An. Şt. Univ. Ovidius Constanţa Vol. 10(1), 2002, 1–34

EXISTENCE PROBLEMS FOR $\omega-$CLOSED ORBITS

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Abstract

Hereafter we denote by $\omega(x)$ the $\omega-$limit set associated to a solution x of a differential system. The orbit corresponding to this solution is called $\omega-$closed if $\omega(x) = \{x(0)\}$. In this paper, the existence problem for such orbits is considered.

1 Introduction

Let us consider the differential system

$$\dot{x} = f(t, x), \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $\mathbb{R}_+ = [0, +\infty)$. We suppose that every solution of this system exists on $\mathbb{R}_+$.

For x a solution of system (1.1), $\omega(x)$ denotes the set

$$\omega(x) := \{\xi \in \mathbb{R}^n,\ (\exists)\ t_m \in \mathbb{R}_+,\ t_m \to \infty,\ x(t_m) \to \xi\}.$$

When studying a differential system, one would like to get an information as rich as possible about the asymptotic behavior of its trajectories. In particular, it is natural to study the structure and the properties of the limit points of a given solution, i.e. for the $\omega-$limit sets.

For planar dynamical systems, the interest in this problem goes back to Poincaré, but surprisingly enough, a complete characterization of their limit set has not been given yet (for details see [2] and its references).

The present paper intends to bring a contribution in this field; more precisely, we intend to obtain some existence results for a special class of orbits, the so-called $\omega-$closed ones.

Key Words: Boundary value problems on infinite interval, degree theory.
Mathematical Reviews subject classification: 34B99, 47H11

Definition 1. *Let x be a solution of the system* (1.1) *and let $\omega(x)$ be its $\omega-$limit set. We say that the correspondent orbit of x is $\omega-$closed if*

$$\omega(x) = \{x(0)\}. \tag{1.2}$$

The condition (1.2) is equivalent to following two:

$$(\exists) \lim_{t\to\infty} x(t) := x(+\infty) \in \mathbb{R}^n, \tag{1.3}$$

$$x(0) = x(\infty). \tag{1.4}$$

A solution which satisfies (1.3) is called a **convergent solution**; existence problems for such convergent solutions have been treated by many authors (see e.g. [1], [3], [4], [6], [7], [8], [9], [15], [16], [17], [18]).

The condition (1.4) can be regarded as a generalization of the boundary condition

$$x(0) = x(T), \ 0 < T < \infty. \tag{1.5}$$

The problem (1.1)+(1.5) plays a central role in the theory of periodic solutions; important results in this field were obtained by J. Mawhin(see [5], [11], [12], [13]) and rely on the topological degree theory. Some of the techniques used by J. Mawhin will be adjusted for the problem (1.1) + (1.2) in this work. The main idea consists in reducing of the problem (1.1) + (1.2) up to a fixed point problem for some operator working in an adequate space.

We end this section with some necessary notations and preliminary results.

The $n-$dimensional space $\mathbb{R}^n$ is endowed with the usual inner product $\langle\cdot\mid\cdot\rangle$ and with the euclidean norm $|\cdot|$.

$\mathrm{C}_c = C_c(\mathbb{R}_+, \mathbb{R}^n)$ is the space of the continuous applications $x : \mathbb{R}_+ \to \mathbb{R}^n$ endowed with the topology of uniform convergence on every compact interval of $\mathbb{R}_+$.

C_R denotes the space

$$C_R = C_R(\mathbb{R}_+, \mathbb{R}^n) := \left\{ x \in C_c, \ (\exists) \int_0^\infty x(t)\,dt := \lim_{A\to\infty} \int_0^A x(t)\,dt \in \mathbb{R}^n \right\}$$

and becomes a Fréchet space, but its topology is complicated enough. Also, let us consider

$$\begin{aligned} C_l \ &: \ = \left\{ x \in C_c, \ (\exists) \ x(\infty) := \lim_{t\to\infty} x(t) \right\}, \\ C_{0l} \ &: \ = \{ x \in C_l, \ x(0) = x(\infty) \}, \end{aligned}$$

endowed with the norm $\|x\|_\infty := \sup_{t\in\mathbb{R}_+} |x(t)|$.

Another space is

$$C_\theta := \{x \in C_c,\ (\exists)\ k = k(x),\ |x(t)| \le k\theta(t),\ t \in \mathbb{R}_+\},$$

where $\theta : \mathbb{R}_+ \to (0,\infty)$, $\theta \in C_R$, with the norm

$$\|x\|_\theta := \sup_{t\in\mathbb{R}_+} \frac{|x(t)|}{\theta(t)}.$$

Clearly, C_l, C_{0l}, C_θ are Banach space having stronger topologies than the topology of C_c. Moreover,

$$(x \in C_\theta) \Longrightarrow (|x(t)| \le \|x\|_\theta \theta(t),\ t \ge 0).$$

By C_l^1, C_c^1 and so on we denote the subsets of C_l, C_c and so on with the derivative $\dot{x}$ in C_l and C_c, respectively.

It is easy to see that

$$\left(x \in C_l^1\right) \Longleftrightarrow (\dot{x} \in C_R).$$

Moreover,

$$(x \in C_l \text{ and } \dot{x} \in C_l) \Longrightarrow \lim_{t\to\infty} \dot{x}(t) = 0.$$

This last remark has an important consequence: if the equation (1.1) is autonomous, i.e. $f(t,x) \equiv f(x)$ and the Cauchy problem for $t = 0$ has a unique solution, then the orbit is $\omega-$closed if and only if it is a rest point (equilibrium). However, simple examples show that for nonautonomous equations the existence of nonconstant $\omega-$closed orbits is possible.

The space C_l has an important compactness property.

Definition 2. *A set $\mathcal{A}$ is called* **equiconvergent** *if*

$$\begin{aligned}(\forall)\ \epsilon > 0,\ (\exists)\ T = T(\epsilon) > 0,\ (\forall)\ t > T,\ (\forall)\ x \in \mathcal{A},\\ |x(t) - x(\infty)| < \epsilon.\end{aligned}$$

Proposition 3 (see [1])**.** *A set $\mathcal{A} \subset C_l$ is relatively compact if and only if it is uniformly bounded in C_l, equicontinuous in C_c and equiconvergent.*

Indeed, the second and the third conditions imply that $\mathcal{A}$ is equicontinuous in C_l. On the other hand, the mapping $x \to y$, with $x \in C_l$ and y given by

$$y(t) = \begin{cases} x\left(\frac{t}{1-t}\right),\ t \in [0,1) \\ x(\infty),\ t = 1 \end{cases}$$

is a linear homeomorphism between C_l and C_0, where

$$C_0 := \{x : [0,T] \to \mathbb{R}^n,\ x \text{ continuous}\},$$

with the usual norm.

Now, the range of $\mathcal{A}$ under this mapping is uniformly bounded and equicontinuous in C_0.

Another result we will use later is due to J. Mawhin (see [1]).

Proposition 4. *Let X, Y be two linear spaces, $L : D(L) \subset X \to Y$ be a linear operator and $G : X \to Y$ be an arbitrary operator.*

Assume that:

a) *there is a projector $P : X \to X$, such that $R(P) = N(L)$;*

b) *there is a projector $Q : Y \to Y$, such that $N(Q) = R(L)$;*

c) *there is an injective mapping $J : R(Q) \to N(L)$.*

Then

i) *there is a linear operator $K : R(L) \to D(L) \cap N(P)$, such that*

$$LKu = u,\ \ (\forall)\ \ u \in R(L);$$

ii) *the equation*

$$Lx = Gx \tag{1.6}$$

is equivalent with the equation

$$x = Mx, \tag{1.7}$$

where

$$M := P + JQG + K(I - Q)G. \tag{1.8}$$

Throughout this paper, $N(T)$ signifies the kernel of T, $R(T)$ is the range of T and $D(T)$ denotes the domain of T.

Let Ω be an open bounded set in a normed space X and $M : \overline{\Omega} \to X$ be a compact operator such that $x \neq Mx$, for every $x \in \partial\Omega$ (herein, $\partial\Omega$ is the boundary of Ω). The Schauder topological degree on M will be referred to be the usual $\deg(I - M, \Omega, 0)$. We presume that this notion along with its main properties are known. In particular, if $\deg(I - M, \Omega, 0) \neq 0$, then M has at least one fixed point. Of course, for a finite dimensional space X, the Schauder degree is replaced by the Brouwer degree and will be denoted by $\deg_B(I - M, \Omega, 0)$.

