An. Şt. Univ. Ovidius Constanţa Vol. 10(2), 2002, 45–62

A METHOD OF SOLVING OPTIMAL CONTROL PROBLEMS

Popoviciu Ioan

Abstract

The method is based on the reformulation of an optimal control problem with square performance criterion in a Hilbert space as an optimal discrete control problem. The optimal conditions for the discrete problem lead to the solving of a linear system with block structure, which can be efficiently solved through iterative methods. At the same time, a discretization way of an optimal control problem in a Hilbert space is presented. This allows us to obtain a solution by solving a linear system.

1.1 General Presentation

We consider the following problem:

$$\min J(u) = \frac{1}{2}\int_0^T \| u(t) \|_U^2 \, dt + \frac{\alpha_1}{2}\int_0^T ||C(t)y(t) - z_1(t)||_z^2 \, dt$$

$$+\frac{\alpha_2}{2}\,||C_T y(T) - z_2||_{z_\tau}^2 , \tag{1.1.1}$$

where y is the solution of a partial differential equation, the *state equation*, which is abstractly written as:

$$\frac{\partial}{\partial t} y(t) + A(t)y(t) = B(t)u(t) + f(t),\ t \in (0,T), \tag{1.1.2a}$$

$$y(0) = y_0. \tag{1.1.2b}$$

We assume that the state equation (1.1.2) admits a unique solution y for each control u and that the optimal control problem (1.1.1), (1.1.2) has a solution. The solution of (1.1.1) is characterized by the optimality conditions which consist of:

Key Words: control problems with square performance criterion

• the state equation (1.1.2);
• the adjoint equation:

$$-\frac{\partial}{\partial t}p(t) + A(t)^*p(t) = \alpha_1 C(t)^*(C(t)y(t) - z_1(t)),\ \ t \in (0,T), \qquad (1.1.3a)$$

$$p(T) = \alpha_2 C_T^*(C_T y(T) - z_2); \qquad (1.1.3b)$$

• the equation:

$$u(t) + B(t)^*p(t) = 0,\ \ t \in (0,T). \qquad (1.1.4)$$

Here M^* denotes the adjoint of the operator M (which is the transpose, if M is a real matrix).

The control strategies split the optimal control problem (1.1.1), (1.1.2) into a set of smaller problems of the same type. These problems are obtained by restricting the original optimal control problem to time intervals (T_i, T_{i+1}), where $0 = T_0 < T_1 < ... < T_{N_T} = T$. These problems are solved sequentially to obtain a **suboptimal control**. The computation of suboptimal controls proceeds as follows. Suppose the suboptimal controls $\widehat{u}_i$ and the corresponding states $\widehat{y}_i$ have been computed on the subintervals (T_i, T_{i+1}), $i = 1, ..., j-1$. Then the optimal control problem (1.1.1),(1.1.2) is restricted to (T_j, T_{j+1}) and the initial condition is replaced by $y(T_j) = \widehat{y}_{j-1}(T_j)$. If $j < N_t - 1$, the last term in the objective function (1.1.1) is dropped. An optimization procedure is applied to compute an approximation of the optimal control $\widehat{u}_j$ for this problem together with the corresponding state $\widehat{y}_j$. The suboptimal control $\hat{u}$ for the original problem (1.1.1), (1.1.2) is defined by connecting the piecewise controls, $\widehat{u}_{[T_i,T_{i+1}]} = \widehat{u}_i,\ i = 0, ..., N_t$.

The control strategies found in literature differ in the way the partition $0 = T_0 < T_1 < ... < T_{N_t} = T$ is chosen, in the optimization method applied to the subproblems, and in the truncation criteria applied in these optimization methods.

1.2 Problem formulation in Hilbert space

We use the following notations and hypotheses:

Given two Banach spaces X and Y, $L(X,Y)$ denotes the space of the bounded linear operators from X to Y; $L(X) := L(X,X)$.

The norm in the Banach space X is denoted by $||\bullet||_X$.

The dual of a Banach space X is denoted by X^*.

The duality pairing between the Hilbert space X and its dual X^*, is denoted by $<< \bullet, \bullet >>_{X^* \times X}$.

Given an operator $M \in L(X,Y)$, its adjoint is denoted by M^*.

$A(t), B(t), C(t), C_T$ are large sparse matrices, where $A(t)$ is obtained from the spatial discretization of an elliptic partial differential equation.

In this setting, $M \in L(X,Y)$ simply reads $M \in R^{m\times n}, M^* = M^T$, and $< y, x >_{X^*\times X}$, is simply $y^T x$.

Let H, V, U be Hilbert spaces with V dense în H. Without loss of generality we assume that:

$$||v||_H \leq ||v||_V,\ \forall v \in V \tag{1.2.1}$$

and $H^* = H$.

