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FUZZY SUBFIELDS

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Abstract

The extensions with all fuzzy intermediate fields having the sup property are characterized. The extensions having the intermediate fields chained are described. We prove that any fuzzy intermediate field of such an extension has the sup property.

Let F/K be a field extension and let $\mathcal{I}(F/K) = \{L | L \text{ subfield of } F, K \subseteq L\}$ be the lattice of its intermediate fields (we also call them subextensions of F/K).

1. Definition. Let F/K be an extension of fields and $\mu : F \to [0,1]$ a fuzzy subset of F. We call μ a *fuzzy intermediate field of* F/K if, $\forall x, y \in F$:

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\};$$
$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\} \text{ if } y \neq 0;$$
$$\mu(x) \leq \mu(k),\ \forall k \in K.$$

If $\mu \in \mathcal{FI}(F/K)$, then μ is a constant on K. Let $\mathcal{FI}(F/K)$ denote the set of all fuzzy intermediate fields of F/K.

For any fuzzy subset $\mu : F \to [0,1]$ and $s \in [0,1]$, define the level set $\mu_s := \{x \in F \mid \mu(x) \geq s\}$. It is clear that $s < t$ implies $\mu_t \subseteq \mu_s$.

It is well known that a *fuzzy subset* $\mu : F \to [0,1]$ *is a fuzzy intermediate field if and only if,* $\forall s \in \mathrm{Im}\mu$*, the level set* μ_s *is an intermediate field of* F/K.

A fuzzy subset μ of F is said to have the *sup property* if, for every nonempty subset A of $\mathrm{Im}\mu$, there exists $x \in \{y \in F | \mu(y) \in A\}$ such that $\mu(x) = \sup A$. In other words, μ has the sup property if and only if $\mathrm{Im}\mu$ is well ordered under $\geq$ (for any nonempty subset A of $\mathrm{Im}\mu$, $\sup A \in A$; equivalently, A has a greatest element).

2. Theorem. *Let* F/K *be a field extension. Then the following assertions are equivalent:*

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a) *Every fuzzy intermediate field of F/K has the sup property.*

b) *There are no infinite strictly decreasing sequences of intermediate fields of F/K.*

Proof. a) $\Longrightarrow$ b) Suppose that b) is false. Then there exists an infinite strictly decreasing sequence

$$F = F_0 \supset F_1 \supset ... \supset F_n \supset ... \supset K$$

of intermediate fields of F/K. Fix $a_0 < a_1 < ... < a_n < ...$, with $a_i \in [0,1]$, $\forall a_i \geq 0$, and let $b = \sup\{a_i \mid i \geq 0\}$. Define the fuzzy subset $\mu : F \to [0,1]$ by:

$$\forall x \in F,\ \mu(x) = \begin{cases} a_i & \text{if } x \in F_i \setminus F_{i+1} \\ b & \text{if } x \in \bigcap\limits_{i \geq 0} F_i \end{cases}.$$

μ is well defined, since $\forall x \in F$, we have either $\exists i \geq 0$ such that $x \in F_i \setminus F_{i+1}$ or $x \in \bigcap\limits_{i\geq 0} F_i$. Indeed, these conditions cannot occur simultaneously. If $x \notin \bigcap\limits_{i\geq 0} F_i$, then $\exists i \geq 0$ such that $x \notin F_i$. We have $i > 0$ since $x \in F_0 = F$, so there exists $i > 0$ minimal with $x \notin F_i$. Then $x \in F_{i-1} \setminus F_i$.

It is easy to see that the level subsets of μ are: $\mu_{a_i} = F_i$, $i \geq 0$ and $\mu_b = \bigcap\limits_{i\geq 0} F_i$. These are intermediate fields, so μ is a fuzzy intermediate field.

$\mu \in \mathcal{FI}(F/K)$ has not the sup property, since $\{a_i \mid i \geq 0\} \subseteq \mathrm{Im}\mu$ and $\sup\{a_i \mid i \geq 0\} = b \notin \{a_i \mid i \geq 0\}$.

b) $\Longrightarrow$ a) Suppose, on the contrary, that $\mu \in \mathcal{FI}(F/K)$ has not the sup property. Then there exists a nonempty $A \subseteq \mathrm{Im}\mu$, with $\sup A \notin A$. Let $a_0 \in A$. Since $a_0 \neq \sup A$, $\exists a_1 \in A$ with $a_0 < a_1$. Inductively, we obtain a sequence $a_0 < a_1 < ... < a_n < ...,$, with $a_i \in A$, $\forall i \geq 0$, $\sup\{a_i \mid i \geq 0\} \notin \{a_i \mid i \geq 0\}$. Thus, there exists a strictly decreasing sequence of intermediate fields of F/K,

$$\mu_{a_0} \supset \mu_{a_1} \supset ... \supset \mu_{a_n} \supset ...$$

This contradicts the assumption. $\square$

3. Remark. This result can be applied, mutatis mutandis, to any algebraic structure for which is defined a notion of "fuzzy substructure". For instance, if $(G, \cdot)$ is a group and 1 is its neutral element, a fuzzy subset $\mu : G \to [0,1]$ is called a *fuzzy subgroup* of G if, $\forall x, y \in G$,

