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ACTIONS OF GROUPS ON LATTICES

Abstract

The aim of this paper is to study the actions of the groups on lattices and to give some connections between the structure of a group and the structure of its subgroup lattice. Moreover, we shall introduce the concept of direct $\vee$-sum of G-sublattices and we shall present a generalization of a result about finite nilpotent groups.

1 Preliminaries

Let $(G, \cdot, e)$ be a monoid and L be a G–set (relative to an action ρ of G on L; for $(g, \ell) \in G \times L$, we denote by $g \circ \ell$ the element $\rho(g)(\ell) \in L$). If L is a poset (reltive to a partial ordering relation "$\leq$") and, for $\ell, \ell' \in L$, $\ell \leq \ell'$ implies $g \circ \ell \leq g \circ \ell'$, for any $g \in G$, then L is called a *G–poset*. Moreover, if $(L, \leq)$ is a lattice and, for $\ell, \ell' \in L$, we have:

$$g \circ (\ell \wedge \ell') = (g \circ \ell) \wedge (g \circ \ell'),$$
$$g \circ (\ell \vee \ell') = (g \circ \ell) \vee (g \circ \ell'),$$

for any $g \in G$, then L is called a *G–lattice*.

A *G–sublattice* of a G–lattice L is a sublattice L' of L satisfying the property:

$$G \circ L' = \{g \circ \ell' \mid g \in G,\ \ell' \in L'\} \subseteq L'.$$

Let L_1 and L_2 be two G–posets (respectively two G–lattices). A monotone map (respectively a lattice homomorphism) $f : L_1 \longrightarrow L_2$ is called a *G–poset homomorphism* (respectively a *G–lattice homomorphism*) if $f(g \circ \ell_1) = g \circ f(\ell_1)$, for any $(g, \ell_1) \in G \times L_1$. Moreover, if f is one-to-one and onto, then it is called a *G–poset isomorphism* (respectively a *G–lattice isomorphism*).

A *G–congruence* on a G–lattice L is a congruence relation "$\sim$" on L which has the property that $\ell \sim \ell'$ $(\ell, \ell' \in L)$ implies $g \circ \ell \sim g \circ \ell'$, for any $g \in G$.

Let L be a G–lattice and "$\sim$" be a G–congruence on L. Then the quotient lattice $L/\!\sim\ = \{[\ell] \mid \ell \in L\}$ of L modulo "$\sim$" is a G–lattice, where $g \circ [\ell] = [g \circ \ell]$, for any $(g, \ell) \in G \times L$.

If $f : L_1 \longrightarrow L_2$ is a G–lattice homomorphism, then the sublattice $\mathrm{Im}\, f = \{f(\ell_1) \mid \ell_1 \in L_1\}$ of L_2 is a G–lattice and there exists a G–congruence "$\sim$" on L_1 such that the G–lattices $L_1/\sim$ and $\mathrm{Im}\, f$ are isomorphic.

Let L be a lattice having the initial element 0. On L is well defined the *height* function: for $\ell \in L$, let $h_L(\ell)$ denote the length of a longest maximal chain in $[0, \ell]$ if there is a finite longest maximal chain; otherwise put $h_L(\ell) = \infty$. If L is of finite length, then the following conditions are equivalent:

i) L is modular.

ii) The height function h_L on L satisfies the property:
$h_L(\ell) + h_L(\ell') = h_L(\ell \wedge \ell') + h_L(\ell \vee \ell')$, for any $\ell, \ell' \in L$.

2 Main results

2.1 Finite G–lattice

Let $(G, \cdot, e)$ be a monoid.

Proposition 1. *Let $(L, \leq)$ be a complete lattice such that L is a G–poset. Then we have:*

$$G = \bigcup_{\ell \in L} \mathrm{Stab}_G(\ell).$$

Proof. Let $g \in G$ and $L_g = \{\ell \in L \mid g \circ \ell \geq \ell\}$. We have $L_g \neq \emptyset$ (L_g contains the initial element of L). Since L is complete, there exists $\bar{\ell} = \vee L_g$. We have $\ell \leq g \circ \ell \leq g \circ \bar{\ell}$, for any $\ell \in L_g$, therefore:

$$\bar{\ell} \leq g \circ \bar{\ell}. \tag{1}$$

Using the relation (1), we obtain that $g \circ \bar{\ell} \leq g \circ (g \circ \bar{\ell})$, thus $g \circ \bar{\ell} \in L_g$. Since $\bar{\ell} = \vee L_g$, it results:

$$g \circ \bar{\ell} \leq \bar{\ell}. \tag{2}$$

The relations (1) and (2) give us $g \circ \bar{\ell} = \bar{\ell}$, so that $g \in \mathrm{Stab}_G(\bar{\ell})$. Thus $G = \bigcup\limits_{\ell \in L} \mathrm{Stab}_G(\ell)$.