The control space φ and the state space ζ are given by:

$$\varphi = L^2(0,T;U) \text{ and } \zeta = \left\{ v | v \in L^2(0,T;V), \frac{\partial}{\partial t} y \in L^2(0,T;V^*) \right\}.$$

We assume that: $A(t) \in L(V,V^*),\ t \in [0,T]$ is a family of continuous linear operators such that $\forall v, w \in V, t \to < A(t)v, w >_{V^*\times V}$ is measurable on $(0,T)$, and there exist $c, \vartheta > 0,\ \lambda \geq 0$ such that, for all $t \in [0,T]$ and for all $v, w \in V,\ \langle A(t)v, w\rangle_{V^*\times V} \leq c||v||_V||w||_V$, and $\langle A(t)v, v\rangle_{V^*\times V} + \lambda||v||_H^2 \geq \vartheta||v||_V^2|$.

$A(t)$ is defined by a bilinear form $\langle A(t)v, w\rangle_{V^*\times V} = a(t; v, w)$. Furthermore we assume that $B(t) \in L(U,V^*),\ f \in L^2(0,T;V^*),\ y_0 \in H$..

The state equation (1.1.2) admits a unique solution ([2]), $y \in \zeta$ which depends continuously on the initial condition and on the right hand side.

To specify the objective function (1.1.1), let Z, Z_T be Hilbert spaces and we assume that $C(t) \in L(V,Z), C_T \in L(H, Z_T),\ z_1 \in L^2(0,T;Z)$ and $z_2 \in Z_T$. Since $\zeta \in C([0,T];H)$([21]), the objective function (1.1.1) is well defined for $u \in \varphi$ and $y \in \zeta$. With the assumption $\alpha_1, \alpha_2 \geq 0$, the optimal control problem (1.1.1), (1.1.2) admits a unique solution u ([2]). The solution is characterized by the optimality conditions (1.1.2) - (1.1.4).

1.3 Decomposition of the optimal control problem

We use a multiple shooting approach to reformulate the optimization problem (1.1.1), (1.1.2). We select a partition: $0 = T_0 < T_1 <...< T_{N_t} = T$ of $[0,T]$ and we introduce the auxiliary variables $\bar{y}_i \in H, i = 0, ..., N_t$,where $\bar{y}_0 \overset{def}{=} y_0$ and we set: $u_i = u_{(T_i,T_{i+1})} \in L^2(T_i, T_{i+1}; U),\ i = 0, ..., N_t - 1$.

The state equations:

$$\frac{\partial}{\partial t} y_i + A(t) y_i = B(t)\bar{u}_i + f,\ t \in (T, T_{i+1}) \tag{1.3.1a}$$

$$y_i(T_i) = \bar{y}_i, \tag{1.3.1b}$$

$i = 0, ..., N_t - 1$, together with the continuity conditions:

$$\bar{y}_{i+1} = y_i(T_{i+1}),\ \ i = 0, ..., N_t - 1,$$

are equivalent to the original state equation (1.1.2). If y solves (1.1.2), then $\bar{y}_i = y(T_i),\ y_i = y_{\Omega \times (T_i \times T_{i+1})}$,
$i = 0, ..., N_t - 1$ solve (1.3.1), (1.3.2) and vice versa. In this case:

$$\frac{1}{2}\int_0^T ||u(t)||_U^2 dt + \frac{\alpha_1}{2}\int_0^T ||Cy(t) - z_1(t)||_Z^2 dt + \frac{\alpha_2}{2}||C_T y(T) - z_2||_{Z_T}^2 =$$
$$= \sum_{i=1}^{N} \left\{ \frac{1}{2}\int_{T_i}^{T_{i+1}} ||u_i(t)||_U^2 dt + \frac{\alpha}{2}\int_{T_i}^{T_{i+1}} ||Cy_i(t) - z_1(t)||_Z^2 dt \right\} + \tag{1.3.3}$$
$$+ \frac{\alpha_2}{2}||C_T y_{N_t - 1}(T_{N_t}) - z_2||_{Z_T}^2.$$

Remark 1.3.1. *In the reformulation (1.3.3) of the objective function, we express the terminal observation $y(T)$ by $y_{N_t-1}(T_{N_t})$. Alternatively, we could have used the continuity condition $\bar{y}_{N_t} := y_{N_t-1}(T_{N_t})$, and express the terminal observation $||C_T y(T) - z_2||_{Z_T}^2$ as $||C_T \bar{y}_{N_t} - z_2||_{Z_T}^2$.*

