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EVALUATING THE BIVARIATE COMPOUND GENERALIZED POISSON DISTRIBUTION

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

This paper presents a method for the evaluation of the compound Generalized Poisson distribution, based on the recursive evaluation of two other distributions.

1. Introduction

In the univariate case, a compound distribution is the distribution of a sum of independent and identically distributed (i.d.) random variables (r.v.-s). The number of terms is itself a r.v. assumed to be independent of the terms. Denoting by p the distribution of the number of terms (the counting distribution), by h the distribution of the terms (the severity distribution) and by g then the compound distribution, then we have

$$g(x) = \sum_{n=0}^{\infty} p\left(n\right) h^{n*}\left(x\right).$$

An important attention was paid lately in actuarial mathematics to the recursive evaluation of bi and multivariate compound distributions. When extending the concept of compound distribution to the multivariate case, we have two directions:

A. The counting distribution is still univariate, but the severities are independent and i.d. random vectors of dimension m. For this case recursions have been studied in [2] and [16].

B. The counting distribution is multivariate and the severities are one-dimensional. See [1], [11], [20] and [22].

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The two approaches can be combined.

In this paper we present a method for the evaluation of the bivariate compound Generalized Poisson distribution obtained when the counting distribution is the Bivariate Generalized Poisson distribution (BGPD) and the severities are one-dimensional.

Consul and Jain [5] provided an alternative to the standard Poisson distribution by introducing the univariate Generalized Poisson distribution (GPD). Details on this distribution can be found in Consul's book [4]. From its applications in the actuarial literature we note [3], [10], [12] etc.

The GPD was extended to the bivariate case and was applied in the insurance field in [9], [14], [19]. A multivariate generalization of the GPD was also studied in [21].

Section 2 of this paper recalls the form of the GPD and BGPD.

In section 3 we present a recursive formula for a multivariate compound distribution obtained when the counting distribution is the univariate GPD and the severity distribution is multivariate. Such multivariate compound distributions were already studied in [2] and [16], but with the counting distribution satisfying Panjer's relation [13], $p(n) = (a + b/n)\, p(n-1),\ n \geq 1$. In section 4 we derive a recursion for a bivariate compound distribution with the bivariate counting distribution given by (N_1, N_2), where N_1, N_2 are independent and each one follows an univariate GPD (e.g. N_i is the number of claims of type i). The bivariate compound GPD is evaluated in section 5 as the convolution of two bivariate compound distributions which can be recursively calculated as in section 3 and 4. Since the complexity of the calculations required for this method is quite important, we also present a criterion for choosing between the bivariate compound GPD and the bivariate compound Poisson distribution.

In the following we will use the notations $\mathbf{x} = (x_1, ..., x_m)'$, $\mathbf{u} = (u_1, ..., u_m)'$. By $\mathbf{u} \leq \mathbf{x}$ we mean that $u_j \leq x_j$ for $j = 1, ..., m$ and by $\mathbf{u} < \mathbf{x}$ that $u_j \leq x_j$, $j = 1, ..., m$, with at least one strict inequality. By $\mathbf{0}$ we denote the $m \times 1$ vector consisting of only zeros and by $\mathbf{e}_j$ the jth $m \times 1$ unit vector. We make the convention $\sum_{i=c}^{d} = 0$ when $d < c$.

2. The Bivariate Generalized Poisson Distribution (BGPD)

2.1 The univariate Generalized Poisson Distribution (GPD)

The GPD has two parameters λ and θ, with $\lambda > 0$ and $max(-1, -\lambda/q) \leq \theta < 1$, where $q \geq 4$ is the largest positive integer for which $\lambda + \theta q > 0$ when $\theta < 0$. The probability function (p.f.) of $N \sim GPD(\lambda, \theta)$ is given in [6]:

$$p(n) = P(N = n) = \begin{cases} \dfrac{1}{n!}\lambda(\lambda + n\theta)^{n-1}\exp\{-\lambda - n\theta\}\ ,\ n = 0, 1, ... \\ 0\ , \qquad otherwise \end{cases} . \tag{2.1}$$