Now, let A be a continuous on $\mathbb{R}_+$, $n \times n$ matrix; we denote by $|A|$ the spectral norm of A and we adopt the following notations:

$$\begin{aligned} X &: = \left\{x \in C_c^1,\ \dot{x} = A(t)\,x\right\}, \\ X^* &: = \left\{x \in C_c^1,\ \dot{x} = A^*(t)\,x\right\}, \\ X_l &: = X \cap C_l,\ X_{0l} := X \cap C_{0l}, \\ X_l^* &: = X^* \cap C_l,\ X_{0l}^* := X^* \cap C_{0l}. \end{aligned}$$

(A^* is the adjoint of A).

By $X(t)$ we denote the fundamental matrix of $\dot{x} = A(t)\,x$, with $X(0) = I$ (the identity).

2 Linear equations

2.1 Preliminaries

The aim of this section is to obtain some information about the $\omega-$closed orbits for the linear case.

2.2 Homogenous equations

Let us consider the equation

$$\dot{x} = A(t)\,x. \tag{2.1}$$

This equation has the degenerate $\omega-$closed orbit $x \equiv 0$. Clearly, a solution x of the equation (2.1) is $\omega-$closed if and only if $x \in X_{0l}$; therefore we can formulate the existence of $\omega-$closed orbits as

$$\dot{x} = A(t)\,x,\ x \in C_{0l}.$$

Denote by X_0 the range of X_l under the mapping $x \to x(0)$ and let $U : \mathbb{R}^n \to \mathbb{R}^n$ be a projector onto X_0.

$$Y(t) := X(t)\,U,$$

then the limit below

$$Y(\infty) := \lim_{t\to\infty} Y(t)$$

does exist and is finite.

Now, it is easy to remark that

$$\begin{aligned} X_l &= \left\{Y(t)\,c,\ c \in \mathbb{R}^n\right\}, \\ X_{0l} &= \left\{Y(t)\,c,\ c \in \mathbb{R}^n,\ [U - Y(\infty)]\,c = 0\right\}. \end{aligned}$$

Consequently, the equation (2.1) has nondegenerate $\omega-$closed orbits if and only if

$$\operatorname{rank}\ [U - Y(\infty)] < n.$$

An important particular case is when

$$U = I,$$

i.e. when

$$X(\infty) := \lim_{t\to\infty} X(t)$$

exists and is finite. In this case,

$$X_{0l} = \{X(t)\,c,\ \ [I - X(\infty)]\,c = 0\}.$$

If $X(\infty)$ is nondegenerate, then

$$\operatorname{rank}\ [X(\infty) - I] = \operatorname{rank}\ [X^*(\infty) - I]$$

and so

$$X = X_l,\ X^* = X_l^*,\ \dim X_{0l} = \dim X_{0l}^*.$$

2.3 Nonhomogeneous equations

We shall consider the existence problem of $\omega-$closed orbits for the system

$$\dot{x} = A(t)\,x + b(t), \tag{2.2}$$

where $b \in C_c$.

Theorem 5. *Suppose that:*
a) $A(t)$ *is bounded on* ; $\mathbb{R}^n$;
b) *the limit from below is finite:*

$$L := \lim_{t\to\infty} \int_0^t X(t)\,X^{-1}(s)\,ds;$$

c) *there exists a constant* $k > 0$, *such that*

$$\int_0^t \left|X(t)\,X^{-1}(s)\right| ds \le k,\ \ 0 \le s \le t < \infty.$$

Then, for every $b \in C_l$, *the equation* (2.2) *has one and only one* $\omega-$*closed orbit.*

Proof. From hypotheses a), b), it follows that

$$\left|X(t)\,X^{-1}(s)\right| \leq \alpha e^{-\beta(t-s)},\ \alpha,\ \beta > 0,\ 0 \leq s \leq t$$

and hence

$$X(\infty) := \lim_{t\to\infty} X(t) = 0.$$

Therefore, we have

$$X_l = X,\ X_{0l} = \{0\}\,.$$

Moreover, if x is a solution of (2.2), with $b \in C_l$, then

$$x(\infty) := \lim_{t\to\infty} x(t) = Lb(\infty)\,.$$

(For details see [1] .)

Because of

$$x(t) = X(t)\,c + \int_0^t X(t)\,X^{-1}(s)\,b(s)\,ds, \tag{2.3}$$

we have $x \in C_{0l}$ if and only if $c = Lb(\infty)\,.$ □

Theorem 6. *Suppose that*

$$\int_0^\infty |A(t)|\,dt < \infty. \tag{2.4}$$

Then:

i) $X_l = X_l^* = X$;

ii) $\dim X_{0l} = \dim X_{0l}^*$;

iii) *for every* $b \in C_\theta$ *and* $\xi \in \mathbb{R}^n$*, the equation* (2.2) *has a unique solution in* C_l*, such that* $x(\infty) = \xi$;

iv) *for every* $b \in C_\theta$ *with*

$$\int_0^\infty \left\langle b(t) \mid \psi_j(t)\right\rangle dt = 0,\ j \in \overline{1,m},$$

where $\left\{\psi_j\right\}_{j\in\overline{1,m}}$ *form an orthonormal base in* X_{0l}^**, the system* (2.2) *has at least one* $\omega-$*closed orbit.*

Proof. Consider $x \in X$; then

$$x(t) \leq |x(0)|\, e^{\int_0^t |A(s)|ds}$$

and, consequently, x is bounded on $\mathbb{R}_+$; since $\dot{x}(t) = A(t)\,x(t)$, it follows that $\dot{x} \in C_R$ and hence $x \in C_l$. Therefore, $X(\infty)$ exists; moreover, we have

$$\det X(\infty) = e^{\int_0^\infty \mathrm{Tr}\ A(s)ds} \neq 0.$$

Consequently, conclusions i), ii) hold.

Let x be a solution for (2.2); from (2.3) it follows

$$x(\infty) = X(\infty)\,c + X(\infty)\int_0^\infty X^{-1}(s)\,b(s)\,ds$$

and hence the equation $x(\infty) = \xi$ has the unique solution

$$c = X^{-1}(\infty)\,\xi - \int_0^\infty X^{-1}(s)\,b(s)\,ds.$$

Similarly, the equation $x(0) = x(\infty)$ has solutions if and only if c satisfies

$$\left[X^{-1}(\infty) - I\right] c = \int_0^\infty X^{-1}(s)\,b(s)\,ds. \tag{2.5}$$

Nevertheless, (2.5) has solutions if and only if

$$\left\langle \int_0^\infty X^{-1}(s)\,b(s)\,ds \mid e \right\rangle = 0, \tag{2.6}$$

for all e satisfying

$$\left[X^{-1}(\infty) - I\right]^* e = 0. \tag{2.7}$$

Clearly, the solutions of (2.7) are exactly the values in $t = 0$ of the functions of X_{0l}^*.

Now, we have

$$\begin{aligned}
\left\langle \int_0^\infty X^{-1}(s)\,b(s)\,ds \mid e \right\rangle &= \int_0^\infty \left\langle X^{-1}(s)\,b(s) \mid e \right\rangle ds = \\
&= \int_0^\infty \left\langle b(s) \mid \left(X^{-1}(s)\right)^* e \right\rangle ds = \\
&= \int_0^\infty \left\langle b(s) \mid \psi(s) \right\rangle ds,
\end{aligned}$$

where $\psi \in X_{0l}^*$. □

Theorem 7. *Assume that there exists a mapping* $\varphi \in C_c^1(\mathbb{R}_+, \mathbb{R})$ *such that* $\dot{\varphi}(t) > 0$, $\varphi(0) = 0$, $\varphi(\infty) = 1$, *while the limit from below is finite:*

$$\Lambda := \lim_{t\to\infty} \frac{1}{\dot{\varphi}(t)} A(t).$$

Then, the conclusions of the previous theorem are true.