Corollary. (The Fixed–Point Theorem of complete lattice)
Any monotone map of a complete lattice L into itself has a fixed point.

Proof. The set G' of all monotone maps of L into itself is a monoid. Moreover, L is a G'-poset, where $f \circ \ell = f(\ell)$, for any $(f, \ell) \in G' \times L$. From

Proposition 1, we obtain $G' = \bigcup_{\ell \in L} \mathrm{Stab}_{G'}(\ell)$, therefore, for any $f \in G'$, there exists $\ell \in L$ such that $f \in \mathrm{Stab}_{G'}(\ell)$, i.e. $f(\ell) = f \circ \ell = \ell$.

In the followings we suppose that $(G, \cdot, e)$ is a group and we denote by $L(G)$ (respectively by $L_0(G)$) the lattice of subgroups of G (respectively the lattice of normal subgroups of G).

Proposition 2. *Let L be a complete G–lattice such that* $\mathrm{Stab}_G(\ell) = \{e\}$*, for any $\ell \in L$. Then the group G is abelian.*

Proof. Let g_1, g_2 be two elements of G and $f_{g_1,g_2} : L \longrightarrow L$ be the map defined by $f_{g_1,g_2} = [g_1, g_2] \circ \ell$, for any $\ell \in L$ (where $[g_1, g_2]$ is the commutator of g_1 and g_2). We have $f_{g_1,g_2}(\ell \wedge \ell') = [g_1, g_2] \circ (\ell \wedge \ell') = ([g_1, g_2] \circ \ell) \wedge ([g_1, g_2] \circ \ell') = f_{g_1,g_2}(\ell) \wedge f_{g_1,g_2}(\ell')$, for any $\ell, \ell' \in L$, thus f_{g_1,g_2} is a monotone map. From the above corollary, we obtain that there exists $\ell_0 \in L$ such that $f_{g_1,g_2}(\ell_0) = \ell_0$. It results $[g_1, g_2] \in \mathrm{Stab}_G(\ell_0)$, i.e. $[g_1, g_2] = e$.

Since any ordered latticeal group G is a G–lattice, from Proposition 2 we obtain the following result:

Corollary. *Any ordered latticeal group complete as lattice is abelian.*

Let L be a finite G–lattice, 0 be the initial element of L and 1 be the final element of L.

Remark. If $L = \{\ell_1 = 0, \ell_2, ..., \ell_m = 1\}$ and $H_i = \mathrm{Stab}_G(\ell_i)$, $i = \overline{1, m}$, then from Proposition 1, we have $G = \bigcup_{i=1}^{m} H_i$. Let I be a maximal subset of $\{1, 2, ..., m\}$ with the property:

$$\begin{cases} G = \bigcup_{i \in I} H_i \\ H_j \not\subseteq \bigcup_{i \in I \setminus \{j\}} H_i, \quad \text{for any } j \in I. \end{cases}$$

Then, for any $g \in G$, there exists $n_g \in \mathbb{N}^*$ such that $g^{n_g} \in \bigcap_{i \in I} H_i$. Since, for any $\ell, \ell' \in L$, $\mathrm{Stab}_G(\ell) \cap \mathrm{Stab}_G(\ell') \subseteq \mathrm{Stab}_G(\ell \wedge \ell')$, we obtain that there exists $\ell_0 \in L$ such that every element of G has a natural power in $\mathrm{Stab}_G(\ell_0)$.

We suppose that G is a finite group, $\mathrm{Stab}_G(0) = \mathrm{Stab}_G(1) = G$ and let $f_L : L \longrightarrow L$ be the map defined by $f_L(\ell) = \bigwedge_{g \in G} g \circ \ell$, for any $\ell \in L$.

Proposition 3. *The map f_L is a G-poset homomorphism which has the following properties:*

a) $f_L(\ell) \leq \ell$, *for any* $\ell \in L$.

b) $\operatorname{Im} f_L = \operatorname{Fix}_G(L)$, *where* $\operatorname{Fix}_G(L) = \{\ell \in L \mid g \circ \ell = \ell$, *for any* $g \in G\}$.

c) $f_L^2 = f_L$.

Proof. a) Since $e \circ \ell = \ell$, we obtain $f_L(\ell) = \ell \wedge \left(\bigwedge\limits_{g \in G \setminus \{e\}} g \circ \ell \right) \leq \ell$, for any $\ell \in L$.

b) Let $\ell' \in \operatorname{Im} f_L$. Then there exists $\ell \in L$ such that $\ell' = f_L(\ell)$. For any $g' \in G'$, we have:

$$g' \circ \ell' = g' \circ f_L(\ell) = g' \circ \left(\bigwedge_{g \in G} g \circ \ell \right) = \bigwedge_{g \in G} g' \circ (g \circ \ell) = \bigwedge_{g \in G} (g'g) \circ \ell = f_L(\ell) = \ell',$$

therefore $\ell' \in \operatorname{Fix}_G(L)$.