It is clear that the solution of the differential equation (1.3.1) is a function of $\bar{y}_i, \bar{u}_i$ and f. Therefore the continuity conditions (1.3.2) and the objective function (1.3.3) can be viewed as functions of $\bar{u}_i,\ i = 0, ..., N_t - 1$ and $\bar{y}_i,\ i = 0, ..., N_t$. This will be formalized in the following. We define

$$\varphi_i = L^2\left(T_i, T_{i+1}\right); U \text{ and } \zeta_i = \left\{ y | y \in L^2\left(T_i, T_{i+1}; V\right), \frac{\partial}{\partial t} y \in L^2\left(T_i, T_{i+1}; V^*\right) \right\}.$$

To express the continuity conditions (1.3.2) in terms of $\bar{y}_i, \bar{u}_i$, we define:

$$\overline{A}_i \in L(H),\ \overline{B}_i \in L(\varphi_i, H),\ \bar{b}_i \in H,\ i = 0, ..., N_t - 1,$$

as follows:

$$\overline{A}_i \bar{y}_i = y_i^y\left(T_{i+1}\right),\ \overline{B}_i \bar{u}_i = y_i^u\left(T_{i+1}\right),\ \bar{b}_i = y_i^f\left(T_{i+1}\right), \tag{1.3.4}$$

where y_i^y is the solution of (1.3.1) with $\bar{u}_i = 0$ and $f = 0$, y_i^u is the solution of (1.3.1) with $\bar{y}_i = 0$ and $f = 0$ and y_i^f is the solution of (1.3.1) with $\bar{y}_i = 0$ and $\bar{u}_i = 0$. Using (1.3.4), the continuity conditions (1.3.2) can be written as:

$$\bar{y}_{i+1} = \overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i,\ \ i = 0, .., N_t - 1. \tag{1.3.5}$$

To express the objective function (1.3.3) in terms of $\bar{y}_i, \bar{u}_i$, we define:

$$\overline{E}_i \in L(H, \zeta_i),\ \overline{F}_i \in L(\varphi_i, \zeta_i),\ \bar{f}_i \in \zeta_i,\ i = 0, ..., N_t - 2,$$

and

$$\overline{E}_i \in L(\varphi_i, \zeta_i \times H),\ \ \overline{F}_i \in L(\varphi_i, \zeta_i \times H),\ \ \overline{f}_i \in \zeta_i \times H,\ i = N_t - 1$$

as follows. For $i=0,...,N_t-1$, let y_i^y be the solution of (1.3.1) with $\bar{u}_i=0$ and $f=0$, let y_i^u be the solution of (1.3.1) with $\bar{y}_i=0$ and $f=0$, and let y_i^f be the solution of (1.3.1) with $\bar{y}_i=0$ and $\bar{u}_i=0$. We set:

$$\overline{E}_i\bar{y}^i=y_i^y,\ \overline{F}_i\bar{u}_i=y_i^u,\ \overline{f}_i=y_i^f,\ i=0,...,N_t-2 \tag{1.3.6}$$

and

$$\overline{E}_i\bar{y}^i=\begin{pmatrix} y_i^y \\ y_i^y(T)\end{pmatrix},\ \overline{F}_i\bar{u}_i=\begin{pmatrix} y_i^u \\ y_i^u(T)\end{pmatrix},\ \ \overline{f}_i=\begin{pmatrix} y_i^f \\ y_i^f(T)\end{pmatrix},\ \ i=N_t-1. \tag{1.3.7}$$

For $i=0,...,N_t-2$, the solution y_i of (1.3.1) is given by:

$$y_i(t)=\left(\overline{E}_i\bar{y}_i\right)(t)+\left(\overline{F}_i\bar{u}_i\right)(t)+\bar{f}_i(t),\ \ t\in(T_i,T_{i+1})\,, \tag{1.3.8}$$

and for $i=N_t-1, \underline{y_i(t)}$ is the first component of $\overline{E}_i+\overline{F}_i+\bar{f}_i$ evaluated at t. It is clear that $\overline{A}_i,\overline{B}_i,\overline{b}_i$are closely related to $\overline{E}_i,\overline{F}_i,\bar{f}_i$. For example $\overline{A}_i\bar{y}_i=\left(\overline{E}_i\bar{y}_i\right)(T_{i+1})$.

