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EULER APPROXIMATION OF NONCONVEX DISCONTINUOUS DIFFERENTIAL INCLUSIONS

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Abstract

In the paper we study two types of time-discretization of one sided Lipschitz differential inclusions which right-hand side is neither upper nor lower semicontinuous. In the first one the original right-hand side is used. In the second one we use its closed graph convex regularization. It is remarkable that the both schemes give $O(h^{1/2})$ approximation of the solution set of the regularized differential inclusion. In the last section we apply these results to investigate some qualitative properties of differential inclusions in Hilbert spaces.

The paper is a natural extension of [6] (see also [8]). Let H be a Hilbert space and let $I = [0,1]$. Consider the following differential inclusion:

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0. \tag{1}$$

Here $x_0 \in H$ and F is a multifunction from $I \times H$ into H with nonempty closed and bounded values. The corresponding to (1) discretized inclusion is:

$$\dot{y}(t) \in F(t, y(t_i)); \; y(t_i) = \lim_{t \uparrow t_i} y(t); \; y(0) = x_0. \tag{2}$$

The mesh points on I are $0 = t_0 < t_1 < \cdots < t_N = 1$.

The main advantage of (2) is that we require only that $F(\cdot, x)$ admits a (strongly) measurable selection. No assumptions for $F(t, \cdot)$ have to be made.

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The following scheme (called Euler's scheme) is commonly used in the literature:

$$\begin{cases} z(t) = y(t_i) + (t - t_i)f_i, \quad z(0) = x_0, \quad z(t_{i+1}) = \lim\limits_{t \to t_{i+1}} z(t) \\ \quad \text{where} \quad f_i \in F(t_i, z(t_i)), \quad i = 0, 1, \ldots, N-1 \,, \end{cases} \tag{3}$$

The problem of the approximation of the solution set of (1) by the solution set of (3) is investigated in a great number of papers. We note only the survey [14] and the refferences therein. The approximation of the reachable set of (1) is considered in [1, 16, 19]. The most general result in case of Lipschitz differential inclusions is obtained in [10]. The so called strengthened one sided Lipschitz condition has been used in [13, 14] to obtain $O(h)$ approximation in case of autonomous differential inclusions. In [7] the one sided Lipschitz condition is used to obtain $C(w(F,h) + \tau(F,h))$ accuracy. Here $w(\cdot,\cdot)$ is the modulus of continuity of F on the state variable, while $\tau(\cdot,\cdot)$ denotes the so called averaged modulus of smoothness (cf. [7, 10]). Similar results (with accuracy $O(h^{1/2})$) are obtained in case of differential inclusions with almost Upper SemiContinuous (USC) right-hand side in [8]. The case of nonconvex right-hand side (without any accuracy estimation) is considered in [15]. In all (to the author knowledge) papers the problem (1) is considered in R^n. In [6] the space is infinite dimensional. The right hand side, however, admits (convex) compact values and is almost LSC.

In the paper we study (mainly) the approximated differential inclusion (2). We let $S(t,x) = \bigcap\limits_{\varepsilon > 0} \overline{co}\, F(t, x + \varepsilon U)$, where U is the open unit ball and $\bar{A}$ is the closure of A.

$$\dot{x}(t) \in S(t, x(t)), \quad x(0) = x_0. \tag{4}$$

Denote by R_i the solution set of the (differential) inclusion (i).

We show that the Hausdorff distance $D_H(R_2, R_4) \le O(h^{1/2})$. Here

$$D_H(A,B) = \max\{\sup_{a \in A} dist(a,B), \sup_{b \in B} dist(b,A)\}, \text{ where } dist(a,B) = \inf_{b \in B} |a - b|.$$

Further we show that R_4 is nonempty, $C(I,H)$ closed and depends in Lipschitz way by x_0 and (depends) continuously on parameters.

Denote by $P_f(H)$ the set of all nonempty, closed and bounded subsets of H and by $P_C(H)$ the set of all convex sets in $P_f(H)$. The support function of the set A is $\sigma(x,A) = \sup\limits_{a \in A} \langle x, a \rangle$.

Definition 1 *The multifunction $G : I \to P_f(H)$ is said to be measurable when the set $\{t \in I : G(t) \bigcap A \ne \emptyset\}$ is measurable for every open $A \subset H$.*

The multifunction $G : I \to P_f(H)$ is said to be strongly measurable when there exists a sequence $G_n : I \to P_f(H)$ of simple functions such that

$$\lim_{n\to\infty} \int_I D_H(G(t), G_n(t))\, dt = 0.$$

The multifunction G is called upper hemicontinuous (UHC) when the support function $\sigma(l, G(\cdot))$ from H equpped with the strong topology is upper semicontinuous as a real valued function.