Conversely, let $\ell' \in \operatorname{Fix}_G(L)$. Then $g \circ \ell' = \ell'$, for any $g \in G$. It results $f_L(\ell') = \bigwedge\limits_{g \in G} g \circ \ell' = \bigwedge\limits_{g \in G} \ell' = \ell'$, thus $\ell \in \operatorname{Im} f_L$.

c) We have $f_L^2(\ell) = f_L(f_L(\ell)) = \bigwedge\limits_{g \in G} g \circ f_L(\ell) = \bigwedge\limits_{g \in G} f_L(\ell) = f_L(\ell)$, for any $\ell \in L$. Thus $f_L^2 = f_L$.

Now, the fact that f_L is a G-poset homomorphism is obvious.

Remark. If L is a fully ordered G–lattice, then f_L is a G–lattice homomorphism. Moreover, the binary relation"$\sim$" on L defined by $\ell \sim \ell'$ if and only if $f_L(\ell) = f_L(\ell')$ is a G-congruence. Therefore, we obtain the G–latice isomorphism:

$$L/\sim \; \cong \operatorname{Fix}_G(L).$$

Let $n = |\operatorname{Fix}_G(L)|$ and $C_1, C_2, ..., C_n$ be the equivalence classes modulo "$\sim$". If $(\ell_i')_{i=\overline{1,n}}$ is a set of representatives for the equivalence classes $(C_i)_{i=\overline{1,n}}$ then $C_i = \{\ell \in L \mid f_L(\ell) = f_L(\ell_i')\} \neq \emptyset$, $i = \overline{1,n}$, $C_i \cap C_j = \emptyset$, for $i \neq j$ and $L = \bigcup\limits_{i=1}^{n} C_i$. Moreover, for any $i \in \{1, 2, ..., n\}$, we have:

$$G \circ \ell_i' = \{g \circ \ell_i' \mid g \in G\} \subseteq C_i.$$

It results that:

$$|G \circ \ell'_i| = \frac{|G|}{|\mathrm{Stab}_G(\ell'_i)|} \leq |C_i|, \ i = \overline{1,n}.$$

This implies the following inequality:

$$(*) \qquad |G| \sum_{i=1}^{n} \frac{1}{|\mathrm{Stab}_G(\ell'_i)|} \leq \sum_{i=1}^{n} |C_i| = |L|.$$

Let $C_{i_1}, C_{i_2}, ..., C_{i_r}$ be the classes having an unique element (i.e. $c_{i_j} = \{\ell'_{i_j}\}$, $j = \overline{1,r}$, where $r \leq n$, $i_r = n$ and $\ell'_n = 1$). Then, for each $s \in \{1, 2, ..., n\} \setminus \{i_1, i_2, ..., i_r\}$, we can suppose that $\ell'_s \notin \mathrm{Fix}_G(L)$. We obtain $|G \circ \ell'_s| \neq 1$, therefore

$$\frac{|G|}{|\mathrm{Stab}_G(\ell'_s)|} \geq p,$$

where p is the smallest prime divisor of $|G|$. Using the inequality $(*)$, it results that:

$$|L| \geq pn - (p-1)r.$$

Taking the particular case $L = L(G)$, it obtains the following results:

Corollary 1. *If G is a finite group and r is the number of equivalence classes modulo "$\sim$" having a unique element, then:*

$$|L(G)| \geq p|L_0(G)| - (p-1)r,$$

where p is the smallest prime divisor of $|G|$.

Corollary 2. *If G is a nonabelian simple finite group, then:*

$$|L(G)| \geq p + 1,$$

where p is the smallest prime divisor of $|G|$.

Remark. Let $\mathrm{Min}(L)$ be the set of all minimal elements of L and $\mathrm{Ker}\, f_L = \{\ell \in L \mid f_L(\ell) = 0\}$. Then the following relations hold:

$$(**) \qquad \mathrm{Min}(L) \subseteq \mathrm{Ker}\, f_L \cup \mathrm{Fix}_G(L).$$

Indeed, if $\ell \in \mathrm{Min}(L)$ and $f_L(\ell) \neq 0$, then, from the inequalities $0 \leq f_L(\ell) \leq \ell$, we obtain $f_L(\ell) = \ell$, i.e. $\ell \in \mathrm{Fix}_G(L)$.

Let k be the length of the finite G-lattice L.

Definition 1. We say that L is *regular* if it satisfies the following conditions:

(i) All maximal chains of L have the same length.

(ii) For any $\ell \in L \setminus (\operatorname{Ker} f_L \cup \{1\})$ with $h_L(\ell) = p$, the equivalence class modulo "$\sim$" of ℓ has at most $k - p$ elements.