Proof. Let us denote $\psi(t) := \varphi^{-1}(t)$ and

$$B(s) = \begin{cases} \dot{\psi}(s)[A\psi(s)], & \text{if } s \in [0,1) \\ \Lambda, & \text{if } s = 1 \end{cases}.$$

Obviously, B is continuous in $[0,1]$. Now, if x satisfies (2.1), then $y(s) := x(\varphi(s))$ is solution of the equation

$$\dot{y} = B(s)\, y, \ \ s \in [0,1]. \tag{2.8}$$

Furthermore, $x(0) = y(0)$. If $Y(t)$ is the fundamental matrix of (2.8) with $Y(0) = I$, then

$$X(t) = Y[\varphi(t)], \ t \geq 0$$

and

$$X(\infty) = Y(1),$$

so that the proof can continue as in the previous theorem. □

Proposition 8. *Suppose that:*
a) *the condition* (2.4) *holds;*
b) $1 \notin \sigma(X(\infty))$.
Let K denote the mapping which associates to each $b \in C_\theta$ the unique solution in C_{0l} of (2.2).
Then,
i) *there exists a positive constant k such that*

$$\|Kb\|_\infty \leq k\, \|b\|_\infty, \ b \in C_\theta; \tag{2.9}$$

ii) *the operator $K : C_\theta \to C_l$ is compact.*

Proof. The assumption b) implies that

$$\text{rank } [I - X(\infty)] = n$$

and, consequently,

$$X_{0l} = \{0\}.$$

This implies that (2.2) has a unique solution in C_{0l} for every $b \in C_R$ and so the operator K is well defined on C_R.

A well-known result shows that this operator, while working in C_c, is closed. Furthermore, since the topologies of C_l and C_θ are stronger than the topology of C_c and by the use of the Closed Graph Theorem, we can conclude that $K : C_\theta \to C_l$ is continuous and so (2.9) holds true.

Let $\Omega \subset C_\theta$ be a bounded set. Then, there exists a positive constant m such that

$$((\forall)\ \ b \in \Omega) \Longrightarrow (\|b\|_\theta \leq m)$$

or, equivalently,

$$|b(t)| \leq m \cdot \theta(t), \ \ (\forall)\ \ t \geq 0.$$

The inequality (2.9) shows that $K\Omega$ is bounded in C_{0l}. So, from

$$|\dot{x}(t)| \leq |A(t)| \cdot |x(t)| + |b(t)|, \tag{2.10}$$

we conclude that $K\Omega$ is equicontinuous in C_c.

Now, from (2.10) again, we have

$$\begin{aligned} |x(t_1) - x(t_2)| &= \left| \int_{t_1}^{t_2} \dot{x}(t)\, dt \right| \leq \\ &\leq \|x\|_\infty \cdot \int_{t_1}^{t_2} |A(s)|\, ds + m \int_{t_1}^{t_2} \theta(s)\, ds, \ \ t_2 > t_1. \end{aligned}$$

Since the hypothesis $|A| \in C_R$, $\theta \in C_R$ implies that the set $K\Omega$ is equiconvergent and by use of Proposition 3, the conclusion of the compactness of $\overline{K\Omega}$ holds.

3 The Poincaré operator

3.1 Preliminaries

In this section, the Cauchy problem

$$\dot{x} = f(t, x), \tag{3.1}$$

$$x(0) = y, \ \ y \in \mathbb{R}^n \tag{3.2}$$

is supposed to have one and only one solution in C_l; we shall denote this solution by $x(t, y)$.

Now, we define an operator $U : \mathbb{R}^n \to \mathbb{R}^n$, by setting

$$Uy := x(\infty, y).$$

We also suppose that the operator U is continuous; for example, this is true if $f(t, x)$ is locally Lipschitz in x and satisfies an inequality of the type

$$|f(t, x)| \leq \theta(t)\, g(|x|),$$

for some continuous function $g : \mathbb{R}^n \to \mathbb{R}^n$.

Clearly, the orbit of the solution $x(t, y)$ is $\omega-$closed if and only if y is a fixed point for U.

3.2 An existence result

Using the Poincaré operator, we can prove the following theorem.

Theorem 9. *Assume that there exists an open bounded set $G \subset \mathbb{R}^n$ such that:*
a) *for each $t \in \mathbb{R}_+$ and $y \in \partial G$ one has $x(t,y) \neq 0$;*
b) *for each $y \in \partial G$ one has $f(0,y) \neq 0$;*
c) $\deg_B [f(0,\cdot), G, 0] \neq 0$.
Then the equation (3.1) *has at least one $\omega-$closed orbit.*

Proof. We start by defining the mapping $h : \overline{G} \times [0,1] \to \mathbb{R}^n$ by

$$h(t,y) = \begin{cases} \frac{1-\lambda}{\lambda}\left[y - x\left(\frac{\lambda}{1-\lambda}, y\right)\right], \text{ if } \lambda \in (01) \\ -f(0,y), \text{ if } \lambda = 0 \\ y - Uy, \text{ if } \lambda = 1 \end{cases}.$$

Clearly,

$$\lim_{\lambda \nearrow 1} h(y,\lambda) = y - Uy$$

and, by L'Hôspital rule,

$$\lim_{t \searrow 0} h(y,\lambda) = -f(0,y).$$

Thus, h is continuous on $\overline{G} \times [0,1]$. Moreover, by assumptions a), b), $h(y,\lambda) \neq 0$, on $\partial G \times [0,1]$. The homotopy invariance of the Brouwer degree along with the assumption c) implies that

$$\begin{aligned} deb_B[h(\cdot,0), G, 0] &= \deg_B[h(\cdot,1), G, 0] = \deg_B[I - 0, G, 0] = \\ &= (-1)^n \deg_B[h(0,\cdot), G, 0] \neq 0. \end{aligned}$$

□

3.3 Some auxiliary results

Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function; let us define the Niemîtzky operator $F : C_c \to C_c$, by

$$(Fx)(t) := f(t, x(t)).$$

Consider also the operator $H : C_c \to C_c$, defined by

$$Hx := \int_0^{(\cdot)} (Fx)(s)\, ds.$$

Proposition 10. *Assume that there are a bounded set $\Omega \subset C_c$ and a function θ such that $F(\Omega)$ is a bounded set of C_θ. Then, the operator $H : \Omega \cap C_l \to C_l$ is compact.*

Proof. Since $F(\Omega)$ is bounded in C_θ, there exists a $k > 0$, such that

$$|(Fx)(t)| \leq k \cdot \theta(t), \ \ (\forall) \ x \in \Omega, \ t \geq 0. \tag{3.3}$$

Next, let $\epsilon > 0$ be given; because of $\theta \in C_R$ and of $\theta(t) > 0$, there exists also a constant $A > 0$ such that

$$\int_A^\infty \theta(t)\, dt < \frac{\epsilon}{3k}. \tag{3.4}$$

We consider x, $x_m \in \Omega$ such that $x_m \to x$ in C_l; we choose a positive number r such that

$$r \geq \max\{\|x\|_\infty, \ \|x_m\|_\infty, \ m \geq 1\}$$

and denote

$$B(r) := \{x \in \mathbb{R}^n, \ |x| \leq r\}.$$

Since $f : [0, A] \times B(r) \to \mathbb{R}^n$ is uniformly continuous, it follows that

$$|f(t, x_m(t)) - f(t, x(t))| < \frac{\epsilon}{3}, \ (\forall) \ t \in [0, A], \ m \geq m_0(\epsilon). \tag{3.5}$$

Now, from

$$\begin{aligned} |(Hx_m)(t) - (Hx)(t)| \ \leq \ & \int_0^A |f(s, x_m(s)) - f(s, x(s))|\, ds + \\ & + \int_A^\infty |Fx_m(s)|\, ds + \int_A^\infty |Fx(s)|\, ds, \end{aligned}$$

according to (3.4) and (3.5), one has $\|Hx_m - Hx\|_\infty < \epsilon$, $(\forall)$ $m \geq m_0(\epsilon)$.