Definition 2 *The multifunction $F : I \times H \to P_f(H)$ is said to be relaxed one-sided Lipschitz (ROSL) with a constant L (not necessarily positive) when*

$$\sigma(x-y, F(t,x)) - \sigma(x-y, F(t,y)) \le L|x-y|^2$$

for every $x, y \in H$ and a.a. $t \in I$.

The last definition has been used in much author's papers. Some properties and applications of this condition are studied in [8] (see also [6, 14]). It is esy to see that $S(t,\cdot)$ is ROSL with a constant L if $F(t,\cdot)$ is ROSL with a constant L.

Notice that all the concepts not discussed in the sequel can be found in [2, 5]. Now we give the main assumptios used in the paper.

A1. $F : I \times H \to P_f(H)$ is bounded on the bounded sets. F is ROSL.

A2. $F(\cdot, x)$ is strongly measurable or $F(\cdot, x)$ is measurable and H is separable. We need the following lemma which is proved in [6].

Lemma 1 *Assume A1 and A2 hold. Then there exist constants M and K such that $|x(t)| \le M - 1$ and $|S(t, x(t) + U) + \bar{U}| \le K$, for every absolutely continuous (AC) $x(\cdot)$ with $x(0) = x_0$ and $\dot{x} \in \overline{co}\ S(t, x + U) + \bar{U}$.*

In the next section we present our main results. In the last one we discuss briefly some applications of the results.

1 Euler approximations.

In this section we consider (mainly) the discretized inclusion (2). Suppose the mesh points are $t_i = ih$ where $h = \dfrac{1}{N}$. Throughout the paper we consider only steps $h > 0$ such that $hK \le 1$, where K is the constant from lemma 1. Given h we denote R_2 by R_h.

Theorem 1 *If A1 and A2 hold then $R_h \neq \emptyset$. Furthermore $R = \lim_{h\to 0^+} R_h$ exists and $R \subset R_4$.*

Proof. The fact that $R_h \neq \emptyset$ is obvious. Indeed if $hK \le 1$, every solution $y(\cdot)$ of (2) is also a solution of

$$\dot{y}(t) \in S(t, y(t) + U) + \bar{U}.$$

Thus Lemma 1 applies and hence $y(\cdot)$ can be extended on the whole I. Fix $h_1 > 0$ and let $x_1(\cdot)$ be a solution of (2) with $h = h_1$. Let $h_2 \neq h_1$. Suppose $y(\cdot)$ is AC function such that

$$\dot{y}(t) \in F(t, y(t^j)); \; y(t^j) = \lim_{t \uparrow t^j} y(t); \; y(0) = x_0.$$

We have denoted by $t^j = jh_2$ the mesh points of the second subdivision (the mesh points of the first subdivision are $t_i = ih_1$). For $t \in [t^j, t^{j+1}]$ we take a strongly measurable $f(t) \in F(t, y^j)$ $(y^j = y(t_j))$ such that

$$\left\langle y^j - x_1(t_i), f(t) - \dot{x}_1(t) \right\rangle \le L|y^j - x_1(t_i)|^2 \text{ when } t \in [t^j, t^{J+1}) \bigcap [t_i, t_{i+1}).$$

Notice first that $|f(t)| \le K - 1$ and $|\dot{x}_1(t)| \le K - 1$. We set

$$y(t) = y^j + \int_{t^j}^t f(s)\, ds.$$

Denote $h = \max\{h_1, h_2\}$. Evidently the following inequalities hold:

$$\left\langle y(t) - x_1(t), f(t) - \dot{x}_1(t) \right\rangle \le L|y(t) - x_1(t)|^2 + |L| \Big| |y^j - x_1(t_i)| - |y(t) - x_1(t)| \Big| +$$

$$+ |y(t) - y^j||\dot{y}(t) - \dot{x}_1(t)| + |x_1(t) - x_1(t_i)||\dot{y}(t) - \dot{x}_1(t)|$$

$$\le L|y(t) - x(t)|^2 + |L| \Big(|y^j - x_1(t_1) + y(t) - x_1(t)| \cdot |(y^j - y(t)) + (x_1(t) - x_1(t_i))| \Big)$$

$$+ 4K^2 h \le L|y(t) - x(t)|^2 + 8|L|MKh + 4K^2 h.$$

Obviously one can extend $y(\cdot)$ over the whole interval I such that $y(\cdot) \in R_{h_1}$; $y(0) = x_0$ and $|y(t) - x(t)|^2 \le r(t)$. Here $r(0) = 0$ and

$$\dot{r}(t) \le 2Lr(t) + 16|L|MKh + 8K^2 h,$$

i.e.