Definition 2. A family $U = (u_i)_{i=\overline{1,k}}$ of elements of L is called a *k-independent minimal system* if it has the properties:

(i) $U \subseteq \operatorname{Min}(L),\ U \cap \operatorname{Fix}_G(L) \neq \emptyset$.

(ii) For any distinct numbers $i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$, we have:

$$\begin{aligned}
&|\{u_{i_1} \vee u_j \mid j \neq i_1\}| = k - 1,\\
&|\{u_{i_1} \vee u_{i_2} \vee u_j \mid j \notin \{i_1, i_2\}\}| = k - 2,\\
&\vdots\\
&|\{u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_{k-2}} \vee u_j \mid j \notin \{i_1, i_2, ..., i_{k-2}\}\}| = 2.
\end{aligned}$$

(iii) For any distinct numbers $i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$ (where $p \in \mathbb{N}^*$, $p \leq k$), if $\{u_{i_1}, u_{i_2}, ..., u_{i_p}\} \cap \operatorname{Fix}_G(L) \neq \emptyset$, then $h_L(u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) = p$.

Proposition 4. *Let L be a finite G–lattice of length k. If L is regular and it has a k-independent minimal system, then there exists a maximal chain of L:*

$$0 = a_0 < a_1 < \cdots < a_k = 1,$$

with $a_i \in \operatorname{Fix}_G(L)$, for any $i = \overline{0,k}$.

Proof. We prove the statement by induction on k. If $k \leq 1$, the statement is trivial. Let us assume the statement to hold for $k-1$ and let $U = (u_i)_{i=\overline{1,k}}$ be a k-independent minimal system of L. Since $U \cap \operatorname{Fix}_G(L) \neq \emptyset$, we can suppose that $u_k \in \operatorname{Fix}_G(L)$. Let $L' = [u_k, 1] = \{\ell \in L \mid u_k \leq \ell \leq 1\}$. L' is a finite G-lattice of length $k - 1$. For any $\ell \in L' \setminus (\operatorname{Ker} f_{L'} \cup \{1\})$ with $h_{L'}(\ell) = p$, we have $h_L(\ell) = p + 1$, therefore the equivalence class modulo "$\sim$" of ℓ has at most $k - 1 - p$ elements. It results that L' is regular.

Now we prove that $V = (v_i)_{i=\overline{1,k-1}}$, where $v_i = u_i \vee u_k$ for any $i = \overline{1, k-1}$, is a $(k-1)$-independent minimal system of L'.

Since $u_k \in \operatorname{Fix}_G(L)$, we have $h_L(v_i) = 2$, $i = \overline{1, k-1}$, thus $h_{L'}(v_i) = 1$, $i = \overline{1, k-1}$, i.e. $V \subseteq \operatorname{Min}(L')$. If we suppose $V \cap \operatorname{Fix}_G(L') = \emptyset$, then, using the remark $(**)$, we obtain that V is containing in the equivalence class modulo "$\sim$" of u_k. It results that the equivalence class modulo "$\sim$" of u_k has at least k elements (u_k and v_i, $i = \overline{1, k-1}$). This contradicts the assumption that L

is regular. The fact that V satisfies the property (ii) of Definition 2 is obvious. For the property (iii), let the distinct numbers $i_1, i_2, ..., i_p \in \{1, 2, ..., k-1\}$ (where $p \in \mathbb{N}^*$, $p \leq k-1$). We have $h_{L'}(v_{i_1} \vee v_{i_2} \vee \cdots \vee v_{i_p}) = h_{L'}(u_k \vee u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) = h_L(u_k \vee u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) - 1 = (p+1) - 1 = p$.

From inductive hypothesis, it results that there exists a maximal chain of L':

$$u_k = a_1 < a_2 < \cdots < a_k = 1,$$

with $a_i \in \mathrm{Fix}_G(L'),\ i = \overline{1,k}$. Thus

$$0 = a_0 < a_1 < \cdots < a_k = 1$$

is a maximal chain of L, with $a_i \in \mathrm{Fix}_G(L),\ i = \overline{0,k}$.

Corollary. *The symmetric group of degree* 3 Σ_3 *and the dihedral group of order* 8 D_8 *have principal series of subgroups.*