We also need the operators:

$$\overline{M}_i^z\in L\left(\zeta_i,\zeta_i^*\right),\ i=0,...,N_t-2,\ \overline{M}_i^z\in L\left(\zeta_i\times H,\zeta_i^*\times H\right),\ i=N_t-1,$$

defined by

$$\left\langle\overline{M}_i^z y_i,w_i\right\rangle_{\zeta_i^*\times\zeta_i}=\int\limits_{T_i}^{T_{i+1}}\alpha_1\left\langle y_i(t),C(t)^*C(t)w_i(t)\right\rangle_{V^*\times V}dt, \tag{1.3.9}$$

$$\forall y_i,w_i\in\zeta_i,\ i=0,...,N_t-2,$$

and

$$\left\langle\overline{M}_i^z\left(\frac{y_i}{y_i}\right),\left(\frac{w_i}{w_i}\right)\right\rangle_{\left(\zeta_i^*\times H\right)\times(\zeta_i\times H)}=\int\limits_{T_i}^{T_{i+1}}\alpha_1\left\langle y_i(t),C(t)^*C(t)w_i(t)\right\rangle_{V^*\times V}dt+$$

$$+\alpha_2\left\langle\bar{y}_iC_T^*C_T\bar{w}_i\right\rangle_H,\ \forall y_i,w_i\in\zeta_i,\bar{y}_i,\bar{w}_i\in H,\ i=N_t-1 \tag{1.3.10}$$

and the vectors $\bar{z}_i=\alpha_1C(\cdot)^*z_1\in\zeta_i,\ \ i=0,...,N_t-2$ and $\bar{z}_i=\begin{pmatrix}\alpha_1C(\cdot)^*z_1\\ \alpha_2C_T^*z_2\end{pmatrix}\in\zeta_i\times H, i=N_t-1$.

We can now express the objective function (1.3.3) in terms of $\bar{y}_i,\bar{u}_i$:

$$\sum_{i=0}^{N_t-1}\int_{T_i}^{T_{i+1}}\frac{1}{2}||u_i(t)||_U^2+\frac{\alpha_1}{2}||C(t)y_i(t)-z_1(t)||_z^2dt+\frac{\alpha_2}{2}||C_Ty_{N_t-1}(T_{N_t})-z_2||_{z_T}^2=$$

$$= \sum_{i=0}^{N_t-1} \left\{ \frac{1}{2} \left\langle \bar{u}_i \left(I + \overline{F}_i^* \overline{M}_i^z \overline{F}_i \right) \bar{u}_i \right\rangle_{\varphi_i} + \left\langle \bar{y}_i, \overline{E}_i^* \overline{M}_i^z \overline{F}_i \bar{u}_i \right\rangle_H + \right. \tag{1.3.11}$$

$$\frac{1}{2} \left\langle \bar{y}_i, \overline{E}_i^* \overline{M}_i^z \overline{E}_i \bar{y}_i \right\rangle_H + \frac{1}{2} \left\langle \bar{y}_i, \overline{E}_i^* \overline{M}_i^z \overline{E}_i \bar{y}_i \right\rangle_H + \left\langle \bar{u}_i, \overline{F}_i^* \left(\overline{M}_i^z \overline{f}_i - \bar{z}_i \right) \right\rangle_{\varphi_i} +$$

$$\left. + \left\langle \bar{y}_i, \overline{E}_i^* \left(\overline{M}_i^z \bar{f}_i - \bar{z}_i \right) \right\rangle_H \right\} + const \overset{def}{=} \sum_{i=0}^{N_t-1} \frac{1}{2} \left\langle \bar{y}_i, \overline{Q}_i \bar{y}_i \right\rangle_H + \left\langle \bar{c}_i, \bar{y}_i \right\rangle_H + \left\langle \bar{y}_i, \overline{R}_i \bar{u}_i \right\rangle_{\varphi_i}$$

$$+ \frac{1}{2} \left\langle \bar{u}_i, \overline{S}_i \bar{u}_i \right\rangle_{\varphi_i} + \left\langle \bar{d}_i, \bar{u}_i \right\rangle_{\varphi_i} + \frac{1}{2} \left\langle \bar{y}_{N_t}, \overline{Q}_{N_t} \right\rangle_H + \left\langle \bar{c}_{N_t}, \bar{y}_{N_t} \right\rangle_H + const,$$

where $\overline{Q}_{N_t} = 0$ and $\bar{c}_{N_t} = 0$.

Remark 1.3.2. *In the definition of operators and vectors, we have to distinguish between the cases: $i = 0, ..., N_t - 2$ and $i = N_t - 1$. This is necessary because of our reformulation (1.3.3) of the objective function. One would obtain the problem(1.3.11) with $\overline{Q}_{N_t} = \alpha_2 C_T^* C_T$, $\bar{c}_{N_t} = -\alpha_2 C_T^* z_2$.*