$$r(t) \le 8(|L|KM + 2K^2) h \exp(2Lt) \int_0^t \exp(-2Ls)\, ds$$

If

$$C = \max_{t \in I} \exp(Lt) \int_0^t \exp(-Ls)\, ds$$

then

$$|x(t) - y(t)| \le 4C\sqrt{|L|MK + K^2/2}h^{1/2}$$

(remind that $h = \max\{h_1, h_2\}$). If we do not use h we can derive

$$|x(t) - y(t)| \le 2C\sqrt{(2|L|MK + K^2/2)(h_1 + h_2)}.$$

Obviously such estimation is valid also when the grids are not uniform and

$$h_1 = \max_i (t_{i+1} - t_i);\ h_2 = \max_j (t_{j+1} - t_j).$$

($h_1K \le 1, h_2K \le 1$!) Therefore $\{R_h\}_{h>0}$ is a Cauchy net of (nonempty) closed subsets of $C(I, H)$. Thus there exists a nonempty $C(I, H)$ closed set $R = \lim_{h\to 0^+} R_h$. Suppose $x(\cdot) \in R$, i.e. $x(\cdot)$ is AC and hence a.e. differentiable function. Furthermore there exists a net $\{x_h(\cdot)\}_{h>0}$ with $x_h(\cdot) \in R_h$ and $\lim_{h\to 0^+} x_h(t) = x(t)$ uniformly on I. Since $|\dot{x}_h(t)| \le K$ for every $h > 0$ and every $x_h(\cdot) \in R_h$, one has that the net $\{\dot{x}_h(\cdot)\}_{h>0}$ is $L_1(I, H)$ weakly precompact. Using standard considerations one can show with the help of Mazur's lemma that $x(\cdot)$ is a solution of (4). ■

Corollary 1 *Assume all the conditions of theorem 1 hold. If* $y(\cdot) \in R_h$ *then*

$$dist(y(\cdot), R_4) \le 2C\sqrt{(2|L|MK + K^2)}h^{1/2}.$$

Proof. Fix $\varepsilon > 0$. One can construct a sequence $\{x_i(\cdot)\}_{i=1}^{\infty}$ of solutions of (2) with $|x_j(t) - x_{j+1}(t)| \le 2C\sqrt{(2|L|MK + K^2/2)(h_j + h_{j+1})} + \dfrac{\varepsilon}{2^j}$. Here $h_i = \max_i (t_{i+1}^j - t_i^j)$ is the step of the subdivision corresponding to $x_j(\cdot)$ and $h_{i+1} = \max_l (t_{l+1}^{j+1} - t_l^{j+1})$ the step of $x_{j+1}(\cdot)$. Obviously choosing appropriately $\{h_j\}_{j=1}^{\infty}$ one will obtain $\displaystyle\sum_{j=0}^{\infty} |h_j + h_{j+1}|^{1/2} \le h_0^{1/2} + \varepsilon$. ■

Theorem 2 *Assume all the conditions of theorem 1 hold. If* $x(\cdot) \in R_4$, *then* $dist(x(\cdot), R_h) \le 2C\sqrt{(2|L|M + K)K}h^{1/2}$.

Proof. Consider the following discretized inclusion:

$$\dot{x}(t) \in S(t, x(t_i)),\ x(t_i) = \lim_{t\uparrow t_i} y(t);\ y(0) = x_0. \quad (5)$$

The inclusion (5) is obtained when $F(\cdot,\cdot)$ in (2) is replaced by $S(\cdot,\cdot)$. Let $x(\cdot)$ be a solution of (4). Define the solution $z(\cdot)$ of (5) as follows:

$$z(t) = z(t_i) + \int_{t_i}^{t} f(\tau)\,d\tau, \text{ where } f(t) \in S(t, z(t_i)) \text{ for } t \in [t_i, t_{i+1})$$

is such that

$\big\langle x(t) - z(t_i), \dot{x}(t) - f(t)\big\rangle \leq L|x(t) - z(t_i)|^2$. Therefore

$$\begin{aligned}
&\big\langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\big\rangle \leq \\
&\Big|\big\langle x(t) - z(t_i), \dot{x}(t) - \dot{z}(t)\big\rangle - \big\langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\big\rangle\Big| + \\
&+\big\langle x(t) - z(t_i), \dot{x}(t) - \dot{z}(t)\big\rangle \leq L|x(t) - z(t_i)|^2 + |x(t) - z(t_i)||\dot{x}(t) - \dot{z}(t)| \\
&\leq L|x(t) - z(t)|^2 + |L|\Big(|x(t) - z(t_i)|^2 - |x(t) - z(t)|^2\Big) + M(t_\varepsilon - t)K^2.
\end{aligned}$$

Using the same fashion as in the previous proof one obtains:

$$|x(t) - z(t)| \leq 2C\sqrt{(2|L|M + K)K}h^{1/2}.$$

We have to prove that the solution set of (5) is the closure of the solution set of (2).