Proof. We have $\Sigma_3 = \{e, \sigma_1, \sigma_2, \sigma_3, \tau, \tau^2\}$ (where $\sigma_1 = (2\ 3)$, $\sigma_2 = (1\ 3)$, $\sigma_3 = (1\ 2)$ and $\tau = (2\ 3\ 1)$) and $D_8 = \{1, \rho, \rho^2, \rho^3, \varepsilon, \rho\varepsilon, \rho^2\varepsilon, \rho^3\varepsilon\}$ (where $\rho^4 = \varepsilon^2 = 1$ and $\varepsilon\rho = \rho^3\varepsilon$). We obtain $L(\Sigma_3) = \{H_0 = \{e\},\ H_1 = \{e, \sigma_1\},\ H_2 = \{e, \sigma_2\},\ H_3 = \{e, \sigma_3\},\ H_4 = \{e, \tau, \tau^2\},\ H_5 = \Sigma_3\}$ and $L(D_8) = \{H'_0 = \{1\},\ H'_1 = \{1, \varepsilon\},\ H'_2 = \{1, \rho^2\varepsilon\},\ H'_3 = \{1, \rho^2\},\ H'_4 = \{1, \rho\varepsilon\},\ H'_5 = \{1, \rho^3\varepsilon\},\ H'_6 = \{1, \rho^3, \rho\varepsilon, \rho^3\varepsilon\},\ H'_7 = \{1, \rho, \rho^2, \rho^3\},\ H'_8 = \{1, \rho^2, \rho\varepsilon, \rho^3\varepsilon\},\ H'_9 = D_8\}$. It is a simple exercise to verify that the Σ_3–lattice $L(\Sigma_3)$ (respectively the D_8–lattice $L(D_8)$) is regular and that $U = \{H_3, H_4\}$ (respectively $U' = \{H'_2, H'_3, H'_4\}$) is a 2-independent minimal system of $L(\Sigma_3)$ (respectively a 3-independent minimal system of $L(D_8)$). Now the statement results from Proposition 4.

2.2 On a property of finite nilpotent groups

Let $(G, \cdot, e)$ be a group.

Definition 1. Let L be a G–lattice having the initial element 0 and $(L_i)_{i \in I}$ be a finite family of G–sublattices of L. We say that L is the *direct* $\vee$*–sum* of the family $(L_i)_{i \in I}$ (and we denote this by $L = \overset{\vee}{\bigoplus\limits_{i \in I}} L_i$) if the following two equalities hold:

i) $L = \bigvee\limits_{i \in I} L_i$.

ii) $L_j \wedge \left(\bigvee\limits_{\substack{i \in I \\ i \neq j}} L_i \right) = \{0\}$, for any $j \in I$.

Examples. 1) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, be the decomposition of the natural number n as a product of prime factors. If, for any $m \in \mathbb{N}^*$, we denote by L_m the lattice of all natural divisors of m and we consider the set $G = \{\sigma \in \mathrm{Aut}(L_n) \mid \sigma(L_{p_i^{\alpha_i}}) = L_{p_i^{\alpha_i}},\ i = \overline{1,k}\}$, then G is a group, L_n is a G–lattice (where $\sigma \circ d = \sigma(d)$, for any $(\sigma, d) \in G \times L_n$) and $L_{p_i^{\alpha_i}}$ is a G–sublattice of L_n, $i = \overline{1,k}$. It is easy to see that L_n is the direct $\vee$–sum of the family $(L_{p_i^{\alpha_i}})_{i=\overline{1,k}}$.

2) Let $m \geq 2,\ n \geq 2$ be two natural numbers, $f : \mathbb{Z}_m \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ be a group homomorphism and $\hat{k}_0 = f(\bar{1})(\hat{1})$. We denote by S the semidirect product of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ with respect to the homomorphism f and by G, respectively H the images of $\mathbb{Z}_m$, respectively $\mathbb{Z}_n$ through the group homomorphisms:

$$\sigma_1 : \mathbb{Z}_m \longrightarrow S,\ \sigma_1(\bar{x}) = (\bar{x}, \hat{0}),\ \text{for any } \bar{x} \in \mathbb{Z}_m,$$

respectively

$$\sigma_2 : \mathbb{Z}_n \longrightarrow S,\ \sigma_2(\hat{y}) = (\bar{0}, \hat{y}),\ \text{for any } \hat{y} \in \mathbb{Z}_n.$$

If $L(S), L(G)$ and $L(H)$ are the subgroup lattices of S, G, respectively H, then we have $L(S) = L(G) \overset{\vee}{\oplus} L(H)$ if and only if $(m, n) = 1$ and $k_0 \equiv 1 (\mathrm{mod}\, n)$ (see [10], Proposition 3).