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \text{ and } \mu(x^{-1}) \geq \mu(x).$$

It is known that: *μ is a fuzzy subgroup of G if and only if, $\forall s \in \mathrm{Im}\mu$, the level set μ_s is a subgroup of G.*

By replacing in the proof of the Theorem 2 "intermediate field" with "subgroup"; K (the base field) with the trivial subgroup $\{1\}$, one obtains a proof of the following fact:

Let G be a group. Then every fuzzy subgroup of G has the sup property if and only if there are no infinite strictly decreasing sequences of subgroups of G.

Similarly, for a commutative unitary ring $(R, +, \cdot)$, a fuzzy subset $\mu : R \to [0, 1]$ is called a *fuzzy ideal of R* if, $\forall x, y \in R$,

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \text{ and } \mu(xy) \geq \max\{\mu(x), \mu(y)\}.$$

The usual characterization with the level subsets is valid: *μ is a fuzzy ideal of R if and only if, $\forall s \in \mathrm{Im}\mu$, the level set μ_s is an ideal of R.* Theorem 2 reads in this case:

Let R be a unitary commutative ring. Then every fuzzy ideal of R has the sup property if and only if R is Artinian.

4. Definition. [2] Let F/K be an extension of fields and $\mu \in \mathcal{FI}(F/K)$. Then μ is called a *fuzzy chain subfield* of F/K if $\forall x, y \in F$, $\mu(x) = \mu(y) \iff K(x) = K(y)$.

Here is a fuzzy characterization of the fact that $\mathcal{I}(F/K)$ is a chain.

5. Theorem. [[2], Th. 3.3]. *The intermediate fields of F/K are chained if and only if F/K has a fuzzy chain subfield.* □

We investigate the property that an extension F/K has $\mathcal{I}(F/K)$ a chain.

6. Theorem. *Let F/K be an extension such that the intermediate fields of F/K are chained. Then:*

a) *F/K is algebraic.*

b) *Any intermediate field L of F/K with $L \neq F$ is a finite simple extension of K.*

c) *$(\mathcal{I}(F/K), \subseteq)$ satisfies the descending chain condition (there is no strictly decreasing sequence of intermediate fields of F/K). Thus, $(\mathcal{I}(F/K), \subseteq)$ is well ordered.*

Proof. a) Suppose F contains an element x transcendental over K. It is easy to check that $K(x^2)$ and $K(x^3)$ are intermediate fields of F/K and are not included one in each other.

b) F is algebraic over K, so F is the union of its intermediate fields that are finite over K. Let $L \in \mathcal{FI}(F/K)$, $L \neq F$. Then L cannot include every finite subextension of F/K (because then $L = F$). So there exists $E \in \mathcal{I}(F/K)$, E/K finite, with $E \nsubseteq L$. Since $\mathcal{I}(F/K)$ is a chain, $L \subseteq E$ and L/K is finite. Then $\mathcal{I}(F/K)$ is also a chain and it is finite (otherwise $[L : K]$ would be infinite). Using the fact that a finite extension is simple if and only if it has a finite number of intermediate fields, we see that L/K is simple.

c) Suppose $F_i \in \mathcal{I}(F/K)$, $\forall i \in N$, such that $F_0 \supset F_1 \supset ... \supset F_n \supset ....$ But then $[F_1 : K]$ is infinite, contradicting ii). $\square$

7. Example. a) There exist algebraic extensions F/K such that $\mathcal{I}(F/K)$ does not satisfy the descending chain condition. Let $P_0 = \{p_1, ..., p_n, ...\}$ be the prime positive integers and let $P_n = P_0\{p_1, ..., p_n\}$, $R_n = \{x^{1/2} \mid x \in P_n\}$, $\forall n \in \mathbb{N}$, $n \geq 1$. Then $\mathbb{Q}(R_0)/\mathbb{Q}$ is algebraic and $\mathbb{Q}(R_0) \supset \mathbb{Q}(R_1) \supset \mathbb{Q}(R_2) \supset ...$ is an infinite strictly descending chain.

b) There exist algebraic infinite extensions F/K with ($\mathcal{I}(F/K)$ a chain. The extension $\mathbb{Q} \subseteq \mathbb{Q}(\{2^{1/2^n} \mid n \in \mathbb{N}\}) = F$ has its intermediate fields chained:

$$\mathbb{Q} \subseteq \mathbb{Q}(2^{1/2}) \subseteq \mathbb{Q}(2^{1/4}) \subseteq \mathbb{Q}(2^{1/8}) \subseteq ... \subseteq F.$$

But $F/\mathbb{Q}$ is not finite.

