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USE OF THE INTERIOR-POINT METHOD FOR CORRECTING AND SOLVING INCONSISTENT LINEAR INEQUALITY SYSTEMS

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

By the correction of an inconsistent linear system we mean avoiding its contradictory nature by means of relaxing the constraints. In $[Va2]$ it was shown that for inconsistent linear equation systems $Ax = b$, the correction of the whole augmented matrix (A, b) using Euclidean norm criterion, is a problem equivalent to finding the least eigenvalues (and corresponding eigenvectors) of the matrix $(-b, A)^T(-b, A)$. In $[Va1]$ Vatolin proposed an algorithm based on linear programming, which finds minimal corrections of the constraint matrix and RHS vector. In $[PM]$ is analyzed correction problem for an inconsistent linear inequality system $Ax \leq b$, using two criteria $\|.\|_1$ and $\|.\|_\infty$. In this paper, we use interior-point techniques for solving the associated linear program.

1. Introduction

Consider the linear inequality system:

$$\begin{cases} \langle a_i, x\rangle \leq b_i,\ i \in M_0 \cup M_1 \\ x_j \geq 0,\ j = 1, ..., n, \end{cases} \tag{1}$$

where a_i^T, $i = 1, ..., m$, forms the i^{th} row of the matrix A, b_i is the i^{th} component of b, M_0, M_1 are finite index sets and $\langle ., .\rangle$ stands for the standard inner

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product in $\mathbf{R}^n$.

With system (1) we associate the *corrected system*:

$$\begin{cases} \langle a_i, x\rangle \le b_i, \ i \in M_0 \\ \langle a_i + h_i', x\rangle \le b_i - h_{i,n+1}, \ i \in M_1 \\ x_j \ge 0, \ j = 1, ..., n, \end{cases} \tag{2}$$

where $h_i' \in \mathbf{R}^n$ and $h_{i,n+1} \in \mathbf{R}$. Let $h_i \in \mathbf{R}^{n+1}$,

$$h_i = (h_i', h_{i,n+1}) = (h_{i1,...,}h_{i,n+1})$$

be the vector correcting the i^{th} row of system (1), $i \in M_1$.
The rows with indices $i \in M_0$ are not corrected (are assumed to be fixed). We can fix also arbitrary columns of the augmented matrix (A, b), with indices $j \in J_0 \subset \{1, ..., n+1\}$. Thus we set $h_{ij} = 0, \ i \in M_1, \ j \in J_0$.
Let $M_1 = \{i_1, ..., i_p\}$ and $J_1 = \{1, ..., n+1\} \setminus J_0$ the complement of $J_{0,}$ i.e. the set of indices of columns to be corrected.

The correction problem of system (1) may be expressed as

$$min\,\{\Phi(H)/H \in S\} \tag{3}$$

where $H(h_{ij})_{p\times(n+1)}$,

$$S = \{H/h_{ij} = 0, \ i \in M_1, \ j \in J_0 \textit{and system (2) is consistent}\}.$$

$\Phi\,(H)$ is the *correction criterion* estimating the quality of correction.

2. The LP-based algorithm for solving correction problem (3)

The main difficulties in solving the problem (3) are that the left-hand sides of system (2) are bilinear in h_i and x. The idea of Vatolin algorithm is to take h_i of the form:

$$h_i = t_i c, \quad i \in M_1, \quad t_i \in \mathbf{R},$$

where $c \in \mathbf{R}^{n+1}$, $c = (c_1,...,c_{n+1})$ is defined bellow.

Thus, the problem (2) is also bilinear, but it can be converted into a linear one by:

a) changing variable $x \in \mathbf{R}^n$ for variable $h_0 \in \mathbf{R}^{n+1}$ so that

$$x = h_{0,n+1}^{-1}(h_{01}, ..., h_{0,n})^T,$$

where it is assumed that $0 \notin M_1$, $h_0 = (h_{01}, ..., h_{0,n}, h_{0,n+1}), h_{0,n+1} > 0$ and by

b) introducing additional constraint

$$\langle c, h_0 \rangle = -1.$$

Consequently, the algorithm reduces solving the correction problem (3) to solving a linear programming problem. If $\Phi(H)$ takes form:

$$\Phi(H) = \max_{i,j} |h_{ij}|, \tag{4}$$

then the vector $c \in \mathbf{R}^{n+1}$ is of the form

$$c_j = \begin{cases} 0, & j \in J_0 \\ -1, & j \in J_1. \end{cases}$$

We have to solve a linear program:

$$\left\{ \begin{array}{l} \quad \min \theta \\ \text{subject to} \\ \quad \langle d_i, h_0 \rangle \leq 0, \ i \in M_0 \\ \quad \langle d_i, h_0 \rangle \leq t_i, \ i \in M_1 \\ \quad 0 \leq t_i \leq \theta, \ i \in M_1 \\ \quad \sum\limits_{j \in J_1} h_{0,j} = 1 \\ \quad h_{0,j} \geq 0, \ j = 1, ..., n+1, \end{array} \right. \tag{5}$$

where $d_i = (a_i, -b_i) \in \mathbf{R}^{n+1}$, $i \in M_0 \cup M_1$. Using the criterion (4), the rows $i \in M_1$ and all columns $j \in J_1$ are effectively corrected.