The GPD reduces to the Poisson distribution for $\theta = 0$. When θ is negative, the GPD model includes a truncation due to the fact that $p(n) = 0$ for all $n > q$. In the following we will consider only the case when $\lambda > 0$, $0 \leq \theta < 1$ and $q = \infty$, situation which gives convenient explicit expressions for the moments (see [4]). The p.f. (2.1) can also be recursively evaluated from

$$p(n; \lambda, \theta) = \frac{\lambda}{\lambda + \theta} \left(\theta + \frac{\lambda}{n} \right) p(n - 1; \lambda + \theta, \theta), \ n \geq 1. \tag{2.2}$$

2.2 The BGPD

Studied in [9] and [19], the BGPD is obtained by the trivariate reduction method as $N_1 = M_1 + M_3$, $N_2 = M_2 + M_3$, where M_1, M_2, M_3 are independent r.v.-s and $M_j \sim GPD(\lambda_j, \theta_j)$, $j = 1, 2, 3$. Then the common p.f. of (N_1, N_2) is given by (see [19])

$$p(n_1, n_2) = P\left(N_1 = n_1, N_2 = n_2\right) = \lambda_1 \lambda_2 \lambda_3 \exp\{-(\lambda_1 + \lambda_2 + \lambda_3) - n_1\theta_1 - n_2\theta_2\} \times$$

$$\sum_{k=0}^{\min(n_1,n_2)} \frac{\exp\{k(\theta_1 + \theta_2 - \theta_3)\}}{(n_1 - k)!(n_2 - k)!k!} (\lambda_1 + (n_1 - k)\theta_1)^{n_1 - k - 1} (\lambda_2 + (n_2 - k)\theta_2)^{n_2 - k - 1} \times$$

$$(\lambda_3 + k\theta_3)^{k-1}, \ n_1, n_2 = 0, 1, ... \tag{2.3}$$

If $\theta_i = 0, i = 1, 2, 3$, (2.3) becomes the usual bivariate Poisson distribution.

3. A multivariate recursion

In the following we will generalize the recursion of [3] for the compound generalized Poisson distribution to a multivariate situation where each claim event generates a random vector.

Let N denote the number of claim events and let $\mathbf{X}_i = (X_{i1}, ..., X_{im})'$ be the m-dimensional claim vector generated by the ith of these events, $i = 1, 2, ...$ Then $\mathbf{S} = (S_1, ..., S_m)' = \sum_{i=1}^{N} \mathbf{X}_i$ is the random vector of the aggregate claims.

As in the univariate case, we assume that $\mathbf{X}_1, \mathbf{X}_2, ...$ are mutually independent and i.d. with p.f. f, and also independent of N. In addition, we assume that the X_{ij}'s are non-negative, but $f(\mathbf{0}) = 0$. Let p and g denote the p.f.-s of N and $\mathbf{S}$, respectively. Then

$$g = \sum_{n=0}^{\infty} p(n)\, f^{*n}. \tag{3.1}$$

Theorem 3.1. Under the assumption that $N \sim GPD(\lambda, \theta)$, the p.f. g satisfies the recursion

$$g(\mathbf{0}; \lambda, \theta) = p(0) = e^{-\lambda}, \tag{3.2}$$

$$x_k\, g(\mathbf{x};\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\le\mathbf{x}}(\theta x_k+\lambda u_k)\, f(\mathbf{u})g(\mathbf{x}-\mathbf{u};\lambda+\theta,\theta),\ \mathbf{x}>\mathbf{0},\, k=1,...,m. \tag{3.3}$$