It remains to show that $H\Omega$ is compact. The inequality (3.3) implies that $H\Omega$ is bounded in C_l; from the same (3.3) we can deduce that

$$|(Hx)(t_2) - (Hx)(t_1)| \leq k \cdot \int_{t_2}^{t_1} \theta(s)\, ds, \ 0 \leq t_1 \leq t_2. \tag{3.6}$$

Let us consider an interval $[0, A]$; then, from (3.6) it results that

$$|(Hx)(t_2) - (Hx)(t_1)| \leq kA \sup_{t \in [0,A]} \cdot |t_1 - t_2|$$

and so $H\Omega$ is equicontinuous. From (3.6) it follows that

$$|(Hx)(t_2) - (Hx)(t_1)| < \epsilon, \ \ (\forall) \ \ t > T(\epsilon)$$

and, consequently, $H\Omega$ is equiconvergent. Now, the compactness of $H\Omega$ is assured by Proposition 8. □

Corollary 11. *Assume that*

$$|f(t,x)| \leq \theta(t) \cdot g(x), \ \ (\forall) \ \ t \geq 0, \ x \in \mathbb{R}^n$$

for some continuous mapping $g : \mathbb{R}_+ \to \mathbb{R}_+$. *Then,* $H : C_l \to C_l$ *is compact.*

Now, we consider the operator

$$S := I - H.$$

One can see straightaway that S is injective when f is locally Lipschitz with respect to x; consequently, for every closed set $B \subset C_l$, the mapping $S : B \to S(B)$ is a homeomorphism. Clearly,

$$\left(y \in \mathbb{R}^n \cap S(b)\right) \Longrightarrow \left(S^{-1}y = x(\cdot, y)\right).$$

If we consider

$$Px := x(\infty), \ P : C_l \to C_l,$$

then P is a projector in C_l and the Poincaré operator U has the formula:

$$U = PS^{-1}.$$

Next we introduce the operator

$$Mx := Px + \int_0^{(\cdot)} (Fx)(s)\, ds. \tag{3.7}$$

It can be easily seen that $x \in C_l$ is a solution for (3.1) in C_{0l} if and only if it is a fixed point of M.

Theorem 12. *Assume that:*
a) $\Omega \subset C_l$ *is a bounded open and connected set;*
b) f *is locally Lipschitz;*
c) $F\overline{\Omega}$ *is bounded in* C_θ;
d) *the equation* (3.1) *has no solution in* $\partial\Omega \cap C_{0l}$.
Then

$$\deg(I - M, \Omega, 0) = \pm \deg_B\left(I - U, S(\Omega) \cap \mathbb{R}^n, 0\right). \tag{3.8}$$

Proof. From the identity below

$$I - M = \left(I - PS^{-1}\right) S$$

and from Leray's theorem on the topological degree of a product, we have

$$\deg\left(I - M, \Omega, 0\right) = \deg\left(S, \Omega, y\right) \cdot \deg\left(I - PS^{-1}, S\left(\Omega\right), 0\right).$$

Notice that the value of this degree is independent of y, because Ω is connected. Since $S : \Omega \to S\left(\Omega\right)$ is one-to-one, it follows that

$$\deg\left(S, \Omega, y\right) = \pm 1.$$

From $R\left(P\right) = \mathbb{R}^n$, it comes to

$$\deg\left(I - PS^{-1}, S\left(\Omega\right), 0\right) = \deg_B\left(I - PS^{-1}, S\left(\Omega\right) \cap \mathbb{R}^n, 0\right).$$

□

Corollary 13. *Assume that:*
a) *there exists an* $r > 0$ *such that, if* $x \in C_{0l}$ *is a solution for* (3.1), *then* $\left\|x\right\|_\infty < r$;
b) $f\left(0, y\right) \neq 0$, $(\forall)$ $y \in \mathbb{R}^n$, $|y| = r$;
c) *there is also a* $k > 0$ *such that*

$$\left|f\left(t, x\right)\right| \leq k \cdot \theta\left(t\right), \quad (\forall) \ \ t \geq 0, \ \ |x| \leq r.$$

Then,

$$\deg\left(I - M, \Sigma\left(r\right), 0\right) = \pm \deg_B\left(f\left(0, \cdot\right), B\left(r\right), 0\right),$$

where

$$\Sigma\left(r\right) = \left\{x \in C_l, \ \ \left\|x\right\|_\infty \leq r\right\}.$$

4 Compact operators whose fixed points are $\omega-$clo sed solutions

4.1 Preliminaries

Literature offers various examples of compact operators whose fixed points are solutions of the problem $(1.1) + (1.5)$. A general method for the construction of such operators based on Proposition 4 can be adjusted to the problem $(1.1) + (1.2)$.

We start by considering $X = C_l$; later on C_{0l} will be used as X.

Only some choices for F, L, P, Q, Q will be taken into account.

4.2 The case $\mathbf{X}=\mathbf{C}_l$

Let us consider $D(L)=C_l^1$. For

$$Lx:=(\dot{x},\ 0), \tag{4.1}$$

it follows that

$$\begin{aligned} \mathcal{N}(L) &= \mathbb{R}^n,\ Y=C_R\times\mathbb{R}^n,\ R(L)=C_R\times\{0\}, \\ Q(y,c) &: =(0,c),\ Gx:=(Fx,x(0)-x(\infty)) \end{aligned}$$

and, consequently,

$$\begin{aligned} R(Q) &= \{0\}\times\mathbb{R}^n,\ N(Q)=C_R\times\{0\}, \\ J &= a\cdot c,\ a\in\mathbb{R},\ c\in\mathbb{R}^n,\ a\neq 0. \end{aligned}$$

A first candidate for P can be

$$Px:=x(b),\ b\in[0,\infty]. \tag{4.2}$$

An easy but tedious calculation shows that, in this case,

$$M_1x=x(b)+a[x(\infty)-x(0)]+\int_b^{(\cdot)}(Fx)(s)\,ds. \tag{4.3}$$

Another choice of P is the following

$$Px:=\int_0^\infty e^{-t}x(t)\,dt. \tag{4.4}$$

Because of $PLx=Px$, the operator corresponding to M has the formula:

$$\begin{aligned} M_2x &= a[x(\infty)-x(0)]+\int_0^\infty e^{-s}[x(s)-(Fx)(s)]\,ds+ \\ &+\int_0^{(\cdot)}(Fx)(s)\,ds. \end{aligned} \tag{4.5}$$

Now, let L be

$$Lx:=(\dot{x},x(0)-x(\infty)). \tag{4.6}$$

This choice implies that

$$\begin{aligned} N(L) &= \mathbb{R}^n,\ Y=C_R\times\mathbb{R}^n,\ Gx:=(Fx,0), \\ R(L) &= \left\{(y,c)\in Y,\ c+\int_0^\infty y(s)\,ds=0\right\}. \end{aligned}$$

The structure of $R(L)$ suggests to choose a Q like below

$$Q(y,c) := \left(0, c + \int_0^\infty y(s)\,ds\right).$$

Consequently,

$$\begin{aligned} R(Q) &= \{0\} \times \mathbb{R}^n,\ N(Q) = R(L), \\ J &= a \cdot c,\ a \in \mathbb{R},\ c \in \mathbb{R}^n,\ a \neq 0. \end{aligned}$$

Using (4.2) for projector, we deduce for M the formula:

$$M_3 x = x(b) + \int_0^\infty (Fx)(s)\,ds + \int_b^{(\cdot)} (Fx)(s)\,ds. \tag{4.7}$$

As for (4.4) as a projector, it comes to

$$\begin{aligned} M_4 x &= \int_0^\infty e^{-s}\left[x(s) - (Fx)(s)\right] ds + a \int_0^\infty (Fx)(s)\,ds + \\ &\quad + \int_0^{(\cdot)} (Fx)(s)\,ds. \end{aligned} \tag{4.8}$$

Finally, we consider

$$Lx := (\dot{x}, x(0)). \tag{4.9}$$

Now,

$$\begin{aligned} N(L) &= \{0,\},\ P = Q = 0, \\ M &= L^{-1},\ Gx := (Fx, x(\infty)). \end{aligned}$$

In this case, for M we can obtain the operators

$$M_5 x = x(\infty) + \int_0^{(\cdot)} (Fx)(s)\,ds \tag{4.10}$$

or

$$M_6 x = x(0) + \int_\infty^{(\cdot)} (Fx)(s)\,ds. \tag{4.11}$$

4.3 The case $X = C_{0l}$

It is necessary to consider

$$Lx = \dot{x},\ D(L) = C^1_{0l},\ Gx = Fx,\ Y = C_R$$

and, consequently,

$$N(L) = \mathbb{R}^n,\ R(L) = \left\{ y \in Y,\ \int_0^\infty y(s)\,ds = 0 \right\}.$$

The first option for Q is

$$(Qy)(t) = e^{-t} \int_0^\infty y(s)\,ds.$$

With $J = ac$ and (4.2) as projector, we conclude that

$$\begin{aligned}(M_7 x)(t) &= x(b) + \left(a + e^{-t} - e^{-b}\right) \int_0^\infty (Fx)(s)\,ds + \\ &\quad + \int_b^{(\cdot)} (Fx)(s)\,ds.\end{aligned} \tag{4.12}$$

With (4.4) as projector,

$$\begin{aligned}(M_8 x)(t) &= \int_0^\infty e^{-s} x(s)\,ds + \int_0^\infty \left(a + e^{-t} - e^{-b}\right)(Fx)(s)\,ds + \\ &\quad + \int_0^{(\cdot)} (Fx)(s)\,ds.\end{aligned} \tag{4.13}$$

Another way to construct operators M is to find some linear operators $\Phi : C_{0l} \to C_R$, for which the operator $L := \dot{x} + \Phi$ becomes invertible; thus $M = L^{-1}(F + \Phi)$.

