ l = \varepsilon'$, putting $G'(l(t_{n+1})) := \xi_{n+1}$. Finally, we get the following commutative diagram:

$$\begin{tikzcd}
\Gamma_n[t_{n+1}]/P'' \arrow[d, hook] \arrow[rr, bend left, "G'"] & \Gamma_n[t_{n+1}]/P'' \arrow[l, "l"'] \arrow[r, "\varepsilon'"] \arrow[d, "\pi"] & {}^*\tilde{K} \arrow[d, equal] \\
\Omega \arrow[d, equal] & (\Gamma_n/P')[t_{n+1}] \arrow[l, "l_1"'] \arrow[r, "\varepsilon_1"] \arrow[d, hook, "j"] & {}^*\tilde{K} \arrow[d, equal] \\
\Omega & \Lambda[t_{n+1}] \arrow[l, "l_2"'] \arrow[r, "\varepsilon'"] & {}^*\tilde{K}
\end{tikzcd}$$

with Λ and Ω from above.

As the extensions $\Lambda \hookrightarrow \Omega$ is algebraic and q is irreducible, it follows that $Ker\ l_2 =< q >$. Using the definition of ε', we can see that $\bar{\varepsilon}'(q) = 0$, so $< q >\subseteq Ker\bar{\varepsilon}'$. But $< q >$ is a maximal ideal (because q is irreducible), so $Ker\ l_2 = Ker\ \bar{\varepsilon}' =< q >$. Going up on the previous diagram, we deduce that $Ker\,\varepsilon_1 = Ker\,l_1$ and $Ker\,l = Ker\,\varepsilon'(= P'')$. So, the morphism G' is *injective*. We proved

Lemma 2.15: $Ker\,\varepsilon' = P''$.

Now, we are ready to end the proof of Theorem 2.7. We recall that we use induction on n. If $P \in Spec(\Gamma_{n+1})$, $0 \neq P \neq \Gamma_{n+1}$, then $P' := P \cap \Gamma_n \in Spec(\Gamma_n)$ and $(\xi_1, \ldots, \xi_n) \in \widehat{hal_n}(0)$ is a generic point for P'. From Lemma 2.12 we have $(\xi_1, \ldots, \xi_{n+1}) \in \widehat{hal}_n(0)$ such that $(\forall)\varphi \in \Gamma_{n+1}$, $(\exists)\rho \in \Gamma_n[t_{n+1}]$ with $\varphi(\xi_1, \ldots, \xi_{n+1}) = \rho(\xi_1, \ldots, \xi_{n+1})$ (Lemma 2.13). But $l(\varphi) = l(\rho)$ (see the proof of Lemma 2.13 and Construction 2.11: $\omega \in P$). So $\varphi \in P$ (i.e. $l(\varphi) = 0$) if and only if $\rho \in P \cap \Gamma_n[t_{n+1}] = P''$ (i.e. $l(\rho) = 0$). But $P'' = Ker\,\varepsilon'$ (Lemma 2.15), so $\rho \in P''$ if and only if $\varepsilon'(\rho) = 0$, i.e. $\rho(\xi_1, \ldots, \xi_{n+1}) = 0$ (see the definition of ε'). So $\varphi \in P$ if and only if $(\xi_1, \ldots, \xi_{n+1}) = 0$, i.e. $(\xi_1, \ldots, \xi_{n+1}) \in \widehat{hal}_{n+1}(0)$ is a generic point for P, q.e.d. $\square$

Theorem 2.16. (Rückert Nullstellensatz, Non-standard version): *If* $I \in Id(\Gamma_n)$ *is an ideal, then* $\mathcal{I}(\mathcal{V}_{\tilde{K}}(I)) = \sqrt{I}$.

Proof. Take $\psi \notin \sqrt{I}$. Put $\mathcal{A}_\psi := \{J \in Id(\Gamma_n)/I \subseteq J \text{ and } \psi \notin \sqrt{J}\}$. Then $\mathcal{A}_\psi \neq \emptyset (I \in \mathcal{A}_\psi)$ and $\mathcal{A}_\psi$ inductive set. By Zorn Lemma, we find $P_\psi \in \mathcal{A}_\psi$ a maximal element. We prove that $P_\psi \in Spec(\Gamma_n)$. Suppose the contrary and let $x, y \in \Gamma_n$ such that $xy \in P_\psi$ and $x \notin P_\psi$, $y \notin P_\psi$. Then $P_\psi+ < x >\supsetneq P_\psi$, $P_\psi+ < y >\supsetneq P_\psi$ so $P_\psi+ < x >\notin \mathcal{A}_\psi$, $P_\psi+ < x >\notin \mathcal{A}_\psi$.

We find that $\psi \in \sqrt{P_\psi + < x >}$, $\psi \in \sqrt{P_\psi + < y >}$ so $\psi^m = \alpha + \lambda x$ and $\psi^P = \beta + \mu y$, $\alpha, \beta \in P_\psi$, $\lambda, \mu \in \Gamma_n$ for suitable $m, p \in \mathbf{N}$. So $\psi^{m+p} \in P_\psi$ (use $xy \in P_\psi$) so $\psi \in \sqrt{P_\psi}$, a contradiction. Because $P_\psi \in Spec(\Gamma_n)$, by Theorem 2.7, P_ψ has a generic point in $\widehat{hal}_n(0)$. Because $I \subseteq P_\psi$ and $\psi \notin P_\psi$ it follows that $f(\xi_1, \ldots, \xi_n) = 0$, $(\forall) f \in I$, but $\psi(\xi_1, \ldots, \xi_n) \neq 0$, where $(\xi_1, \ldots, \xi_n) \in \widehat{hal}_n(0)$ is the generic point of P_ψ. So $\psi \notin \mathcal{I}(\mathcal{V}_{\tilde{K}}(I))$. We obtained $\mathcal{I}(\mathcal{V}_{\tilde{K}}(I)) \subseteq \sqrt{I}$. The converse inclusion being always true, we obtain the desired equality. □

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