Proposition 1. *If L is a distributive G–lattice having the initial element 0 and $(L_i)_{i \in I}$ is a finite family of G–sublattices of L such that $L_j \wedge \left(\bigvee_{\substack{i \in I \\ i \neq j}} L_i \right) = \{0\}$, for any $j \in I$, then the following two conditions are equivalent:*

i) $L = \overset{\vee}{\bigoplus_{i \in I}} L_i$.

ii) *Every element $\ell \in L$ can be written uniquely as $\bigvee_{i \in I} \ell_i$, where $\ell_i \in L_i$, for any $i \in I$.*

Proof. i)$\Longrightarrow$ii) Since $L = \overset{\vee}{\bigoplus_{i \in I}} L_i$, we have $L = \bigvee_{i \in I} L_i$, therefore every element $\ell \in L$ can be written as $\bigvee_{i \in I} \ell_i$, where $\ell_i \in L_i$, $i \in I$. If $\ell = \bigvee_{i \in I} \ell_i = \bigvee_{i \in I} \ell'_i$ with $\ell_i, \ell'_i \in L_i$, $i \in I$, then, for any $j \in I$, we have $\ell'_j = \ell'_j \wedge \ell = \ell'_j \wedge \left(\bigvee_{i \in I} \ell_i \right) =$

$\ell'_j \wedge \left[\ell_j \vee \left(\bigvee\limits_{\substack{i\in I\\ i\neq j}} \ell_i\right)\right] = (\ell'_j \wedge \ell_j) \vee \left[\ell'_j \wedge \left(\bigvee\limits_{\substack{i\in I\\ i\neq j}} \ell_i\right)\right] = \ell'_j \wedge \ell_j$, thus $\ell'_j \leq \ell_j$. In the same way, we obtain $\ell_j \leq \ell'_j$, therefore $\ell'_j = \ell_j$, $j \in I$.

ii)$\Longrightarrow$i) Obvious.

Next aim is to establish connections between the direct product of G–lattices and the direct $\vee$-sum of G-sublattices.

Proposition 2. *If $(L_i)_{i\in I}$ is a finite family of G–lattices having initial elements (denoted all by 0), $0 \in \mathrm{Fix}_G(L_i)$, $i \in I$, and L is the direct product of the family $(L_i)_{i\in I}$, then there exists a family $(L'_i)_{i\in I}$ of G–sublattices of L which satisfies the following properties:*

i) $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L'_i$.

ii) $L'_i \cong L_i$ (*isomorphism of G–lattices*), *for any $i \in I$.*

Proof. It is easy to see that the sets $L'_i = \{(a_j)_{j\in I} \in L \mid a_j = 0$, for any $j \in I \setminus \{i\}\}$, $i \in I$, are G-sublattices of L and $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L'_i$. Moreover, the maps

$$\begin{array}{l} f_i : L_i \longrightarrow L'_i \\ f_i(\ell_i) = (a_j)_{j\in I}, \text{ where } a_i = \ell_i \text{ and } a_j = 0, \text{ for } j \neq i, \end{array}$$

are isomorphism of G–lattices, $i \in I$.

Let L be a finite G–lattice with the initial element denoted by 0 such that $0 \in \mathrm{Fix}_G(L) = \{\ell \in L \mid g \circ \ell = \ell$, for any $g \in G\}$. If $(\ell_i)_{i=\overline{1,k}}$ is a family of elements of L, then we make the following notations:

$$\begin{array}{l} L_i = [0, \ell_i] = \{\ell \in L \mid 0 \leq \ell \leq \ell_i\}, \\ G \circ L_i = \{g \circ \ell \mid g \in G,\ \ell \in L_i\}, \end{array}$$

where $i \in \{1, 2, ..., k\}$.

Definition 2. The family $(\ell_i)_{i=\overline{1,k}}$ is called a *maximal system* of L if it satisfies the properties:

i) $L = \bigvee\limits_{i=1}^{k} G \circ L_i$.

ii) $G \circ L_j \wedge \left(\bigvee_{\substack{i=I \\ i \neq j}}^{k} G \circ L_i \right) = \{0\}$, for any $j = \overline{1,k}$.

Remark. If $(\ell_i)_{i=\overline{1,k}}$ is a maximal system of L, then, for $i \in \{1, 2, ..., k\}$, the sublattice L_i of L is not necessarily a G–sublattice. A sufficient condition for this fact holds is $\ell_i \in \mathrm{Fix}_G(L)$. In the case when $(\ell_i)_{i=\overline{1,k}} \subseteq \mathrm{Fix}_G(L)$, we have $G \circ L_i = L_i$ for any $i = \overline{1,k}$ and $L = \overset{\vee}{\bigoplus_{i=\overline{1,k}}} L_i$.

Definition 3. Let $U, V \in L(G)$.

(i) We say that U and V form a *permutable pair* if $[U \cup V] = UV = VU$ (where $[U \cup V]$ denotes the subgroup of G generated by $U \cup V$).

(ii) We say that U and V form a *modular pair* if

$W \cap [U \cup V] = [U \cup (W \cap V)]$ for any $W \in L(G)$ with $U \subseteq W$

and

$W \cap [U \cup V] = [V \cup (W \cap U)]$ for any $W \in L(G)$ with $V \subseteq W$.

Remarks. 1) Any permutable pair of subgroups is a modular pair (see [7], Theorem 5, page 5).