From (1.3.5), (1.3.3) and (1.3.11) we see that the linear quadratic optimal control problem (1.1.1), (1.1.2) is equivalent to the problem:

$$\min \frac{1}{2} \left\langle \bar{y}_{N_t}, \overline{Q}_{N_t} \bar{y}_{N_t} \right\rangle + \left\langle \bar{c}_{N_t}, \bar{y}_{N_t} \right\rangle_H + \sum_{i=0}^{N_t-1} \frac{1}{2} \left\langle \bar{y}_i, \overline{Q}_i \bar{y}_i \right\rangle_H + \left\langle \bar{c}_i, \bar{y}_i \right\rangle_H + \left\langle \bar{y}_i, \overline{R}_i \bar{u}_i \right\rangle_{\varphi_i} +$$

$$+ \frac{1}{2} \left\langle \bar{u}_i, \overline{S}_i \bar{u}_i \right\rangle_{\varphi_i} + \left\langle \bar{d}_i, \bar{u}_i \right\rangle_{\varphi_i}, \tag{1.3.12a}$$

$$\bar{y}_{i+1} = \overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i, \; i = 0, ..., N_t, \tag{1.3.12b}$$

$$\bar{y}_0 = y_0.$$

The problem (1.3.12) is a **discrete-time optimal control problem** in a Hilbert space. From the definition of $\overline{S}_i, \overline{Q}_i,\ i = 0, ..., N_t - 1$ and $\overline{Q}_{N_t}$, in (1.3.11) we obtain the following result.

Theorem 1.3.1. *The operators $\overline{S}_i,\ i = 0, ..., N_t - 1$, are strictly positive,*

$$\left\langle \bar{u}_i, \overline{S}_i \bar{u}_i \right\rangle_{\varphi_i} \geq ||\bar{u}_i||^2_{\varphi_i}, \; \forall \bar{u}_i \in \varphi_i,$$

and the operators $\overline{Q}_i,\ i = 0, ..., N_t - 1$, are positive.

The augmented Lagrange function for (1.3.12) is given by:

$$L_p \left(\bar{\bar{y}}, \bar{\bar{u}}, \bar{\bar{p}} \right) = \frac{1}{2} \left\langle \bar{y}_{N_t}, \overline{Q}_{N_t} \bar{y}_{N_t} \right\rangle_H + \left\langle \bar{c}_{N_t}, \bar{y}_{N_t} \right\rangle_H +$$

$$+ \sum_{i=0}^{N_t-1} \frac{1}{2} \left\langle \bar{y}_i, \overline{Q}_i \bar{y}_i \right\rangle_H + \left\langle \bar{c}_i, \bar{y}_i \right\rangle_H + \left\langle \bar{y}_i, \overline{R}_i \bar{u}_i \right\rangle_{\varphi_i} + \frac{1}{2} \left\langle \bar{u}_i, \overline{S}_i \bar{u}_i \right\rangle_{\varphi_i} + \left\langle \bar{d}_i, \bar{u}_i \right\rangle_{\varphi_i} +$$

$$+\sum_{i=0}^{N_t-1}\left\langle \bar{p}_{i+1}, -\bar{y}_{i+1}+\overline{A}_i\bar{y}_i+\overline{B}_i\bar{u}_i+\bar{b}_i\right\rangle_H+\frac{\rho}{2}\sum_{i=0}^{N_t-1}||\overline{A}_i\bar{y}_i+\overline{B}_i\bar{u}_i+\bar{b}_i-\bar{y}_{i+1}||_H^2,$$

where $\overline{\overline{y}}=(\bar{y}_1,...,\bar{y}_{N_t})$, $\overline{\overline{u}}=(\bar{u}_1,...,\bar{u}_{N_t})$, $\overline{\overline{p}}=(\bar{p}_1,...,\bar{p}_{N_t})$ *and* $\rho\geq 0$ *is the augmentation parameter. The optimality conditions for (1.3.12) consist of the following equations*:

a) the state equation:

$$\bar{y}_{i+1}=\overline{A}_i\bar{y}_i+\overline{B}_i\bar{u}_i+\bar{b}_i,\ i=0,...,N_t-1,\ \ \bar{y}_0=y_0; \tag{1.3.13a}$$

b) the adjoint equation:

$$\bar{p}_{N_t}=\overline{Q}_{N_t}\bar{y}_{N_t}+\bar{c}_{N_t}-\rho\left(\overline{A}_{N_t-1}\bar{y}_{N_t-1}+\overline{B}_{N_t-1}\bar{u}_{N_t-1}+\bar{b}_{N_t-1}-\bar{y}_{N_t}\right), \tag{1.3.13b}$$

$$\bar{p}_i=\overline{A}_i^*\bar{p}_{i+1}+\overline{Q}_i\bar{y}_i+\overline{R}_i\bar{u}_i+\bar{c}_i+\rho\overline{A}_i^*\left(\overline{A}_i\bar{y}_i+\overline{B}_i\bar{u}_i+\bar{b}_i-\bar{y}_{i+1}\right)-$$
$$-\rho\left(\overline{A}_{i-1}\bar{y}_{i-1}+\overline{B}_{i-1}\bar{u}_{i-1}+\bar{b}_{i-1}-\bar{y}_i\right),\ i=1,...,N_t-1;$$

c) gradient equation:

$$\overline{S}_i\bar{u}_i+\overline{R}_i^*\bar{y}_i+\overline{B}_i^*\bar{p}_{i+1}+\bar{d}_i+\rho\overline{B}_i^*\left(\overline{A}_i\bar{y}_i+\overline{B}_i\bar{u}_i+\bar{b}_i-\bar{y}_{i+1}\right), \tag{1.3.13c}$$

$i=0,...,N_t-1$.