First we will show that given $x, y \in H$, a (strongly) measurable $f(t) \in S(t,x)$ and $\varepsilon > 0$ there exists a (strongly) measurable $g(t) \in F(t,y)$ such that $\big\langle x - y, f(t) - g(t)\big\rangle < L|x - y|^2 + \varepsilon$.

Let $f_1 \in S(t,x)$ be such that $\big\langle x-y, f_1\big\rangle = \sigma(x-y, S(t,x))$. Thus there exist $l_i \to 0$ and $f_i \in F(t, x+l_i)$ such that $\big\langle x-y, f_i\big\rangle \to \big\langle x-y, f_1\big\rangle$. Furthermore for every f_i there exists $g_i \in F(t,x)$ with $\big\langle x + l_i - y, f_i - g_i\big\rangle < L|x - y + l_i|^2 + \dfrac{\varepsilon}{i}$. Hence to $\delta > 0$ there exists $g_\delta \in F(t,y)$ such that $\big\langle x-y, f-g_\delta\big\rangle < L|x-y|^2+\delta$. Therefore $\sigma(x - y, S(t,x)) - \sigma(x - y, F(t,y)) \leq L|x - y|^2$ because $\delta > 0$ is arbitrary. Furthermore the multivalued map

$$F_\delta(t) := \{g \in F(t,y) : \big\langle x(t) - y, f(t) - g\big\rangle \leq L|x(t) - y|^2 + \delta$$

is obviously (strongly) measurable and hence admits a (strongly) measurable selection.

Fix $\varepsilon > 0$ and consider the solution $y_\varepsilon(\cdot)$ of (2) defined as follows:

$$\big\langle y_\varepsilon(t_i) - x(t), \dot{y}_\varepsilon(t) - \dot{x}(t)\big\rangle < L|y_\varepsilon(t_i) - x(t)|^2 + \frac{\varepsilon}{2^i} \text{ for } t \in [t_i, t_{i+1}).$$

One can easily show that $|x(t) - y_\varepsilon(t)| \leq 2C\sqrt{(2|L|M + K)K}h^{1/2} + \alpha\varepsilon^{1/2}$, where α is a constant (not depending on h and ε). Theorem is proved because $\varepsilon > 0$ is arbitrary. ∎

Corollary 2 *Under the assumptions of theorem 1*

$$D_H(R_h, R_4) \leq 2C\sqrt{(2|L|+K)K}h^{1/2}.$$

We have proved that $D_H(R_4, R_h) = O(h^{1/2})$. Now we consider the approximation scheme (3). If $F(\cdot,\cdot)$ in (3) is replaced by $\overline{co}\ F(\cdot,\cdot)$ then denoting the solution set by R_{3co} one has $D_H(R_{3co}, R_6) = O(h)$ where R_6 is the solution set of

$$\dot{x}(t) \in F(t_i, x(t_i)) \text{ on } [t_i, t_{i+1}),\ i = 0, 1, \ldots, N-1;\ x(0) = x_0. \tag{6}$$

Recall the definition of the averaged modulus of continuity (see [6, 8] for instance).

Let $\Delta = \{t_0, t_1, \cdots, t_n\}$ be a partition of I. Denote $I_k = [t_{k-1}, t_k], k = 1, \cdots, m$. Consider the vectors $\vec{y} = (y_1, y_2, \cdots, y_n) \in H^n$. If there exist $A \subset H$ such that $y_i \in A \subset H$ for $i = 1, \cdots, n$, we write $\vec{y} \in A$.

Given partition Δ, $h \in (0,1)$ and $x \in H$ we denote

$$\omega(F, \Delta, x, h, t) = \sup\{D_H(F(s,x), F(r,x)) : s, r \in [t - \frac{h}{2}, t + \frac{h}{2}] \cap I_k\}$$

Let $1 \leq p < \infty$, $h \in (0,1)$, the partition Δ and the vector $\vec{y} = (y_1, y_2, \cdots, y_n)$ be fixed. We denote

$$\rho(F, \Delta, \vec{y}, h)_p = \Big\{\sum_{k=1}^{m} \int_{I_k} \omega(F, \Delta, y_k, h, t)^p\, dt\Big\}^{\frac{1}{p}}.$$

The global L_p-averaged modulus of continuity of F with the step h is

$$\rho(F, A, h)_p = \sup_{\Delta} \sup_{\vec{y} \in A} \rho(F, \Delta, \vec{y}, h)_p$$

Here we have denoted $A = KU$. The following theorem holds true:

Theorem 3 *If all the assumptions of theorem 1 hold, then*

$$D_H(R_4, R_6) \leq C\big(h^{1/2} + \rho(\overline{co}\ F, h)_2\big).$$

The proof is omitted since it is very similar to the proof of Theorem 3 of [6] (see also lemma 4 of [8]).