Foe example, one can set by Φ the mapping

$$(\Phi x)(t) = e^{-t} x(0).$$

It follows that

$$\left(L^{-1} y\right)(t) = e^{-t} \int_0^\infty y(s)\,ds + \int_0^t y(s)\,ds$$

and so

$$(M_9 x)(t) = e^{-t} \int_0^\infty (Fx)(s)\,ds + \int_0^t (Fx)(s)\,ds. \tag{4.14}$$

Generally, if we take

$$\Phi x = h(\cdot)\, x$$

for a continuous mapping $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 < \int_0^\infty h(s)\, ds < \infty,$$

then

$$\begin{aligned} M_{10}x &= \frac{1}{a} e^{\int_0^{(\cdot)} h(s)ds} \cdot \int_0^\infty e^{\int_s^\infty h(s)ds} \left[(Fx)(s) + h(s)\, x(s)\right] ds + \\ &\quad + \int_0^{(\cdot)} e^{\int_s^{(\cdot)} h(u)du} \left[(Fx)(s) + h(s)\, x(s)\right] ds, \end{aligned} \tag{4.15}$$

where

$$a := 1 - e^{\int_0^\infty h(s)ds}.$$

4.4 Comments

If $F\overline{\Omega}$ is bounded in C_θ, where $\Omega \subset C_l$ is a bounded set, we can prove in the same manner as in Proposition 11 the compactness of the operator M_i.

Now, if for some i we have

$$x \neq M_i x, \ \ (\forall)\ \ x \in \partial\Omega \tag{4.16}$$

and

$$\deg(I - M_i, \Omega, 0) \neq 0, \tag{4.17}$$

then the equation $\dot{x} = f(t, x)$ admits $\omega-$closed orbits.

Finally, notice that there are operators M which can be used equally in C_l and C_{0l}; an example is the following

$$(Mx)(t) = x(0) + \frac{1+t}{1+t^2} \int_0^\infty (Fx)(s)\, ds + \int_0^t (Fx)(s)\, ds. \tag{4.18}$$

5 The continuation method

5.1 Introduction

In many cases it is extremely difficult to verify for one of the M_i operators the condition (4.17). Moreover, in the autonomous case only a negative answer can be obtained for the operators M_i, $i \in \overline{7,10}$. The result is the following: if the Cauchy problem for $t = 0$ has a unique solution and if

$$\Omega \cap \{x,\ f(x) = 0\} = \emptyset,$$

then

$$\deg(I - M_i, \Omega, 0) = 0,\ i \in \overline{7,10}.$$

Indeed, if the above is not true, then M_i has a fixed point in Ω. This point is a solution of the problem $\dot{x} = f(x)$, $x \in C_{0l}$ and so x is a rest point. Clearly, $\Omega \cap \{x,\ f(x) = 0\} \neq \emptyset$, contradiction.

By the continuation method which is based on the homotopy invariance of the topological degree, we may replace the condition (4.17) with a similar one which will work for another operator, if possible a better one.

Consider again the Cauchy problem

$$\dot{x} = f(t, x), \tag{5.1}$$

$$x(0) = x(\infty), \tag{5.2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, satisfying the condition

$$|f(t, x)| \leq \theta(t) \cdot g(|x|), \tag{5.3}$$

where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions and

$$\int_0^\infty \theta(t)\, dt < \infty. \tag{5.4}$$

X denote, as above, the space C_l or C_{ll} and $\Omega \subset X$ is a bounded and open set.

As we have seen in the previous section, we can associate to the problem (5.1) + (5.2) a compact operator

$$U : \overline{\Omega} \to X,$$

whose fixed points coincide with the solutions of the problem (5.1)+(5.2). This operator will be called the **associated operator** to the problem (5.1) + (5.2) on $\overline{\Omega}$.

If for an associated operator U we have

$$x \neq Ux, \ \ (\forall) \ x \in \partial\Omega, \tag{5.5}$$

then the relation (5.5) is true for every another associated operator U.

In the case when (5.5) holds, we can define the topological degree of U, denoted by

$$\deg\left(I - U, \ \Omega, \ 0\right),$$

where I denotes the identity operator in X.

It is known that if the compact operator U satisfies (5.5) and

$$\deg\left(I - U, \ \Omega, \ 0\right) \neq 0, \tag{5.6}$$

then U admits fixed points and consequently the problem $(5.1) + (5.2)$ admits solutions.

Theoretically the problem of the existence of solutions for $(5.1) + (5.2)$ is solved; practically the effective computation of the degree (and implicitly the verifications of (5.6)) is a very difficult problem and effectively possible only in some particular cases.

The essence of the continuation method consists in the following.

By using the invariance property of the topological degree with respect to a homotopy, we can replace the condition (5.6) with a similar condition

$$\deg\left(I - V, \ \Omega, \ 0\right) \neq 0, \tag{5.7}$$

where V is homotopic with U, such that the condition (5.7) is easier than (5.6) (this happens in particular if V is a finite rank operator since in this case the degree appearing in (5.7) is a Brouwer one).

We state below the continuation principle in the particular case of the problem $(5.1) + (5.2)$.

Let $h : \mathbb{R}_+ \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be a continuous function; consider the problem

$$\dot{x} = h\left(t, x, \lambda\right), \ x\left(0\right) = x\left(\infty\right) \tag{5.8}$$

and denote by U_λ the associated operator to the problem (5.8), supposing that it exists.

Proposition 16. *Assume that:*

i) *there exists* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, α *continuous*, $\int_0^{+\infty} \alpha\left(t\right) dt < \infty$, *such that*

$$\left|h\left(t, x\left(t\right), \lambda\right)\right| \leq \alpha\left(t\right),$$

$(\forall)$ $x \in \overline{\Omega}$, $(\forall)$ $\lambda \in [0,1]$ *and* $(\forall)$ $t \in \mathbb{R}_+$;

ii) *for every* $\lambda \in [0,1]$ *the problem* (5.8) *does not admit solutions* $x(\cdot)$ *with* $x \in \partial\Omega$;

Then:

a) *there exists* U_λ, $\lambda \in [0,1]$, *the associated operators to the problem* (5.8), *which are, in addition, compact.*

b) *the following equality holds:*

$$\deg\left(I - U_0,\ \Omega,\ 0\right) = \deg\left(I - U_1,\ \Omega,\ 0\right). \tag{5.9}$$

In particular, if

$$h(t,x,0) \equiv f(t,x) \tag{5.10}$$

and

$$\deg\left(I - U_1,\ \Omega,\ 0\right) \neq 0, \tag{5.11}$$

then the problem (5.1) + (5.2) *admits solutions.*

By choosing conveniently the function h, one can obtain concrete existence results. Obviously, hypothesis ii) is more difficult to be checked; this hypothesis can be formulated under the following forms.

"A priori estimates": for every possible solution $x(\cdot)$ *of the problem* (5.8) *with* $x \in \overline{\Omega}$ *we have* $x \in \Omega$.

Another form of the same condition is the next.