8. Proposition. *A finite purely inseparable extension F/K has $\mathcal{I}(F/K)$ chained if and only if F/K is simple.*

Proof. Since $\mathcal{I}(F/K)$ is a chain and F/K is finite, F/K is simple by Theorem 6. If F/K is simple, purely inseparable of characteristic $p > 0$, let $F = K(a)$ and let $g = X^n - a^n$ be the minimal polynomial of a over K. Then $n = p^e$, where $e \in \mathbb{N}$ is minimal with the property $a^{p^e} \in K$. Let $L \in \mathcal{I}(F/K)$. Then a is purely inseparable over L and let d be minimal with $a^{p^d} \in L$. We claim that $L = K(a^q)$, where $q = p^d$. Obviously, $K(a^q) \subseteq L$ and $[K(a) : K(a^q)] \leq q$. But $q = [K(a) : L] \leq [K(a) : K(a^q)]$ and this forces $L = K(a^q)$. Thus $\mathcal{I}(F/K) = \{K(a^{p^d}) \mid \leq d \leq e\}$ and this is a chain. $\square$

9. Corollary. *Let F/K be a field extension.*

a) *Assume that any proper intermediate field of F/K is a finite extension of K. Then every $\mu \in \mathcal{FI}(F/K)$ has the* sup *property.*

b) *If the intermediate fields of F/K are chained, then every $\mu \in \mathcal{FI}(F/K)$ has the* sup *property.*

c) *If every $\mu \in \mathcal{FI}(F/K)$ has the sup property, then F/K is algebraic.*

Proof. a) Suppose there exists $(F_i)_{i\geq 0}$, $F_i \in \mathcal{I}(F/K)$, $\forall i \in \mathbb{N}$, such that $F_0 \supset F_1 \supset ... \supset F_n \supset ....$ But then $[F_1 : K]$ is infinite, contradiction. Thus, the condition b) in Theorem 2 is satisfied.

b) It follows from a) and Theorem 6.c).

c) Assume F/K is not algebraic and let $x \in F$, transcendental over K. Then $K(x) \supset K(x^2) \supset K(x^4) \supset ... \supset K(x^{2^n}) \supset ...$ is an infinite strictly descending chain, contradiction with Theorem 2.b). $\square$

These results allow us to generalize some results in [2].

10. Definition. [1] Let $\mu \in \mathcal{FI}(F/K)$. Then μ is said to be *fuzzy simple* if it has a minimal generating set φ over K such that if c_s and $d_t \in \varphi$ with $t > s$, then $d \in K(c)$.

11. Theorem. [1] *Let $\mu \in \mathcal{FI}(F/K)$. Then μ is fuzzy simple if and only if μ_s is a simple extension of K, $\forall s \in \mathrm{Im}\mu$.*

12. Definition. [2] Let $\mu \in \mathcal{FI}(F/K)$. Suppose μ has the sup property. If $s \in \mathrm{Im}\mu$, let s' denote the predecessor of s in the well ordered set $\mathrm{Im}\mu$ with $\geq$. Then μ is said to be of *maximal chain* if $\forall s \in \mathrm{Im}\mu \setminus \{(0)\}$ and and $\forall L \in \mathcal{I}(F/K)$, $\mu_{s'} \subseteq L \subseteq \mu_s$ implies either $L = \mu_s$ or $L = \mu_{s'}$.

μ is said to be *reduced* in F over K if $\forall s \in \mathrm{Im}\mu$ and $\forall c, d \in \mu_s \setminus \mu_{s'}$, $\mu(c) = \mu(d)$ implies $K(c) = K(d)$.

The following result characterizes extensions having $\mathcal{I}(F/K)$ chained. It generalizes [[2], Th. 3.7]: Theorem 9 allows to drop the assumption that μ has the sup property.

13. Theorem. *Let $\mu \in \mathcal{FI}(F/K)$. The following conditions are equivalent:*

i) *μ is a fuzzy chain subfield of F/K.*

ii) *$\mathcal{I}(F/K)$ is a chain.*

iii) *$\mu_{\mu(c)} = K(c)$, $\forall c \in F$.*

iv) *μ is fuzzy simple, reduced, and of maximal chain.* $\square$

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