Proof. Formula (3.2) follows from (3.1) and from
$f^{*n}(\mathbf{0})=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$. In order to prove (3.3) we will use conditional expectations as in the proof of Panjer's formula in [15]. When $\mathbf{x}>\mathbf{0}$, considering (3.1) and the recursion (2.2) of the GPD, we have

$$x_k\, g(\mathbf{x};\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{n=1}^{\infty}x_k\left(\theta+\frac{\lambda}{n}\right)p(n-1;\lambda+\theta,\theta)\, f^{*n}(\mathbf{x})=$$

$$\frac{\lambda}{\lambda+\theta}\sum_{n=1}^{\infty}p(n-1;\lambda+\theta,\theta)\, E\left[\theta x_k+\lambda X_{1k}\left|\sum_{i=1}^{n}\mathbf{X}_i=\mathbf{x}\right.\right]\, f^{*n}(\mathbf{x})=$$

$$\frac{\lambda}{\lambda+\theta}\sum_{n=1}^{\infty}p(n-1;\lambda+\theta,\theta)\sum_{0\le\mathbf{u}\le\mathbf{x}}(\theta x_k+\lambda u_k)\ f(\mathbf{u})\, f^{*(n-1)}(\mathbf{x}-\mathbf{u})=$$

$$\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\le\mathbf{x}}(\theta x_k+\lambda u_k)\, f(\mathbf{u})\sum_{n=1}^{\infty}p(n-1;\lambda+\theta,\theta)\, f^{*(n-1)}(\mathbf{x}-\mathbf{u})=$$

$$\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\le\mathbf{x}}(\theta x_k+\lambda u_k)\, f(\mathbf{u})g(\mathbf{x}-\mathbf{u};\lambda+\theta,\theta).\ \Box$$

Rem. When $x_k>0$, dividing (3.3) by x_k gives

$$g(\mathbf{x};\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\le\mathbf{x}}\left(\theta+\lambda\frac{u_k}{x_k}\right)f(\mathbf{u})g(\mathbf{x}-\mathbf{u};\lambda+\theta,\theta),\ \mathbf{x}\ge\mathbf{e}_k. \tag{3.4}$$

Starting with (3.2) and using (3.4) we can recursively evaluate g.

Corollary 3.1. Under the assumptions of Theorem 3.1, if $\mathbf{c}$ is an $m\times 1$ constant vector, then for $\mathbf{x}>\mathbf{0}$

$$g(\mathbf{x};\lambda,\theta)\,\mathbf{c}'\mathbf{x}=\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\le\mathbf{x}}(\theta\,\mathbf{c}'\mathbf{x}+\lambda\,\mathbf{c}'\mathbf{u})\ f(\mathbf{u})\, g(\mathbf{x}-\mathbf{u};\lambda+\theta,\theta). \tag{3.5}$$

Proof. Formula (3.5) can be obtained if we multiply (3.3) by c_k and sum over k. $\Box$

Corollary 3.2. Under the assumptions of Theorem 3.1 we also have

$$g(\mathbf{x};\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{0<\mathbf{u}\leq\mathbf{x}}\left(\theta+\lambda\left(\sum_{k=1}^{m}u_k\right)\left(\sum_{k=1}^{m}x_k\right)^{-1}\right)\times$$
$$\times f(\mathbf{u})\,g(\mathbf{x}-\mathbf{u};\lambda+\theta,\theta),\ \mathbf{x}>\mathbf{0}. \tag{3.6}$$

Proof. In Corollary 3.1 we take $c_j=1$, $j=1,...,m$. Then $\mathbf{c}'\mathbf{x}=\sum_{k=1}^{m}x_k>0$ for $\mathbf{x}>\mathbf{0}$ and we can divide (3.5) by $\mathbf{c}'\mathbf{x}$ to get (3.6). □

Particular cases

1. When $\theta=0$, the GPD becomes the standard Poisson distribution, i.e. $N\sim\mathcal{P}(\lambda)$. Then Theorem 3.1 becomes a particular case of Theorem 1 in [16] for $f(\mathbf{0})=0$.