Lemma 2 *If all the assumptions of theorem 1 hold, then there exists a constant C such that $D_H(R_3, R_{3co}) \leq Ch^{1/2}$.*

Proof. (Compare with theorem 6 of [6]) Consider the interval $[t_i, t_{i+1}]$. Let $y(\cdot) \in R_{3co}$ and let $y(t) = y_i + f_i(t - t_i)$, where $f_i \in \overline{co}\ F(t_i, y_i)$. Let $x(\cdot) \in R_3$ on $[0, t_i]$. Since $F(t, \cdot)$ is OSL one has that for every $\varepsilon > 0$ there exists $g_i \in F(t_i, x_i)$ such that $\langle x_i - y_i, f_i - g_i \rangle \leq L|x_i - y_i|^2 + \varepsilon$. We let $x(t) = x_i + (t - t_i)g_i$ on $[t_i, t_{i+1}]$. Consequently

$$\Big\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \Big\rangle \leq L|x(t) - y(t)|^2 + |L| \Big| |x_i - y_i|^2 - |x(t) - y(t)|^2 \Big|$$

$$+ \Big[|x_i - x(t)| + |y_i - y(t)| \Big] |\dot{x}(t) - \dot{y}(t)|. \text{ However,}$$

$$\Big[|x_i - x(t)| + |y_i - y(t)| \Big] |\dot{x}(t) - \dot{y}(t)| \leq 2K(t - t_i)2K = 4K^2(t - t_i)$$

$$\Big| |x_i - y_i|^2 - |x(t) - y(t)|^2 \Big| \leq$$

$$|(x_i + x(t)) - (y_i - y(t))| |(x_i - x(t)| + |y_i - y(t)|$$

$$\leq 2K(t_i - t)4M = 8KM(t - t_i).$$

Hence

$$\Big\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \Big\rangle \leq L|x(t) - y(t)|^2 + 8K(K + 2M|L|)h.$$

Obviously one can determine $x(\cdot) \in R_3$ such that the inequality above holds on the whole interval I. Thus

$$\frac{d}{dt}|x(t) - y(t)|^2 \leq 2L|x(t) - y(t)|^2 + 8K(K + 2M|L|)(h + \varepsilon). \text{ Consequently}$$

$$|x(t) - y(t)|^2 \leq \exp(2Lt) \Big(\int_0^t \exp(-2L\tau)\, d\tau \Big) 8K(K + 2M|L|)(h + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary one has that $D_H(R_3, R_{3co}) \leq Ch^{1/2}$. ∎

From Theorem 2 and Lemma 2 one obtains:

Theorem 4 *If all the assumptions of theorem 1 hold, then there exists a constant $C > 0$ such that $D_H(R_3, R_4) \leq C(h^{1/2} + \rho(\overline{co}\ F, h)_2)$.*

Corollary 3 *Consider the system:*

$$\dot{x}(t) \in F(t, x, u(t)),\ x(0) = x_0;\ u(t) \in V \text{- metric compact.} \tag{7}$$

Let $F(\cdot, x, u)$ be (strongly) measurable, $F(t, \cdot, u)$ be UHC with convex weakly compact values, $F(t, x, \cdot)$ be continuous. If F is OSL with a constant L non-depending on u, then the solution set of (7) is dense in the solution set of

$$\dot{x}(t) \in \overline{co}\ F(t, x, V),\ x(0) = x_0.$$

Remark 1 It seems strange but due to Theorem 3 the accuracy of the approximation scheme (3) is the same no matter $F(\cdot,\cdot)$ or $\overline{co}\ F(\cdot,\cdot)$ is used. For example replace $F(\cdot,\cdot)$ by $\overline{ext}F(\cdot,\cdot)$. The last multimap is neither USC nor LSC on the state variable. The solution set of

$$\dot{x} \in \overline{ext}\ F(t,x),\ x(0) = x_0$$

will be empty in general case. However the approximation scheme (2) applied to the last differential inclusion approximate with $O(h^{1/2})$ accuracy R_4.

2 Concluding remarks.

In this section we discuss briefly some applications of the previous results. In some cases the stile will be extremely concise, because we give only overview of the problems. The detail investigation of the examples below is not the topic of the paper.

Proximal cones and strong invariance. First we consider the problem (1) and the corresponding differential inclusion (4). We are looking for the solutions of (4), belonging to a given closed set D. We will follow closedly [3] (see also [4] for more details in case $H = R^n$).