2) If the group G is finite and it satisfies the property tht any two subgroups $U, V \in L(G)$ with $(|U|, |V|) = 1$ form a permutable pair, then, for any $H_1, H_2, ..., H_k \in L(G)$ with $(|H_i|, |H_j|) = 1$, $i \neq j$, we have:

$$H_1 H_2 ... H_k = \left[\bigcup_{i=1}^{k} H_i \right] \in L(G)$$

and

$$|H_1 H_2 ... H_k| = \prod_{i=1}^{k} |H_i|.$$

Proposition 3. *For a finite group G which satisfies the property that any two subgroups $U, V \in L(G)$ with $(|U|, |V|) = 1$ form a permutable pair, the G–lattice $L(G)$ has a maximal system.*

Proof. Let $n = |G|$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, is the decomposition of n as a product of prime factors, then, for any $i = \overline{1,k}$, let H_i be Sylow p_i–subgroup of G.

We prove that $\{H_1, H_2, ..., H_k\}$ is a maximal system for $L(G)$. Let $H \in L(G)$ and $m = |H|$. Then m/n, therefore there exist the numbers $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$, such that $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$. For any $i = \overline{1,k}$, let U_i be a Sylow p_i–subgroup of H and, using the Theorems of Sylow, let $x_i \in G$ such that $U_i \leq H_i^{x_i}$, i.e. $U_i^{x_i^{-1}} \in [\{e\}, H_i]$ (where e is the identity of G). From Remark 2), we obtain $H = U_1 U_2 ... U_k = \left(U_1^{x_1^{-1}}\right)^{x_1} \left(U_2^{x_2^{-1}}\right)^{x_2} ... \left(U_k^{x_k^{-1}}\right)^{x_k}$, thus:

$$L(G) = \bigvee_{i=1}^{k} G \circ [\{e\}, H_i].$$

Let $j \in \{1, 2, ..., k\}$ and $K \in G \circ [\{e\}, H_j] \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} G \circ [\{e\}, H_i] \right)$. Then $K = V_j^{x_j} \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} V_i^{x_i} \right)$ (where $V_s \leq H_s$ and $x_s \in G$, for any $s = \overline{1,k}$).

Since $\left(|V_j^{x_j}|, \left| \bigvee_{\substack{i=1 \\ i \neq j}}^{k} V_i^{x_i} \right| \right) = 1$, it follows that $K = \{e\}$, thus:

$$G \circ [\{e\}, H_j] \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} G \circ [\{e\}, H_i] \right) = \{e\}.$$

Remark. Let G be a finite group of order n, $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of n as a product of prime factors and H_i be a Sylow p_i–subgroup of G, $i = \overline{1,k}$. If $(H_i)_{i=\overline{1,k}}$ is a maximal system of $L(G)$, then, for any $x_i \in G$, $i = \overline{1,k}$, $(H_i^{x_i})_{i=\overline{1,k}}$ is a maximal system of $L(G)$.

Let L be a modular finite G–lattice with the initial element denoted by 0 and $(\ell_i)_{i=\overline{1,k}}$ be a maximal system of L.

Lemma. *The following equality holds:*

$$h_L \left(\bigvee_{i=1}^{k} \ell_i \right) = \sum_{i=1}^{k} h_L(\ell_i).$$

Proof. We prove the above equality by induction on k. For $k=2$, we have $h_L(\ell_1 \vee \ell_2) = h_L(\ell_1) + h_L(\ell_2) - h_L(\ell_1 \wedge \ell_2) = h_L(\ell_1) + h_L(\ell_2) - h_L(0) = h_L(\ell_1) + h_L(\ell_2)$. Let us assume the equality to hold for $k=1$. We obtain $h_L\left(\bigvee_{i=1}^{k} \ell_i\right) =$
$h_L\left(\left(\bigvee_{i=1}^{k-1} \ell_k\right) \vee \ell_j\right) = h_L\left(\bigvee_{i=1}^{k-1} \ell_i\right) + h_L(\ell_k) - h_L\left(\left(\bigvee_{i=1}^{k-1} \ell_i\right) \wedge \ell_k\right) =$
$= \sum_{i=1}^{k-1} h_L(\ell_i) + h_L(\ell_k) - h_L(0) = \sum_{i=1}^{h} h_L(\ell_i).$

For any $i = \overline{1,k}$, let $\alpha_i = h_L(G \circ L_i)$, (i.e. $\alpha_i = \max\{h_L(g \circ \ell) \mid g \in G, \ell \in L_i\}$), $\mathbb{N}_{\alpha_i} = \{0, 1, ..., \alpha_i\}$ and $h_i : G \circ L_i \to \mathbb{N}_{\alpha_i}$ be the restriction of the height function h_L on the set $G \circ L_i$. We suppose that is well defined the function:

$$h' : L \longrightarrow \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i},$$

$$h'\left(\bigvee_{i=1}^{k} g_i \circ \ell_{ii}\right) = (h_1(\ell_{11}), h_2(\ell_{22}), ..., h_k(\ell_{kk})),$$

where $g_i \in G$, $\ell_{ii} \in L_i$, for any $i = \overline{1,k}$ (it is easy to see that a sufficient condition for this fact holds is "L = distributive lattice").