2. When $m=1$, (3.4) reduces to the univariate recursion (see [3])

$$g(x;\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{u=1}^{x}\left(\theta+\lambda\frac{u}{x}\right)f(u)\,g(x-u;\lambda+\theta,\theta),\ x>0, \tag{3.7}$$

3. When $m=2$, (3.4) gives for $x_1\geq 1$, $x_2\geq 0$,

$$g(x_1,x_2;\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{u_1=0}^{x_1}\left(\theta+\lambda\frac{u_1}{x_1}\right)\sum_{u_2=0}^{x_2}f(u_1,u_2)\,g(x_1-u_1,x_2-u_2;\lambda+\theta,\theta), \tag{3.8}$$

and a similar formula for $x_1\geq 0$, $x_2\geq 1$. For $(x_1,x_2)\neq(0,0)$, from (3.6) we also have

$$g(x_1,x_2;\lambda,\theta)=\frac{\lambda}{\lambda+\theta}\sum_{u_1=0}^{x_1}\sum_{u_2=0}^{x_2}\left(\theta+\lambda\frac{u_1+u_2}{x_1+x_2}\right)f(u_1,u_2)\,g(x_1-u_1,x_2-u_2;\lambda+\theta,\theta). \tag{3.9}$$

The fact that we have included $(u_1,u_2)=(0,0)$ in these formulas in order to simplify their display does not create any problem since we have assumed that $f(0,0)=0$.

In table 1 we present the order of the recursive evaluation based on (3.2) and e.g. (3.9). The columns are labeled with the parameters of the distribution of (S_1,S_2) and the evaluation must be done line by line.

TABLE 1. Recursive evaluation of g

$\Delta_1=(\lambda,\theta)$	$\Delta_2=(\lambda+\theta,\theta)$	$\Delta_3=(\lambda+2\theta,\theta)$	$\Delta_4=(\lambda+3\theta,\theta)$
$g(0,0;\Delta_1)$	$g(0,0;\Delta_2)$	$g(0,0;\Delta_3)$	$g(0,0;\Delta_4)$
$g(0,1;\Delta_1);g(1,0;\Delta_1)$	$g(0,1;\Delta_2);g(1,0;\Delta_2)$	$g(0,1;\Delta_3);g(1,0;\Delta_3)$	
$g(0,2;\Delta_1);g(2,0;\Delta_1)$ $g(1,1;\Delta_1)$	$g(0,2;\Delta_2);g(2,0;\Delta_2)$ $g(1,1;\Delta_2)$		
$g(0,3;\Delta_1);g(3,0;\Delta_1)$ $g(1,2;\Delta_1);g(2,1;\Delta_1)$			

4. A bivariate recursion

We will now present a recursion for the bivariate compound distribution with bivariate counting distribution obtained for example when the policies of a portfolio are submitted to claims of two kinds, whose frequencies are independent. Let N_i denote the number of claims of type $i, i = 1, 2$, and $(X_j)_{j\geq 1}, (Y_j)_{j\geq 1}$ the amounts of type 1 and type 2 claims, respectively. Then denoting by S_1, S_2 the corresponding aggregate amounts of the claims of type 1 and 2, we have

$$\mathbf{S} = (S_1, S_2) = \left(\sum_{j=1}^{N_1} X_j, \sum_{j=1}^{N_2} Y_j \right). \tag{4.1}$$

We consider the following hypotheses for this model:

(H1) The r.v.-s $(X_j)_{j\geq 1}$ are independent, i.d. and defined only on positive integers, with the common p.f. f_1. Same hypotheses for $(Y_j)_{j\geq 1}$, but their common p.f. is f_2.

(H2) The r.v.-s $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ are independent.

(H3) For $i = 1, 2$, the r.v.-s N_i and $(X_j)_{j\geq 1}$ are independent, and so are N_i and $(Y_j)_{j\geq 1}$.

(H4) The r.v.-s N_1, N_2 are independent and $N_i \sim GPD(\lambda_i, \theta_i)$, $i = 1, 2$.

Denoting the p.f. of $\mathbf{S}$ by g and of S_i by $g_i, i = 1, 2$, from the independence assumptions we have for $\Lambda = (\lambda_1, \lambda_2), \Theta = (\theta_1, \theta_2)$,

$$g(x_1, x_2; \Lambda, \Theta) = g_1(x_1; \lambda_1, \theta_1)\, g_2(x_2; \lambda_2, \theta_2). \tag{4.2}$$

It is easy to see that S_i follows an univariate compound GPD, so g_i can be recursively calculated using (3.7), $i = 1, 2$. The following theorem gives a recursion for g which avoids the evaluation of g_i.