The optimality conditions (1.3.13) are obtained by setting the partial gradients of L_ρ to zero:

$$\left[\frac{\partial L_\rho}{\partial\overline{p}_{i+1}},\ \frac{\partial L_\rho}{\partial\overline{y}_i},\ \frac{\partial L_\rho}{\partial\overline{u}_i}\right]=0.$$

We need the adjoints of the operators $\overline{A}_i,\overline{B}_i,\overline{E}_i,\overline{F}_i,\ i=0,...,N_t-1$.

Consider the differential equation

$$-\frac{\partial}{\partial t}p_i(t)+A(t)^*p_i(t)=g(t),\ t\in[T_i,T_{i+1}], \tag{1.3.14a}$$

$$p(T_{i+1})=\bar{p}_{i+1},$$

where $g\in L^2(T_i,T_{i+1};V^*)$ and $\bar{p}\in H$.

Theorem 1.3.2. *i) The adjoints* $\overline{A}_i^*,\overline{B}_i^*, i=0,...,N_t-1$, *of the operators defined in (1.3.4) are given by:*

$$\overline{A}_i^*\bar{p}_{i+1}=p_i(T_i),\overline{B}_i^*\bar{p}_{i+1}=B(t)^*p_i(t),$$

where $p_i(t)$ *is the solution of (1.3.14) with* $g=0$.

ii) The adjoints $\overline{E}_i^*, \overline{F}_i^*,\ i = 0, ..., N_t - 2$, *of the operators defined in (1.3.6) are given by:*

$$\overline{E}_i^* g = p_i(T_i),\ \overline{F}_i^* g = B(t)^* p_i(t),$$

where $p_i(t)$ *is the solution of (1.3.14) with* $\bar{p}_{i+1} = 0$.

iii) The adjoints $\overline{E}_i^*, \overline{F}_i^*,\ i = 0, ..., N_t - 1$, *of the operators defined in (1.3.7) are given by:*

$$\overline{E}_i^* \begin{pmatrix} g \\ \bar{p}_{i+1} \end{pmatrix} = p_i(T_i),\ \overline{F}_i^* \begin{pmatrix} g \\ \bar{p}_{i+1} \end{pmatrix} = B(t) p_i(t),$$

where p_i *is the solution of (1.3.14).*

Proof. We only prove *iii)*, all other statements can be shown similarly.

Let $i = N_t - 1$ i.e. $T_{i+1} = T$ and let y_i and p_i be the solutions of (1.3.1), and (1.3.14), respectively. Then:

$$\int\limits_{T_i}^{T_{i+1}} \left\langle \frac{\partial}{\partial t} y_i(t), p_i(t) \right\rangle_H + \langle A(t) y_i(t), p_i(t) \rangle_{V^* \times V} -$$

$$- \langle B(t) \bar{u}_i(t) + f(t), p_i(t) \rangle_{V^* \times V} \, dt = 0,$$

$$\int\limits_{T_i}^{T_{i+1}} \left\langle -\frac{\partial}{\partial t} p_i(t), y_i(t) \right\rangle_H + \langle A(t)^* p_i(t), y_i(t) \rangle_{V^* \times V} - \langle g(t), y_i(t) \rangle_{V^* \times V} \, dt = 0.$$

Substracting both equations and using (1.3.1b), (1.3.14b) gives:

$$\langle y_i(T_{i+1}), \bar{p}_{i+1} \rangle_H - \langle \bar{y}_i, p_i(T_i) \rangle_H =$$

$$= \int\limits_{T_i}^{T_{i+1}} \langle B(t) \bar{u}_i(t) + f(t), p_i(t) \rangle_{V^* \times V} - \langle g(t), y_i(t) \rangle_{V^* \times V} \, dt. \tag{1.3.15}$$