"A priori bound": there exists a number $r > 0$ *such that the problem* (5.8) *does not admit solutions* $x(\cdot)$ *with* $\|x\|_\infty = r$.

In this case we set $\Omega := \{x \in X,\ \ \|x\| < r\}$.

Finally, let $D \subset \mathbb{R}^n$ be a bounded and open set. Another variant of the same condition is the following.

"Bounded set condition": for every $\lambda \in [0,1]$ *for which* (1.11) *has solutions* $x(\cdot)$ *with* $x(t) \in \overline{D}$, $t \in \mathbb{R}_+$, *we have* $x(t) \in D$, $(\forall)\ t \in [0,\infty]$.

5.2 Some remarks on the problem (5.1) + (5.2)

Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be two continuous functions. Consider the problems (5.1) + (5.2) and

$$\dot{y} = g(t,y) \tag{5.12}$$

$$y(T) = y(0). \tag{5.13}$$

We try to find the link between the existence of solutions of these two problems in hypothesis of a relation between the functions f and g.

Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a continuous and strictly positive function, such that

$$T := \int_0^{+\infty} \theta(t)\ dt < +\infty. \tag{5.14}$$

Set

$$\psi(t) := \int_0^t \theta(s)\ ds,\ \ \varphi := \psi^{-1}.$$

Theorem 17. *Assume that there exists the limit*

$$\lim_{t\to+\infty} \frac{1}{\theta(t)} f(t,y) := \gamma(y),\ \ y \in \mathbb{R}^n,$$

the convergence being uniform with respect to y *on every compact of* $\mathbb{R}^n$. *Let*

$$g(t,y) := \begin{cases} \dot{\varphi}(t) f(\varphi(t), y), \textit{if}\ t \in [0,T),\ y \in \mathbb{R}^n \\ \gamma(y), \textit{if}\ t = T,\ y \in \mathbb{R}^n \end{cases}.$$

Then one can establish a one-to-one link between the solutions of the problems $(5.1)+(5.2)$ *and* $(5.12)+(5.13)$.

Proof. Indeed, if $x(\cdot)$ is a solution for the problem $(5.1)+(5.2)$, then $y := x(\varphi(\cdot))$ will be a solution for $(5.12)+(5.13)$ and conversely.

If we define the operator $\Phi : C_l \to C([0,T], \mathbb{R}^n)$ by the equality

$$(\Phi x)(t) := \begin{cases} x(\varphi(t)),\ t \in [0,T) \\ x(+\infty),\ t = T, \end{cases}$$

this represents an isomorphism from C_l to $C([0,T], \mathbb{R}^n)$, which in addition transforms the solutions of the problem $(5.1)+(5.2)$ into solutions of the problem $(5.12)+(5.13)$; Φ^{-1} will give the link between the solutions of $(5.12)+(5.13)$ and the solutions of $(5.1)+(5.2)$. Hence, $(5.1)+(5.2)$ admits solutions if and only if $(5.12)+(5.13)$ admits solutions. □

Let us prove that we can construct, for the problem $(5.1)+(5.2)$, the associated operators; for this aim we should show that

$$(\forall)\ \ x \in D,\ \ \int_0^{+\infty} (Fx)(t)\ dt < +\infty, \tag{5.15}$$

where $D \subset C_l$ is a bounded set.

Since D is a bounded set, it results

$$(\exists)\ \ \varepsilon > 0,\ \ (\forall)\ \ x \in D,\ \ |x(t)| \le A.$$

Let $\varepsilon_0 > 0$ be fixed; there exists B such that (2.6) implies

$$|f(t,x)| \le (\varepsilon_0 + |\gamma(x)|) \cdot \theta(t),\ \ t \ge B,\ \ x \in \mathbb{R}^n.$$

Setting
$$c := \sup\{|\gamma(x)|\,,\ \ |x| \le A\}\,,$$
one obtains for every $x \in D$,
$$|f(t,x(t))| \le (\varepsilon_0 + c)\cdot\theta(t)\,, \tag{5.16}$$
which assures the relation (5.15).

Consider in C_l the associated operator to the problem $(5.1)+(5.2)$ of the easiest type,
$$Ux := x(+\infty) + \int_0^{(\cdot)} (Fx)(s)\ ds. \tag{5.17}$$

One easily deduces that (5.16) assures the compactness of U; therefore, by using the isomorphism Φ defined above, to U one corresponds an operator defined on $\Phi(D) \subset C([0,T],\mathbb{R}^n)$, which is compact, too.

Let $\Omega \subset C_l$ be a bounded open set; the set $\Omega_\Phi := \Phi(\Omega)$ will be bounded and open, too; furthermore,
$$\partial\Omega_\Phi = \Phi(\partial\Omega)\,.$$

To operator $U : \overline{\Omega} \to C_l$ given by (5.17) one corresponds the operator $U_\Phi : \overline{\Omega_\Phi} \to C([0,T],\mathbb{R}^n)$ given by
$$U_\Phi := \Phi U \Phi^{-1}.$$

An easy computation shows us that
$$U_\Phi y = y(T) + \int_0^{(\cdot)} g(s, y(s))\ ds.$$

But U_Φ is the associated operator to the problem $(5.12)+(5.13)$. Therefore, one can say that
$$\left\{x \in \overline{\Omega},\ Ux = x\right\} \simeq \left\{y \in \overline{\Omega_\Phi},\ U_\Phi y = y\right\}.$$

In particular, suppose that
$$x \ne Ux,\ x \in \partial\Omega.$$

Then,
$$y \ne U_\Phi y,\ y \in \partial\Omega_\Phi$$
and since Φ is an isomorphism, one gets
$$\deg(I - U,\ \Omega,\ 0) = \deg(I - U_\Phi,\ \Omega_\Phi,\ 0)\,.$$

Hence we obtain:

Corollary 18. *Assuming that the mentioned hypotheses are fulfilled and*

$$\deg\left(I - U_\Phi,\ \Omega_\Phi,\ 0\right) \neq 0,$$

then the problem (5.1) + (5.2) *admits solutions.*

An important particular case is the one when

$$f\left(t, x\right) = \theta\left(t\right) \cdot g\left(x\right), \tag{5.18}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function; in this case the problem (5.12)+ (5.13) becomes

$$\dot{y} = g\left(y\right),\ y\left(0\right) = y\left(T\right). \tag{5.19}$$

But in [5] one shows that for the operator U_Φ the following equality is true

$$\deg\left(I - U_\Phi,\ \Omega_\Phi,\ 0\right) = \pm \deg_B\left(g,\ \Omega_\Phi \cap \mathbb{R}^n,\ 0\right).$$

Since

$$\Omega_\Phi \cap \mathbb{R}^n = \Omega \cap \mathbb{R}^n,$$

we obtain the following

Theorem 19. *Assume that*
i) *the problem*

$$\dot{x} = \theta\left(t\right) \cdot g\left(x\right),\ x\left(0\right) = x\left(+\infty\right) \tag{5.20}$$

does not admit solutions $x\left(\cdot\right)$ *with* $x \in \partial\Omega$;
ii) *the following relation is fulfilled*

$$\deg_B\left(g,\ \Omega \cap \mathbb{R}^n,\ 0\right) \neq 0. \tag{5.21}$$

Then the problem (5.20) *admits solutions.*

□

Remark that the topological degree appearing in (5.21) is a Brouwer degree, which simplifies the problem in a certain way.

5.3 Homotopy with a linear equation

Consider throughout this section $X = C_l$.

Let $A : \mathbb{R}_+ \to M_n(\mathbb{R})$ be a continuous matrix; denote by $X(t)$ a fundamental matrix of the system

$$\dot{x} = A(t)\,x.$$

Consider the problems

$$\dot{x} = f(t, x), \ x(+\infty) = x(0), \tag{5.22}$$

$$\dot{x} = (1-\lambda)\,A(t)\,x + \lambda f(t, x), \ x(+\infty) = x(0), \ \lambda \in [0,1]. \tag{5.23}$$

Theorem 20. *Assume that:*
i) *the following condition holds*

$$\int_0^{+\infty} |A(t)| \ dt < +\infty; \tag{5.24}$$

ii) *the following equality holds*

$$\operatorname{rank}\ [X(+\infty) - X(0)] = n; \tag{5.25}$$

iii) $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function fulfilling the condition*

$$|f(t, x)| \le \theta(t) \cdot \omega(|x|), \tag{5.26}$$

where θ, $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous functions and*

$$\int_0^{+\infty} \theta(t) \ dt < +\infty; \tag{5.27}$$

iv) *there exists* $r > 0$ *such that for every* $\lambda \in [0,1]$ *the problem* (5.23) *has no solution* $x(\cdot)$ *such that* $\|x\|_\infty = r$.