If $\Phi(H)$ takes form:

$$\Phi(H) = \sum_{i,j} |h_{ij}|,$$

then the number of linear programming problems which will be solved is $|J_1|$. At each linear programming problem, only a column of augmented matrix (A, b) is corrected (see [Po], [PM]).

Let K be the set of feasible solutions (θ, t, h_0) of problem (5), where vector t is composed of components t_i, $i \in M_1$. If $K = \phi$ then S=ϕ. Else, for each optimal solution (θ, t, h_0) of the problem (5), there are obtained the optimal value $\sigma = \theta$, the optimal correction matrix $H = H(t)$ with (i, j)component

$$h_{ij} = \begin{cases} 0, & j \in J_0 \\ -t_i, & j \in J_1 \end{cases}, \ i \in M_1$$

and the solution x of the corrected system

$$x = h_{0,n+1}^{-1}(h_{01}, ..., h_{0,n})^T.$$

3. Interior-point method for solving linear programming problem (5)

The linear program (5) admits an equivalent program in the standard form, obtainable by adding slack variables: $s_i,\ i \in M_0,\ v_i,\ z_i,\ i \in M_1$:

$$\begin{cases} & \min\ \theta \\ \text{subject to} & \\ & \langle d_i, h_0\rangle + s_i = 0,\ i \in M_0 \\ & \langle d_i, h_0\rangle - t_i + v_i = 0,\ i \in M_1 \\ & -\theta + t_i + z_i = 0,\ i \in M_1 \\ & \langle c, h_0\rangle = 1 \\ & \theta, h_0, t, s, v, z \geq 0, \end{cases} \tag{6}$$

where the vectors s, v and z are composed of components $s_i,\ i \in M_0$ and $v_i,$ $z_i,\ i \in M_1$. Note that the strict inequality $h_{0,n+1} > 0$ in (6) was replaced by $h_{0,n+1} \geq 0$.

We introduce the notations: $f = (1, 0, ..., 0)^T$, $g = (0, ..., 1)^T$. Also, $y = (\theta, h_0, t, s, v, z)^T$ denotes vector composed of $\theta \in \mathbf{R}$ and vectors h_0, t, s, v and z. The coefficient matrix in the linear problem (6) is:

$$G = \begin{pmatrix} 0 & d_i & 0 & I_{m-p} & 0 & 0 \\ 0 & d_i & -I_p & 0 & I_p & 0 \\ -e & 0 & I_p & 0 & 0 & I_p \\ 0 & c^T & 0 & 0 & 0 & 0 \end{pmatrix},$$

where e stands for the all-one vector, $e = (1, 1, ..., 1)^{T\cdot}$ Matrix G has a full row rank. Using these notations, the problem (6) becomes:

$$(P) \qquad \begin{cases} \min f^T y \\ \textit{subject to} \\ Gy = g \\ y \geq 0. \end{cases}$$

We define the feasible set $\mathcal{P}$ to be the set of vectors y satisfying the constraints, i.e.

$$\mathcal{P} = \{y /\ Gy = g \text{ and } y \geq 0\},$$

and the associated set $\mathcal{P}^+$ to be the subset of $\mathcal{P}$ satisfying strict nonnegativity constraints

$$\mathcal{P}^+ = \{y /\ Gy = g \text{ and } y > 0\}.$$

Interior-point methods are iterative methods that compute a sequence of iterates belonging to $\mathcal{P}^+$ and converging to an optimal solution.
This is completely different from the simplex method which explores the vertices of the polyhedron $\mathcal{P}$ and an exact optimal solution is obtained after a finite number of steps (see fig. 0.1).

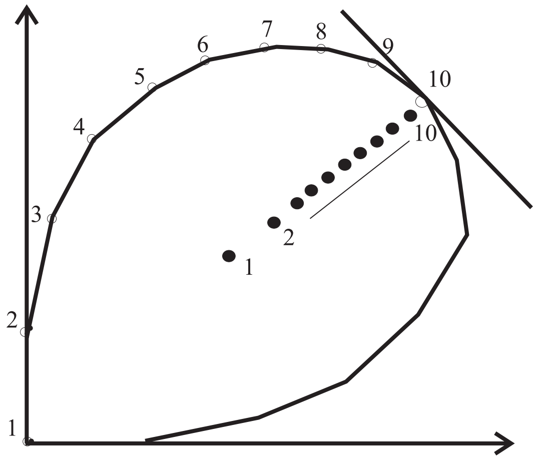

Figure 1:

Interior-point iterates tend to an optimal solution but never attain it (since the optimal solutions do not belong to $\mathcal{P}^+$ but to $\mathcal{P}\backslash\mathcal{P}^+$). Yet an approximate solution (with e.g. 10^{-8} relative accuracy) is sufficient for our purpose. In addition, these methods are practically efficient and can be used to solve large-scale problems. For such of problems, the chances that the system is self-contradictory (inconsistent) are high.