The (possibly empty) set af all closest points to x in D is denoted $proj_D(x) = \{s \in D : |x - s| = dist(x, D)\}$. Given $\delta > 0$ we let $proj_D^\delta = \{s \in D : |x - s|^2 < dist^2(x, D) + \delta^2\}$. Obviously the last set is always nonempty. If $x \notin D$ and $s \in proj_D(x)$ we call the vector $x - s$ a perpendicular to D at s. The set of all nonnegative multipliers of such perpendiculars is called proximal normal cone to D at s and is denoted by $N_D^P(s)$. If $s \in int(D)$ or no perpendicular to D at s exists, then we set $N_D^P(s) = \{0\}$. We will use the following

Proposition 1 *(*Proposition 2.2 of [3]*) Let $x \in H \setminus D, \delta > 0$ and $s_\delta \in proj_D^\delta(x)$. Then there exists $y_\delta \in H\backslash D$ and $\bar{s}_\delta \in D$ such that $y_\delta - \bar{s}_\delta \in N_D^P(\bar{s}_\delta)$, $|(y_\delta - \bar{s}_\delta) - (x - s_\delta)| \leq 2\delta$ and $|s_\delta - \bar{s}_\delta| \leq \delta$.*

The AC function $x(\cdot)$ is said to be *ε-solution* of (1) when $\dot{x}(t) \in F(t, x(t) + \varepsilon U)$ for a.a. $t \in I$. Given the closed set $D \subset H$ we let $x_0 \in D$.

Definition 3 *(c.f. [3, 4]) The system (1) is said to be approximately weakly invariant (with respect to D) when for any $\varepsilon > 0$ and any $x_0 \in D$ there exists a ε-solution $x(\cdot)$ of (1) on $[0,1]$ such that $dist(x(t), D) \leq \varepsilon\ \forall t \in I$. The system (1) is said to be weakly invariant if there exists a solution $x(\cdot)$ of (1) such that $x(t) \in D$. The system (1) is said to be approximately strongly invariant if for every $\lambda > 0$ and any $x_0 \in D$ there exists $\varepsilon(x_0, \lambda) > 0$ such that every ε-solution $x(\cdot)$ remains in D when $\varepsilon < \varepsilon(x_0, \delta)$. Analogously (1) is called strongly invariant when every solution $x(\cdot)$ of (1) satisfies $x(t) \in D$.*

Theorem 5 *Let all the assumptions of Theorem 1 hold. Given a closed set $D \subset H$, the system (4) is strongly invariant if there exits a null set $A \subset I$ such that $\sigma(p, F(t,x)) \le 0$ $\forall p \in N_D^P(x)$ $\forall x \in D$, $\forall t \in I \setminus A$.*

Proof. Let $\delta \in (0, \frac{1}{4})$ be given. For $x \in H$ we choose $s_\delta \in proj_D^\delta(x)$. Since $|F(t,x)| \le K$, one has that every strongly measurable $F_\delta(\cdot, x)$ with $f_\delta(t,x) \in F(t,x)$ satisfies $|f_\delta(t,x)| \le K$. Due to proposition 1 there exist $(s_\delta(x), \bar{s}_\delta(x))$ and strongly measurable $f_\delta(t,x) \in F(t, \bar{s}_\delta(x) + \delta U)$ satisfying $\big\langle f_\delta(t,x), x - \bar{s}_\delta(x)\big\rangle \le 2K\delta$ and $|f_\delta(t,x)| \le K$. Thus $\big\langle f_\delta(t,x), x - s_\delta(x)\big\rangle \le 4K\delta$.

Given a subdivision $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ we define $y(t) = y(t_i) + \displaystyle\int_{t_i}^{t} f_\delta(s, y_i)\, ds$, where $y(t_i) = \lim_{t \uparrow t_i} y(t)$. Obviously:

$$d_D^2(y_{i+1}) \le |y_{i+1} - s_i|^2 = |y_{i+1} - y_i|^2 + |y_i - s_i|^2 + 2\big\langle y_{i+1} - y_i, y_i - s_i \big\rangle$$

$$\le K^2 |t_{i+1} - t_i|^2 + d_D^2(y_i) + \delta^2 + 2\int_{t_i}^{t_{i+1}} \big\langle f_\delta(t, y_i), y_i - s_i \big\rangle\, dt. \text{ Here}$$

$y_i = y(t_i), s_i = s_\delta(y_i)$. Therefore

$$d_D^2(y_{i+1}) - d_D^2(y_i) \le \big(K^2 d(\Delta) + d(\Delta) + 8Kd(\Delta)\big)(t_{i+1} - t_i) \le \tilde{\varepsilon}(t_{i+1} - t_i),$$

where $d_D(y_i) = dist(y_i, D)$ and $d(\Delta) = \max_i |t_{i+1} - t_i|$. Consequently

$$d_D(y_{i+1}) \le \frac{\tilde{\varepsilon}}{2} \text{ for } i = 0, 1, 2, \ldots, N-1.$$