Proposition 4. *The function h' is onto. Moreover, for any $(\beta_1, \beta_2, ..., \beta_k) \in \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i}$, we have:*

$$(h')^{-1}(\beta_1, \beta_2, ..., \beta_k) \cap \left\{\ell \in L \mid h_L(\ell) = \sum_{i=1}^{k} \beta_i\right\} \neq \emptyset.$$

Proof. For each $i \in \{1, 2, ..., k\}$, the function h_i is onto.

Let $(\beta_1, \beta_2, ..., \beta_k) \in \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i}$ and $\ell_{ii} \in G \circ L_i$ such that $h_i(\ell_{ii}) = \beta_i$, $i = \overline{1,k}$. Using the above lemma, it is a simple exercise to verify that

$$\bigvee_{i=1}^{k} \ell_{ii} \in (h')^{-1}(\beta_1, \beta_2, ..., \beta_k) \cap \left\{\ell \in L \mid h_L(\ell) = \sum_{i=1}^{k} \beta_i\right\}.$$

Proposition 5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of the natural number n as a product of prime factors, L_n be the lattice of all natural divisors of n and G be a finite group of order n which satisfies the following properties:*

(i) *$L(G)$ is a modular lattice.*

(ii) *There exists a maximal system $(H_i)_{i=\overline{1,k}}$ of $L(G)$ such that H_i is a Sylow p_i–subgroup of G, $i=\overline{1,k}$.*

(iii) *For any $H \in L(G)$, $|H| = \prod_{i=1}^{k} p_i^{x_i}$ implies $h_{L(G)}(H) \geq \sum_{i=1}^{k} x_i$.*

Then the function $\operatorname{ord} : L(G) \to L_n$, $\operatorname{ord}(H) = |H|$, *for any $H \in L(G)$, is onto.*

Proof. If $m \in L_n$, then $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$, where $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$. For each $i \in \{1,2,...,k\}$, let $U_i \subseteq H_i$ be a subgroup of G having order $p_i^{\beta_i}$. Since $(H_i)_{i=\overline{1,k}}$ is a maximal system of $L(G)$, it results that $h_{L(G)}\left(\bigvee_{i=1}^{k} U_i\right) = h_{L(G)}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \sum_{i=1}^{k} h_{L(G)}(U_i) = \sum_{i=1} \beta_i$. This fact implies the equality $\operatorname{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \prod_{i=1}^{k} p_i^{\beta_i} = m$. Indeed, if we suppose that $\operatorname{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) \neq m$, then $\operatorname{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \prod_{i=1}^{k} p_i^{\gamma_i}$, where $\beta_i \leq \gamma_i \leq \alpha_i$, for any $i = \overline{1,k}$ and there exists $i_0 \in \{1,2,...,k\}$ such that $\beta_{i_0} < \gamma_{i_0}$ (this fact holds because $U_q \leq \left[\bigcup_{i=1}^{k} U_i\right]$ (so that $|U_q| / \left|\left[\bigcup_{i=1}^{k} U_i\right]\right|$) for any $q = \overline{1,k}$ and $(|U_q|, |U_{q'}|_= 1$ for $q \neq q'$). From property (iii), we obtain $h_{L(G)}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) \geq \sum_{i=1}^{k} \gamma_i \geq \sum_{i=1}^{k} \beta_i + 1$; contradiction.

Corollary 1. *For any finite group G of order n which satisfies the property that any two subgroups $U, V \in L(G)$ with $(|U|,|V|) = 1$ form a permutable pair, the function* $\operatorname{ord} : L(G) \longrightarrow L_n$ *is onto.*

Proof. The statement results from Proposition 3 and Proposition 5 or, directly, making the next reasoning.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of n as a product of prime factors and $m \in L_n$. Then m/n, therefore $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$, where $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$. For each $i \in \{1,2,...,k\}$, let U_i be a subgroup of G having the order $p_i^{\beta_i}$. From hypothesis, it results tht the subgroup $\left[\bigcup_{i=1}^{k} U_i\right] \in L(G)$ has the order m.

Corollary 2. *For any finite nilpotent group G of order n, the function* ord : $L(G) \longrightarrow L_n$ *is onto.*

Proof. The statement results from Corollary 1, using the fact that, for a finite nilpotent group, any two subgroups of relative prime orders form a permutable pair.

Corollary 3. *For any finite abelian group G of order n, the function* ord : $L(G) \longrightarrow L_n$ *is onto.*

Proof. Since any abelian group is nilpotent, the statement results from Corollary 2.

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