Theorem 4.1. Under the assumptions (H1)-(H4) we have

$$g(0, 0; \Lambda, \Theta) = e^{-(\lambda_1+\lambda_2)}, \tag{4.3}$$

$$g(x_1, 0; \Lambda, \Theta) = \frac{\lambda_1 e^{\theta_2}}{\lambda_1 + \theta_1} \sum_{y_1=1}^{x_1} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1} \right) f_1(y_1)\, g(x_1 - y_1, 0; \Lambda + \Theta, \Theta),\ x_1 \geq 1, \tag{4.4}$$

$$g(0, x_2; \Lambda, \Theta) = \frac{\lambda_2\, e^{\theta_1}}{\lambda_2 + \theta_2} \sum_{y_2=1}^{x_2} \left(\theta_2 + \lambda_2 \frac{y_2}{x_2} \right) f_2(y_2)\, g(0, x_2 - y_2; \Lambda + \Theta, \Theta),\ x_2 \geq 1, \tag{4.5}$$

$$g(x_1, x_2; \Lambda, \Theta) = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \theta_1)(\lambda_2 + \theta_2)} \sum_{y_1=1}^{x_1} \sum_{y_2=1}^{x_2} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1} \right) \left(\theta_2 + \lambda_2 \frac{y_2}{x_2} \right) \times$$

$$f_1(y_1)\, f_2(y_2)\, g(x_1 - y_1, x_2 - y_2; \Lambda + \Theta, \Theta),\ x_1, x_2 \geq 1. \tag{4.6}$$

Proof. (4.3) is immediate from (4.2) and (3.2). In order to prove (4.4)-(4.6) we will use (4.2) and (3.7):

$$g(x_1, 0; \Lambda, \Theta) = g_1(x_1; \lambda_1, \theta_1)\, e^{-\lambda_2} =$$

$$= \frac{\lambda_1 e^{\theta_2}}{\lambda_1 + \theta_1} \sum_{y_1=1}^{x_1} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1}\right) f_1(y_1)\, g_1(x_1 - y_1; \lambda_1 + \theta_1, \theta_1)\, g_2(0; \lambda_2 + \theta_2, \theta_2) =$$

$$= \frac{\lambda_1 e^{\theta_2}}{\lambda_1 + \theta_1} \sum_{y_1=1}^{x_1} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1}\right) f_1(y_1)\, g(x_1 - y_1, 0; \Lambda + \Theta, \Theta).$$

Similarly for (4.5). For the last recursion we have

$$g(x_1, x_2; \Lambda, \Theta) =$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_1 + \theta_1)\,(\lambda_2 + \theta_2)} \left[\sum_{y_1=1}^{x_1} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1}\right) f_1(y_1)\, g_1(x_1 - y_1; \lambda_1 + \theta_1, \theta_1)\right] \times$$

$$\left[\sum_{y_2=1}^{x_2} \left(\theta_2 + \lambda_2 \frac{y_2}{x_2}\right) f_2(y_2)\, g_2(x_2 - y_2; \lambda_2 + \theta_2, \theta_2)\right] =$$

$$\frac{\lambda_1 \lambda_2}{(\lambda_1 + \theta_1)\,(\lambda_2 + \theta_2)} \sum_{y_1=1}^{x_1} \sum_{y_2=1}^{x_2} \left(\theta_1 + \lambda_1 \frac{y_1}{x_1}\right) \left(\theta_2 + \lambda_2 \frac{y_2}{x_2}\right) f_1(y_1)\, f_2(y_2) \times$$

$$g_1(x_1 - y_1; \lambda_1 + \theta_1, \theta_1)\, g_2(x_2 - y_2; \lambda_2 + \theta_2, \theta_2),$$

and using (4.2) once more we get (4.6). □

In table 2 we present the order of the recursive evaluation based on (4.3)-(4.6). The columns are labeled with the parameters of the distribution of (S_1, S_2) and the evaluation must be done line by line.