We have proved that the system (4) is approximately weakly invariant. Consider the sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ with $\varepsilon_i > \varepsilon_{i+1} \to 0^+$. From the proof of Theorem 1 we know that there exists a constant C such that for every ε_i-solution $x^i(\cdot)$ of (4) there exists a ε_{i+1}-solution $x^{i+1}(\cdot)$ with $|x^i(t) - x^{i+1}(t)| \le C\sqrt{\varepsilon_i + \varepsilon_{i+1}}$ on I. Thus $x^i(\cdot) \to x(\cdot)$ uniformly on I (for appropriately chosen ε_i). Furthermore $x(\cdot)$ is obviously a solution of (4) and $x(t) \in D$. Consequently the system (4) is weakly invariant. Evidently there exists a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ such that $|x^i(t) - x(t)| \le 2C\sqrt{\varepsilon_i}$. Suppose $y(\cdot)$ is a solution of (4) such that $y(t) \notin D$ for some $t \in I$. Denote $\varepsilon = \max_{t \in I} d_D(y(t)) > 0$. Obviously for every $\delta > 0$ there exists a δ -solution $x^\delta(\cdot)$ such that $|x^\delta(t) - y(t)| \le C\sqrt{\delta} < \frac{\varepsilon}{3}$. As it was shown there exists a solution $x(\cdot)$ of (4) such that $|x^\delta(t) - x(t)| \le 2C\sqrt{\delta} < \frac{2\varepsilon}{3}$, i.e. $|x(t) - y(t)| < \varepsilon$ - contradiction. The theorem has been proved. ∎

Remark 2 In [3, 4] the conditions of Theorem 5 are given under the Hamiltonians.

Namely for $p \in H$ define the upper Hamiltonian $H(t,x,p) = \sup\{\langle p, v\rangle : v \in F(t,x)\}$ and assume $H(t,x,p) \le 0$. Obviously $H(t,x,p) = \sigma(p, F(t,x))$. Furthermore one can consider the case when $F(t,\cdot)$ is defined only on D. The existence of solutions of (4) can be proved also under the conditions of theorem 4 since no compactness conditions are required. However, when $F(t,\cdot)$ is defined on the whole H and $S(t,\cdot)$ is USC with compact values one can prove the existence of a (viable) solution of (4) when $h(t,x,p) \le 0$, where $h(t,x,p) = \min\{\langle v,p\rangle, v \in S(t,x)\}$.

When $F(\cdot,\cdot)$ is almost continuous one can replace if in theorem 5 by if and only if.

Averaging of differential inclusions. The averaging technique is comprehensively studied in [17] (see also [18]).

$$\dot{x} \in F(t,x) + \varepsilon G(t,x) \text{ on } [0,\infty), \quad x(0) = x_0. \tag{8}$$

Denote

$$R(x) = \lim_{T\to\infty} cl \frac{1}{T}\int_0^T G(t,x)\,dt. \tag{9}$$

Let the limit (9) exist in sufficiently large neighborhood of x_0. According to (8) we obtain the averaged differential inclusion:

$$\dot{x} \in F(t,x) + \varepsilon R(x(t)) \text{ on } [0,\infty), \quad x(0) = x_0. \tag{10}$$

Suppose the limit (9) exists uniformly on a domain B.
Let A1, A2 hold. Assume $F(\cdot,\cdot)$ satisfies A1 and A2 with constant $L_F \le 0$. Let $G(\cdot,\cdot)$ satisfies A1 and A2 (L_G may be positive). A typical averaging theorem is the following:

Theorem 6 *Let F and G be UHC convex and weakly compact values. Suppose there exist a subset $B' \subset B$ and $\mu > 0$, such that for every $\varepsilon > 0$ and every solution $x(\cdot)$ of (10) with $x_0 \in B$ one has $x(t) + l \in \Delta$ $\forall t \in [0,\varepsilon^{-1}]$ and $\forall l \in \mu U$.*

Then for every $\eta > 0$ there exists $\varepsilon(\eta) > 0$ such that $D_H(S_1,S_2) \le \eta$ on $[0,\varepsilon^{-1}]$ for $\varepsilon < \varepsilon(\eta)$, where S_1 and S_2 are the solution sets of (8) and (10) respectively.