Then the problem (5.22) *admits solutions.*

Proof. Set

$$h(t, x, \lambda) := (1-\lambda)\,A(t)\,x + \lambda f(t, x)$$

and

$$\Omega := \{x \in C_{ll}, \ \|x\|_\infty < r\}.$$

Since

$$|h(t, x, \lambda)| \le r\,|A(t)| + \rho\theta(t),$$

it results that hypothesis i) of Proposition 16 is fulfilled with

$$\alpha(t) = r\,|A(t)| + \rho\theta(t)\,,$$

where

$$\rho := \sup_{|u|\le r} \omega(u)\,.$$

The second hypothesis of Proposition 16 results by hypothesis iv) of our theorem.

Consequently, the equality (5.9) holds; but U_0 is an associated operator to the problem

$$\dot{x} = A(t)\,x,\ \ x(+\infty) = x(0)\,. \tag{5.28}$$

As it has been showed in Proposition 8, the problem (5.6) admits only the zero solution, which allows us to conclude that $U_0 : \overline{\Omega} \to C_{ll}$ is a linear, compact and injective operator; since $0 \in \Omega$, a known result leads us to the equality

$$\deg(I - U_0,\ \Omega,\ 0) = \pm 1,$$

which ends the proof. □

Consider now the problem

$$\dot{x} = A(t)\,x + g(t,x) + p(t)\,,\ \ x(+\infty) = x(0)\,, \tag{5.29}$$

where $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $p : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

Theorem 21. *Assume that:*
i) *the conditions* (5.24), (5.25) *are fulfilled;*
ii) *the following condition holds*

$$\int_0^{+\infty} |p(t)|\ dt < +\infty; \tag{5.30}$$

iii) *there exists* $\beta \in (0,1)$ *such that*

$$g(t,kx) = k^{\beta} g(t,x)\,,\ \ (\forall)\ \ k>0,\ \ (\forall)\ \ t \in \mathbb{R}_+,\ \ (\forall)\ \ x \in \mathbb{R}^n; \tag{5.31}$$

iv) the following inequality holds

$$|g(t,x)| \le \theta(t)\,,\ \ (\forall)\ \ t \in \mathbb{R}_+,\ \ (\forall)\ \ x \in \mathbb{R}^n,\ |x| \le 1, \tag{5.32}$$

where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies condition* (5.27).

Then the problem (5.29) *admits solutions.*

Proof. Apply Proposition 16 by taking

$$\begin{array}{l} h(t,x,\lambda) := (1-\lambda)\,A(t)\,x + \lambda\,(A(t)\,x + p(t) + h(t,x)) \\ \Omega := \{x \in C_{ll},\ \ \|x\|_\infty < r\} \end{array} . \tag{5.33}$$

The conditions (5.31) , (5.32) leads us to the inequality

$$|g(t,x)| \le r^\beta \theta(t),\ \ |x| \le r,$$

which implies

$$|h(t,x,\lambda)| \le 2r\,|A(t)| + r^\beta\theta(t) + |p(t)|\,.$$

Hence the first hypothesis of Proposition 16 is fulfilled. It rests to check that the hypothesis ii) of this proposition is satisfied for a certain Ω. For this aim we show that there exists $r_0 > 0$ such that for every $r > r_0$ and for every $\lambda \in [0,1]$, the problem

$$\dot{x} = h(t,x,\lambda)\,,\ x(+\infty) = x(0)\,, \tag{5.34}$$

with h given by (5.33) does not admit solutions $x(\cdot)$ with $\|x\|_\infty = r$.

Indeed, if this is not true, then one can find two sequences $\lambda_k \in [0,1]$, $r_k \in \mathbb{R}_+$, $r_k \to +\infty$, such that the problem (5.34) has solutions $x_k(\cdot)$ with $\|x_k\|_\infty = r_k$; by setting

$$u_k := \frac{1}{r_k}\cdot x_k,$$

we have

$$\|u_k\|_\infty = 1.$$

Therefore, the sequence $u_k(\cdot)$ is uniformly bounded on $\mathbb{R}_+$. But

$$\begin{array}{rcl} \dot{u}_k(t) & = & (1-\lambda_k)\,A(t)\,u_k(t) + \\ & & +\lambda_k\left[A(t)\,u_k(t) + r_k^{\beta-1}g(t,u_k) + r_k^{-1}p(t)\right] \end{array} \tag{5.35}$$

and

$$u_k(+\infty) = u_k(0)\,. \tag{5.36}$$

Since $r_k \to +\infty$ there exists $\gamma > 0$ such that $r_k^{-1} < \gamma$, $r_k^{\beta-1} < \gamma$; hence

$$|u_k(t)| \le 2\,|A(t)| + \gamma\,(\theta(t) + |p(t)|) \tag{5.37}$$

and so by Proposition 3, the sequence u_k is relatively compact.

One can suppose, without loss of generality, that

$$u = \lim_{k\to\infty} u_k.$$

By (5.35), it results that the sequence $\dot{u}_k$ is uniformly convergent on $\mathbb{R}_+$. Now, from (5.35) and (5.36) one gets

$$\dot{u}(t) = A(t)\,u(t)\,,\ u(+\infty) = u(0)\,,$$

which leads us to $\|u\|_\infty = 0$.

By other hand, from $\|u_k\|_\infty = 1$ it results $\|u\|_\infty = 1$. The proof is now complete. □

5.4 Nonlinear perturbations

In this section we consider the problem

$$\dot{x} = f(t,x) + p(t)\,,\ x(+\infty) = x(0)\,. \tag{5.38}$$

Suppose the following hypotheses:

a$_1$) $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$\int_0^{+\infty} \varphi(t)\ dt < +\infty, \tag{5.39}$$

where

$$\varphi(t) := \sup\{|f(t,x)|\,,\ \ |x| \le 1\}\,;$$

a$_2$) there exists $\beta \in (0,1)$ such that $f(t,kx) = k^\beta f(t,x)$, $(\forall)\ k > 0$, $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}^n$;

a$_3$) $p : \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous function such that

$$\int_0^{+\infty} |p(t)|\ dt < +\infty. \tag{5.40}$$

Let in addition $\theta : \mathbb{R}_+ \to \mathbb{R}^n$ be a continuous function such that

$$\int_0^{+\infty} \theta(t)\ dt = 1. \tag{5.41}$$

Set

$$g(x) := \int_0^{+\infty} f(s,x)\ ds. \tag{5.42}$$

Theorem 22. *Suppose that hypotheses* a$_1$), a$_2$) *and* a$_3$) *are fulfilled. Consider for $\lambda \in [0,1]$ the problem*

$$\dot{x} = (1-\lambda)\,\theta(t)\,g(x) + \lambda\,[f(t,x) + p(t)]\,,\ x(+\infty) = x(0)\,. \tag{5.43}$$

Then:
1) *if*

$$g(y) \neq 0,\ y \in \mathbb{R}^n,\ \ \|y\| = 1, \tag{5.44}$$

it results that there exists $r_0 > 0$ *such that for every* $\lambda \in [0,1]$ *the problem* (5.43) *does not admit solutions* $x(\cdot)$ *with* $\|x\|_\infty = r_0$;
2) *if for this* r_0 *we have*

$$\deg_B(g,\ \Sigma(r_0),\ 0) \neq 0,$$

where $\Sigma(r_0) := \{x \in \mathbb{R}^n,\ \ |x| < r_0\}$, *then the problem* (5.43) *admits solutions.*

Proof. Suppose by means of contradiction that the first conclusion is not true; then one can construct three sequences $r_k \in \mathbb{R}_+$, $x_k \in X$, $\lambda_k \in [0,1]$ such that

$$\begin{aligned} \dot{x}_k &= (1-\lambda_k)\,\theta(t)\,g(x_k) + \lambda_k\left[f(t,x_k) + p(t)\right], \\ x_k(+\infty) &= x_k(0),\ \ \|x_k\|_\infty = r_k,\ r_k \to \infty. \end{aligned} \tag{5.45}$$

Setting again

$$u_k := \frac{1}{r_k} \cdot x_k,$$

we have

$$\dot{u}_k = r_k^{\beta-1}\left[(1-\lambda_k)\,\theta(t)\,g(u_k) + \lambda_k f(t,u_k)\right] + r_k^{-1}\lambda_k p(t), \tag{5.46}$$

$$\|u_k\| = 1,\ u_k(+\infty) = u_k(0). \tag{5.47}$$

By using the conclusion of Proposition 3, we can establish that $(u_k)_k$ is relatively compact in C_l. By passing to sequences, one can suppose that

$$u_k \to u,\ \text{in } C_l \tag{5.48}$$

and

$$\lambda_k \to \lambda \in [0,1]. \tag{5.49}$$

By (5.46) it results that the sequence $\dot{u}_k$ is uniformly convergent in $\mathbb{R}_+$; more precisely,

$$\dot{u}_k \to 0, \tag{5.50}$$

hence u is constant on $\mathbb{R}_+$.