TABLE 2. Recursive evaluation of g

$\Delta_1 = (\Lambda, \Theta)$	$\Delta_2 = (\Lambda + \Theta, \Theta)$	$\Delta_3 = (\Lambda + 2\Theta, \Theta)$	$\Delta_4 = (\Lambda + 3\Theta, \Theta)$
$g(0,0;\Delta_1)$	$g(0,0;\Delta_2)$	$g(0,0;\Delta_3)$	$g(0,0;\Delta_4)$
$g(0,1;\Delta_1); g(1,0;\Delta_1)$ $g(1,1;\Delta_1)$	$g(0,1;\Delta_2); g(1,0;\Delta_2)$ $g(1,1;\Delta_2)$	$g(0,1;\Delta_3); g(1,0;\Delta_3)$ $g(1,1;\Delta_3)$	
$g(0,2;\Delta_1); g(2,0;\Delta_1)$ $g(1,2;\Delta_1); g(2,1;\Delta_1)$ $g(2,2;\Delta_1)$	$g(0,2;\Delta_2); g(2,0;\Delta_2)$ $g(1,2;\Delta_2); g(2,1;\Delta_2)$ $g(2,2;\Delta_2)$		
$g(0,3;\Delta_1); g(3,0;\Delta_1)$ $g(3,1;\Delta_1); g(1,3;\Delta_1)$ $g(3,2;\Delta_1); g(2,3;\Delta_1)$ $g(3,3;\Delta_1)$			

5. A method for the evaluation of the bivariate compound GPD

5.1 The method

We consider the bivariate compound distribution given in (4.1), but we replace the hypothesis (H4) by

(H4') The random vector $(N_1, N_2) \sim BGPD\,(\lambda_1, \lambda_2, \lambda_3; \theta_1, \theta_2, \theta_3)$.

The other hypotheses are the same, and so are the notations. From section 2 it follows that $N_1 = M_1 + M_3$ and $N_2 = M_2 + M_3$, with M_1, M_2, M_3 independent and $M_j \sim GPD(\lambda_j, \theta_j), j = 1, 2, 3$.

The example 3.1 in [2] gives a method for the evaluation of a bivariate compound distribution whose counting distribution is obtained as above. The method consists in writting $\mathbf{S} = \mathbf{U}_1 + \mathbf{U}_2$, where:

- $\mathbf{U}_1$ is a bivariate compound random vector as in section 4, having the counting distribution given by (M_1, M_2) and the claim sizes p.f.-s f_X and f_Y, respectively;

- $\mathbf{U}_2$ is also a bivariate compound random vector, but as in section 3, with the univariate counting distribution given by the r.v. M_3 and same claim sizes as $\mathbf{U}_1$, so that $f(x_1, x_2) = f_X(x_1)\, f_Y(x_2)$.

We denote the p.f. of $\mathbf{U}_j$ by $g_{\mathbf{U}_j}$, $j = 1, 2$. Then $g_{\mathbf{S}} = g_{\mathbf{U}_1} * g_{\mathbf{U}_2}$, i.e.

$$g_{\mathbf{S}}(x_1, x_2) = \sum_{(y_1, y_2)=(0,0)}^{(x_1, x_2)} g_{\mathbf{U}_1}(y_1, y_2)\, g_{\mathbf{U}_2}(x_1 - y_1, x_2 - y_2),$$

where $g_{\mathbf{U}_1}$ can be recursively evaluated as in Theorem 4.1 and the same for $g_{\mathbf{U}_2}$, using (3.8) or (3.9).

5.2 A criterion for choosing between the bivariate compound GPD and the bivariate compound Poisson distribution

It is easy to see that the complexity of the calculations involved in the evaluation of the bivariate compound GPD as described before is quite important. Since the bivariate compound Poisson distribution can be recursively evaluated with less calculations (see [11], [19]), it is natural to try to find out if there is an important difference between the two distributions before starting the evaluation of the bivariate compound GPD. In order to do this, we propose a criterion based on the distance between two bivariate distributions g and h, distance defined in [17] as