Proof. Fix $\varepsilon > 0$. Taking into account the uniform convergence in (9) one can choose $m_\varepsilon \to \infty$ with $\varepsilon m_\varepsilon \to 0$ as $\varepsilon \to 0^+$ such that

$$\lim_{\varepsilon\to 0^+} D_H\Big(R(x), \frac{1}{q}\int_t^{t+q} G(s,x)\,ds\Big) = 0,\ [0,\varepsilon^{-1}] \times B,$$

here $q = \dfrac{1}{\varepsilon m_\varepsilon}$. It is easy to see that $R(\cdot)$ is also OSL with a constant L_G. Furthermore $\forall \delta > 0 - \exists m_\varepsilon$ such that $D_H(\tilde{S}(\varepsilon, G), S(\varepsilon, G)) \le \delta(\varepsilon)$ and $D_H(\tilde{S}(\varepsilon, R), S(\varepsilon, R)) \le \delta(\varepsilon)$. Here $\tilde{S}(\varepsilon, G)$ is a solution set of

$$\dot{x}(t) \in F(t, x) + \varepsilon G(t, x_i),\ x_i = x(t_i), \quad t_i = iq.$$

Using standard computations one can show that $\delta^2(\varepsilon) \le 2r_\varepsilon(t)$, where $\dot{r}_\varepsilon(t) = L_F r_\varepsilon(t) + 4\varepsilon(|L_G| + 1)MNq,\ r_\varepsilon(0) = 0$. The constants $N \ge |F(t, x(t))|$ for every solution $x(\cdot)$ of (8) and $M \ge \max\{\tilde{S}(\varepsilon, G), \tilde{S}(\varepsilon, R), S(\varepsilon, G), S(\varepsilon, R)\}$ exist thanks to lemma 1. Since $L_F \le 0$, one has that $r_\varepsilon(t) \le \dfrac{4(|L_G| + 1)MN}{m_\varepsilon}$. Hence $\delta(\varepsilon) \le \sqrt{\dfrac{8(|L_G| + 1)MN}{m_\varepsilon}}$. One can finish the proof following closedly the proof of theorem 1 of [18]. ∎

Singularly perturbed differential inclusions. The averaging procedure of singularly perturbed differential inclusions and contron systems has been investigated in large number of papers. We note [9, 11, 12] and the refences therein. The following Cauchy problem is a typical singularly perturbed system.

$$\begin{aligned} &\dot{x}(t) \in F(x, y, u),\ x(0) = x^0,\ u \in V \text{ - metric compact} \\ &\varepsilon\dot{y}(t) \in G(x, y, u),\ y(0) = x^0,\ t \in I = [0, 1]. \end{aligned} \tag{11}$$

Here $F : H_1 \times H_2 \times V \to P_c(H_1)$ and $G : H_1 \times H_2 \times V \to P_c(H_2)$. Notice that H_1 and H_2 are infinite dimensional and F, G are not necessarily compact valued. So we can not obtain a result similar to theorem 4 of [9].

The following assumption is crucial.

B1. There exist positive constants A, B, D, μ such that

$$\sigma(x_1 - x_2, F(x_1, y_1, u)) - \sigma(x_1 - x_2, F(x_2, y_2, u)) \le A|x_1 - x_2|^2 + B|y_1 - y_2|^2,$$
$$\sigma(y_1 - y_2, G(x_1, y_1, u)) - \sigma(y_1 - y_2, G(x_2, y_2, u)) \le D|x_1 - x_2|^2 - \mu|y_1 - y_2|^2,$$

uniformly on $u \in V$. Furthermore F and G are UHC. They have nonempty, convex and weakly compact values and are bounded on bounded sets.

Given x the associated system is:

$$\dot{y}(\tau) \in G(x, y(\tau), u(\tau)),\ y(0) \in Q \subset H_2,\ u \in V. \tag{12}$$

Denote $\bar{W}(x, S, Q) = \overline{\left\{\dfrac{1}{S}\displaystyle\int_0^S F(x, Y(\tau, x, S, Q), u(\tau))\, d\tau :\ u(\tau) \in V\right\}}$, where $Y(\tau, x, S, Q)$ is the solution set of (12) on the interval $[0, S]$. Assume there exists a ROSL (with closed bounded values) multifunction $\bar{W}(x)$ such that $graph(\bar{W}(x)) = \lim_{S\to\infty} graph(\bar{W}(x,S,Q))$. Then under certain conditions on F and G one can prove that the "slow" part of the solution set of (11) converges to the solution set of

$$\dot{x}(t) \in \bar{W}(x),\ x(0) = x^0,\ t \in I. \tag{13}$$

With the help of corollary 3 one can easily prove that the solution set of (13) is dense in the solution set of:

$$\dot{x}(t) \in \overline{co}\ \bar{W}(x),\ x(0) = x^0.$$

Now following (with essential modifications) the proofs of Lemmas 3.5 and 3.6 of [9] one would be able to prove:

Theorem 7 *Under the assumptions above* $\lim_{\varepsilon\to 0^+} D_H(X(\varepsilon), X(0)) = 0$*, where* $X(\varepsilon)$ *is the slow part of the solution set of (11) and* $X(0)$ *is the solution set of (13).*

Here we have described briefly the problem. The comprehensive investigation will be subject to other paper.

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