Let $\bar{y}_i, \bar{p}_{i+1} \in H,\ g \in L^2(T_i, T_{i+1}; V^*)$ be arbitrary, $u_i, f = 0$ and let y_i, p_i solve (1.3.1) and (1.3.14). The definition (1.3.7) of $\overline{E}_{N_t - 1}$ and (1.3.15) imply:

$$\left\langle \begin{pmatrix} g \\ p_{i+1} \end{pmatrix}, \overline{E}_i \bar{y}_i \right\rangle_{(\zeta_i^* \times H) \times (\zeta_i \times H)} = \int_{T_i}^{T_{i+1}} \langle g(t), y_i(t) \rangle_{V^* \times V} \, dt + \langle y_i(T_{i+1}), \bar{p} \rangle_H =$$

$$= \langle \bar{y}_i, p_i(T_i) \rangle_H = \left\langle \overline{E}_i^* \begin{pmatrix} g \\ \overline{p} \end{pmatrix}, \bar{y}_i \right\rangle_H.$$

This proves the first part of *iii)*.

To prove the second part of *iii)*, we let $\bar{u}_i \in \varphi_i, \bar{p}_{i+1} \in H,\ g \in L^2(T_i, T_{i+1}; V^*)$ be arbitrary, $\bar{u}_i, f = 0$ and let y_i, p_i solve (1.3.1), and (1.3.14) respectively. The definition (1.3.7) of $\overline{E}_{N_t} - 1$ and (1.3.15) imply:

$$\left\langle \left(\begin{array}{c} g \\ \overline{p}_{i+1} \end{array} \right), \overline{F}_i \bar{u}_i \right\rangle_{(\zeta_i^* \times H) \times (\zeta_i \times H)} = \int_{T_i}^{T_{i+1}} \langle g(t), y_i(t) \rangle_{V^* \times V} \, dt + \langle y_i \left(T_{i+1} \right), \bar{p} \rangle_H =$$

$$= \int_{T_i}^{T_{i+1}} \langle B(t)^* p_i(t), \bar{u}_i(t) \rangle_{U^* \times U} \, dt = \left\langle \overline{F}_i^* \left(\begin{array}{c} g \\ \overline{p} \end{array} \right), \bar{u}_i \right\rangle_{\varphi_i^* \times \zeta_i}. \quad \square$$

Remark 1.3.3. Let $A_i, B_i, b_i, i = 1, ..., N_t - 1$ be defined by (1.3.4) and let $\overline{Q}_i, \overline{R}_i, \overline{S}_i, c_i, d_i, i = 1, ..., N_t - 1,\ \overline{Q}_N, \bar{c}_N$ be defined by (1.3.11).

a) Computation of $\bar{y}_{i+1} = \overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i,\ i = 0, ..., N_t - 1:$

- from (1.3.8) and (1.3.4), we see that $\bar{y}_{i+1} = y_i(T_{i+1})$, where y_i is the solution of (1.3.1);

- $\overline{A}_i \bar{y}_i, \overline{B}_i \bar{u}_i$ and $\bar{b}_i$ can be computed by setting the appropriate parts of the input variables $\bar{y}_i, \bar{u}_i$ and f to zero.

b) Computation of $\bar{p}_i = \overline{A}_i \bar{p}_{i+1} + \overline{Q}_i \bar{y}_i + \overline{R}_i \bar{u}_i + c_i,\ i = 1, ..., N_t - 1:$

– from (1.3.11), we see that:

$$\overline{A}_i^* \bar{p}_{i+1} + \overline{Q}_i \bar{y}_i + \overline{R}_i \bar{u}_i + c_i = \overline{A}_i^* \bar{p}_{i \div 1} + \overline{E}_i^* \left(\overline{M}_i^z \left(\overline{E}_i \bar{y}_i + \overline{F}_i \bar{u}_i + \bar{f}_i \right) - \bar{z}_i \right) =$$

$$= \overline{A}_i^* \bar{p}_{i \div 1} + \overline{E}_i^* \left(\overline{M}_i^z w_i - \bar{z}_i \right),$$

where w_i is the solution of (1.3.1);

– the definitions (1.3.9), (1.3.10) of M_i^z and Theorem 1.3.2 imply $\bar{p}_i = p_i(T_i)$ where p_i is the solution of:

$$-\frac{\partial}{\partial t} p_i(t) + A(t)^* p_i(t) = \alpha_1 C(t)^* \left(C(t) w_i(t) - z_1 \right), \ t \in [T_i, T_{i+1}], \quad (1.3.16a)$$

with final condition:

$$p_i \left(T_{i+1} \right) = \begin{cases} \bar{p}_{i+1} & i = 0, ..., N_t - 2 \\ \bar{p}_{i+1} + \alpha_2 C_T^* \left(C_T w_i(T) - z_2 \right), & i = N_t - 1. \end{cases} \quad (1.3.16b)$$

c) For $i = N_t, \bar{p}_{N_t} = \overline{Q}_{N_t} \bar{y}_{N_t} + \bar{c}_{N_t} = 0$, since $\overline{Q}_{N_t} = 0, \bar{c}_{N_t} = 0$.