$$\varepsilon(g, h) = \sum_{x_1, x_2} |g(x_1, x_2) - h(x_1, x_2)|\,.$$

Let $g_{\mathbf{S}}$ be the p.f. of the bivariate compound GPD as above and $h_{\mathbf{S}'}$ the p.f. of the bivariate compound Poisson distribution given by $\mathbf{S}' = \left(\sum_{j=1}^{N_1'} X_j, \sum_{j=1}^{N_2'} Y_j \right)$, with (N_1', N_2') following a bivariate Poisson distribution with parameters $\left(\lambda_1', \lambda_2', \lambda_3'\right)$. Its p.f. is then given by

$$P\left(N_1' = n_1, N_2' = n_2\right) = e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{k=0}^{\min\{n_1,n_2\}} \frac{\lambda_1^{n_1-k}\ \lambda_2^{n_2-k}\ \lambda_3^{k}}{(n_1-k)!(n_2-k)!k!}\ ,\ n_1, n_2 \geq 0.$$

Since the bivariate Poisson distribution can also be obtained by the trivariate reduction method from three independent r.v.-s M_1', M_2', M_3', $M_j' \sim \mathcal{P}(\lambda_j')$, $j = 1,2,3$, then we can apply Theorem 3.1 in [18], special case (ii), which together with Theorem 4.1 same paper gives

$$\varepsilon\left(g_{\mathbf{S}}, h_{\mathbf{S}'}\right) \leq \sum_{i=1}^{3} \varepsilon\left(p_{M_i}, p_{M_i'}\right), \tag{5.1}$$

where by p_M we denoted the p.f. of the r.v. M. It is interesting that the upper bound does not depend on the severity distribution.

Imposing now the usual condition of equal means $E(M_i) = E(M_i'), i = 1,2,3$, since the GPD is infinitely divisible, we have from [8]

$$\varepsilon\left(p_{M_i}, p_{M_i'}\right) \leq 2\left(\lambda_i' - \frac{p_{M_i}(1)}{p_{M_i}(0)}\right). \tag{5.2}$$

From $E(M_i) = E\left(M_i'\right)$ we obtain $\lambda_i' = \lambda_i\left(1-\theta_i\right)^{-1}$ and $p_{M_i}(1)/p_{M_i}(0) = \lambda_i\, e^{-\theta_i}$, so that (5.1) becomes

$$\varepsilon\left(g_{\mathbf{S}}, h_{\mathbf{S}'}\right) \leq 2\sum_{i=1}^{3} \lambda_i \left(\frac{1}{1-\theta_i} - e^{-\theta_i}\right). \tag{5.3}$$

Then the criterion consists in evaluating an upper bound for $\varepsilon\left(g_{\mathbf{S}}, h_{\mathbf{S}'}\right)$ from (5.3) and decide wether it is little enough, so that we can take the standard bivariate Poisson as counting distribution.

Numerical example

Since the upper bound in (5.3) does not depend on the severity distribution, we will consider only a numerical data set for the bivariate counting distribution of (N_1, N_2).

Vernic [21] fitted the BGPD to the accident data of [7], with N_1 as the accidents in the first period and N_2 in a second period. The parameters estimates obtained by the method of moments are

$$\widetilde{\lambda}_1 = 0.6206, \widetilde{\lambda}_2 = 0.8653, \widetilde{\lambda}_3 = 0.2987;\ \widetilde{\theta}_1 = 0.1057, \widetilde{\theta}_2 = 0.1200, \widetilde{\theta}_3 = 0.0286,$$

and from (5.3) we have $\varepsilon\left(g_{\mathbf{S}}, h_{\mathbf{S}'}\right) \leq 0.737$. In [21] is also considered the situation when $\theta_1 = \theta_2 = \theta_3 = \theta$ and in this case the parameters estimates are $\widetilde{\lambda}_1' = 0.6300, \widetilde{\lambda}_2' = 0.8925, \widetilde{\lambda}_3' = 0.2778;\ \widetilde{\theta} = 0.0935$. The upper bound in (5.3) becomes $\varepsilon\left(g_{\mathbf{S}}, h_{\mathbf{S}'}\right) \leq 0.6927$. Both distributions fit the data and since the bivariate Poisson distribution is not adequate, we should use one of the BGPD-s.

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