On the other hand,

$$(u_k(0) = u_k(+\infty)) \Longrightarrow \left(\int_0^{+\infty} \dot{u}_k(t)\ dt = 0\right) \tag{5.51}$$

and so, by (5.42), we obtain

$$\begin{aligned} 0 &= (1-\lambda_k)\int_0^{+\infty}\theta(t)\,g(u_k(t))\;dt+\lambda_k\int_0^{+\infty}f(s,u_k(s))\;ds+ \\ &\quad +r_k^{-1}\lambda_k p(t). \end{aligned} \tag{5.52}$$

Therefore, as $k\to\infty$,

$$g(u)=0,\ u\in\mathbb{R}^n,\ \ \|u\|=1, \tag{5.53}$$

which contradicts (5.44).

The second conclusion of theorem follows by Proposition 16 for $\Omega:=\{x\in C_l,\ \|x\|_\infty<r_0\}$. □

5.5 Small perturbations

Consider the problem

$$\dot{x}=\theta(t)\,g(x)+e(t,x),\ x(0)=x(+\infty), \tag{5.54}$$

where $g:\mathbb{R}^n\to\mathbb{R}^n$, $\theta:\mathbb{R}_+\to\mathbb{R}_+^*$, $e:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous functions and

$$\int_0^{+\infty}\theta(t)\;dt<+\infty.$$

Attach to problem (5.54) the problem

$$\dot{x}=\theta(t)\,g(x),\ x(0)=x(+\infty). \tag{5.55}$$

Let $D\subset\mathbb{R}^n$ be a bounded open set and take

$$\Omega:=\{x\in C_l,\ x(t)\in D,\ t\in\mathbb{R}_+\}.$$

We state now the following hypotheses:

b$_1$) for every $x(\cdot)$ solution of (5.55) for which $x(t)\in\overline{D}$, $(\forall)\ t\in\mathbb{R}_+$, it results $x(t)\in D$, $(\forall)\ t\in\overline{\mathbb{R}_+}$;

b$_2$) there exists a continuous function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\int_0^{+\infty}\alpha(t)\;dt<+\infty$ such that, for every $x\in\overline{\Omega}$, we have $|e(t,x(t))|\le\alpha(t)$, $(\forall)\ t\in\mathbb{R}_+$.

Consider in addition the problem

$$\dot{x}=\theta(t)\,g(x)+\lambda e(t,x),\ x(0)=x(+\infty),\ \lambda\in[0,1]. \tag{5.56}$$

Theorem 23. *If the hypotheses* b_1) *and* b_2) *are fulfilled, there exists* $\varepsilon_0 > 0$ *such that, if*

$$\|e(\cdot, y)\|_\infty < \varepsilon_0, \ \ (\forall) \ \ y \in \partial D, \tag{5.57}$$

then, for every solution $x(\cdot)$ *of the problem* (5.56) *for which* $x(t) \in \overline{D}$, $(\forall)$ $t \in \mathbb{R}_+$, *it results* $x(t) \in D$, $(\forall)$ $t \in \overline{\mathbb{R}_+}$.

If in addition

$$\deg_B(g, \ D, \ 0) \neq 0, \tag{5.58}$$

then for every $e(\cdot, \cdot)$ *satisfying* (5.57)*, the problem* (5.54) *admits solutions.*

Proof. The first part of the theorem is proved as above. If the conclusion is not true, then for every $k \in \mathbb{N}^*$ there exists a function $e_k(\cdot, \cdot)$ with $\|e_k(\cdot, y)\|_\infty < \frac{1}{k}$, $(\forall)$ $y \in \overline{D}$ and a function $x_k : \mathbb{R}_+ \to \mathbb{R}^n$ such that

$$\dot{x}_k = \theta(t) g(x_k) + \lambda_k e(t, x_k), \ x_k(0) = x_k(+\infty), \ \lambda_k \in [0, 1],$$

with $x_k(t) \in \overline{D}$, $(\forall)$ $t \in \overline{\mathbb{R}_+}$ and $x_k(t_k) \notin \partial D$, for an $t_k \in \mathbb{R}_+$.

As above, we can show that $(x_k)_k$ is compact in C_l; if $x_k \to x$ in C_l and $\lambda_k \to \lambda$, then one contradicts the hypothesis b_1).

The second part follows by Proposition 16 for $\Omega :=\{x \in C_l, \ x(t) \in D, \ (\forall) \ t \in \mathbb{R}_+\}$. □

5.6 Asymptotically homogenous perturbations

Consider again the problems

$$\dot{x} = \theta(t) g(x) + e(t, x), \ x(0) = x(+\infty), \tag{5.59}$$

$$\dot{x} = \theta(t) g(x), \ x(0) = x(+\infty), \tag{5.60}$$

$$\dot{x} = \theta(t) g(x) + \lambda e(t, x), \ x(0) = x(+\infty), \ \lambda \in [0, 1]. \tag{5.61}$$

Assume the following hypotheses:

c_1) $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$g(kx) = g(x), \ x \in \mathbb{R}^n, \ k > 0; \tag{5.62}$$

c_2) the following equality holds

$$\lim_{|x| \to \infty} \frac{e(t, x)}{|x|} = 0, \text{ uniformly with respect to } t \in \mathbb{R}_+; \tag{5.63}$$

c_3) $(\forall)$ $\rho > 0$, $\sup_{|x| \leq \rho} \{|e(t, x)|\} \in L^1(\mathbb{R}_+)$;

c_4) $\theta : \mathbb{R}_+ \to \mathbb{R}_+^*$, $\theta \in L^1(\mathbb{R}_+)$;

c_5) the problem

$$\dot{x} = \theta(t)\, g(x)\,,\ x(0) = x(+\infty) \tag{5.64}$$

admits only the zero solution.

Theorem 24. *Assume that the hypotheses* c_1), c_2), c_3), c_4) *and* c_5) *are fulfilled.*

Then there exists $r_0 > 0$, *such that for every solution* $x(\cdot)$ *of the problem* (5.61) *and for every* $\lambda \in [0,1]$ *we have*

$$\|x\|_\infty < r_0. \tag{5.65}$$

If in addition

$$\deg_B(g,\ \Sigma(r_0)\,,0) \neq 0, \tag{5.66}$$

then the problem (5.59) *admits solutions.*

Proof. The proof is analogous with those from the previous theorems. If the first conclusion is not true, then one finds $\lambda_k \in [0,1]$ and $x_k \in X$ satisfying

$$\dot{x}_k = \theta(t)\, g(x_k) + \lambda_k e(t, x_k)\,,\ x_k(0) = x_k(+\infty)\,,$$

$$\|x_k\|_\infty \to \infty.$$

Setting again $u_k = \frac{1}{\|x_k\|_\infty} \cdot x_k$, we deduce that $(u_k)_k$ is compact in X and it contains an uniformly convergent subsequence on $\mathbb{R}_+$ to a function u satisfying (5.64) and so $u = 0$; on the other hand, since $\|u_k\|_\infty = 1$, it results $\|u\|_\infty = 1$.

The second part is an immediate consequence of Proposition 16 and of condition (5.66). □

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