Since 1984, when Karmarkar introduced this new class of methods in [Ka], many different interior-point methods have been developed. For solving linear program (P), we will use the *affine-scaling method* which has been previously proposed by Dikin, 17 years before Karmarkar. This is in fact a projective gradient method. Also, at each iteration, the y variable is simply scaled by $y = Dw$, where D is a positive diagonal matrix (this scaling operation is responsible for the denomination of the method).

Let us consider the current iterate $y_{k>0}$ and $D = Y_k$, where Y_k is diagonal matrix made up with vector y_k. Choosing this special matrix, which maps the current iterate y_k to e $(Y_k^{-1} y_k = e)$, we obtain the following problem:

$$(P_D) \qquad \begin{cases} \qquad \min(Y_k f)^T w \\ \textit{subject to} \\ \qquad GY_k w = g \\ \qquad w \geq 0. \end{cases}$$

It is easy to show that problem (P_D) is equivalent to (P). We introduce the notations $G_k = GY_k$ and $f_k = Y_k f$.

The scaled current iterate w_k is located far inside of the polyhedron (see fig. 0.2), at equal distance to each face. The iterate is centered because by this, a significant shift toward the optimal solution y^* can be executed (without touching the faces of the polyhedron).

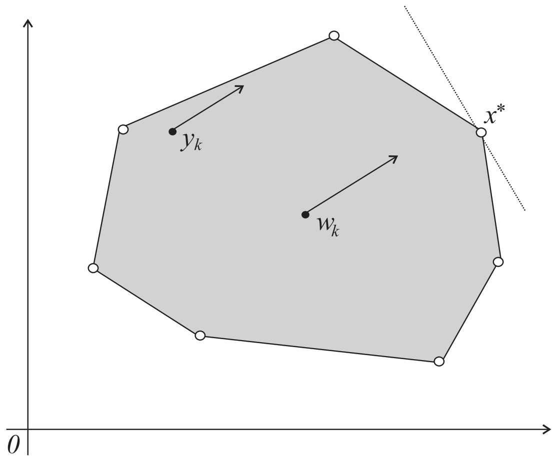

Figure 2:

The displacing direction for the scaled problem $\triangle w_k$ is defined as the projection of the scaled problem gradient, with sign changed $-f_k$, onto $ker(G_k)$. The projection matrix onto $ker(G_k)$ is:

$$P_{G_k} = I - G_k^T \left(G_k G_k^T\right)^{-1} G_k.$$

Then,

$$\triangle w_k = -P_{G_k} f_k = -Y_k[f - G^T (GY_k^2 G^T)^{-1} GY_k^2 f].$$

Using the notation

$$u_k = (GY_k^2 G^T)^{-1} GY_k^2 f, \tag{7}$$

the displacing direction $\triangle w_k$ becomes:

$$\triangle w_k = -Y_k \left(f - G^T u_k \right).$$

Back in the original space we obtain

$$\triangle y_k = Y_k \triangle w_k = -Y_k^2 (f - G^T u_k),$$

where the vector u_k is the solution of the linear system:

$$(GY_k^2 G^T) u_k = GY_k^2 f. \tag{8}$$

The next iterate $y_{k+1} = y_k + \triangle y_k$ is expected to be closer to the optimal solution than y_k.

Since the iterates must always satisfy the strict nonnegativity conditions, we will reduce the step with a factor $\alpha_k < 1$ in order to make it stay within the strictly feasible region $\mathcal{P}^+$:

$$y_{k+1} = y_k + \alpha_k \triangle y_k.$$

The sequence of iterates is converging to the optimal solution (see [An]). While the simplex method may potentially make a number of moves that grows exponentially with the problem size, interior-point methods need a number of iterations that is polynomially bounded by the problem size to obtain a given accuracy.

The stopping criterion is usually a small predefined duality gap ε :

$$\frac{\| f^T y_k - g^T u_k \|}{\| f^T y_k \| + 1} < \varepsilon,$$

where u_k, defined in (7), are the dual variables.

The resolution of the system (8) takes up most of the computing time in affine-scaling algorithm (80-90% of the total CPU-time). It should be therefore very carefully implemented (with a Cholesky factorization, taking advantage of the fact that matrix $GY_k^2 G^T$ is positive definite and with application of sparse matrix techniques).

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