d) Computation of $\bar{v}_i = \overline{S}_i \bar{u}_i + \overline{R}_i^* \bar{y}_i + \overline{B}_i^* \bar{p}_{i+1} + \bar{d}_i,\ i = 0, ..., N_t - 1:$

- from (1.3.11), we see that:

$$\overline{S}_i \bar{u}_i + \overline{R}_i^* \bar{y}_i + \overline{B}_i^* \bar{p}_{i+1} + \bar{d}_i = \overline{B}_i^* \bar{p}_{i+1} + \bar{u}_i + \overline{F}_i^* \left(\overline{M}_i^z \left(\overline{E}_i, \bar{y}_i + \overline{F}_i \bar{u}_i + \bar{f}_i \right) - \bar{z}_i \right) =$$

$$= \overline{B}_i^* \bar{p}_{i+1} + \bar{u}_i + \overline{F}_i^* \left(\overline{M}_i^z w_i - \bar{z}_i \right),$$

where is the solution of (1.3.1);

- the definitions (1.3.9), (1.3.10) of M_i^z and Theorem 1.3.2 imply that $\bar{v}_i(t) = B(t)^* p_i(t) + \bar{u}_i(t)$, where p_i solves (1.3.14).

1.4 Iterative solution of the discrete optimality system

We group the equations (1.3.13) in the following way:

$$\overline{S}_0 u_0 + \overline{R}_0^* \bar{y}_0 + \overline{B}_0^* \bar{p}_1 + \bar{d}_0 + \rho \overline{B}_0^* \left(\overline{A}_0 \bar{y}_0 + \overline{B}_0 \bar{u}_0 + \bar{b}_0 - \bar{y}_1 \right) = 0, \tag{1.4.1a}$$

$$\overline{A}_0 \bar{y}_0 + \overline{B}_0 \bar{u}_0 + \bar{b}_0 - \bar{y}_1 = 0,$$

$$-\bar{p}_i + \overline{A}_i^* \bar{p}_i + \overline{Q}_i^* \bar{y}_i + \overline{R}_i \bar{u}_i + \bar{c}_i + \rho \overline{A}_i^* \left(\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i - \bar{y}_{i+1} \right) - \tag{1.4.1b}$$

$$-\delta \left(\overline{A}_i \bar{y}_{i-1} + \overline{B}_{i-1} \bar{u}_{i-1} + \bar{b}_{i-1} - \bar{y}_i \right) = 0,$$

$$\overline{S}_i \bar{u}_i + \overline{R}_i^* \bar{y}_i + \overline{B}_i^* \bar{p}_{i+1} + \bar{d}_i + \rho \overline{B}_i^* \left(\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i - \bar{y}_{i+1} \right) = 0,$$

$$\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i - \bar{y}_{i+1} = 0, \ i = 1, ..., N_t - 1,$$

$$-\bar{p}_{N_t} + \overline{Q}_{N_t} \bar{y}_{N_t} + c_{N_t} - \delta \left(\overline{A}_{N_{t-1}} + \overline{B}_{N_{t-1}} + \bar{b}_{N_{t-1}} - \bar{y}_{N_t} \right) = 0. \tag{1.4.1c}$$

In (1.3.13), we used $\rho = \delta \geq 0$. Here we introduce the second parameter δ to better distinguish the terms in (1.3.13) corresponding to ρ, δ respectively. We assume that $\rho, \delta \geq 0$.

Now we arrange the equations (1.4.1) into a block system:

Ax=b

where the variables *x* and the term *b* are given by Fig. 1.4.1 and *A* is given by Fig. 1.4.2.

$$
x = \begin{pmatrix} \bar{y}_1 \\ \bar{u}_0 \\ --- \\ \bar{y}_2 \\ \bar{u}_1 \\ \bar{p}_1 \\ --- \\ \bar{y}_3 \\ \bar{u}_2 \\ \bar{p}_2 \\ --- \\ \vdots \\ --- \\ \bar{y}_{N_t} \\ \bar{u}_{N_t-1} \\ \bar{p}_{N_t-1} \\ --- \\ \bar{p}_{N_t} \end{pmatrix}, \quad b = - \begin{pmatrix} \bar{d}_0 + \left(\overline{R}_0^* + \rho \overline{B}_0^* \overline{A}_0\right) y_0 + \rho \overline{B}_0^* \bar{b}_0 \\ \bar{b}_0 + \overline{A}_0 y_0 \\ ----------------- \\ \bar{c}_1 + \rho \overline{A}_1^* \bar{b}_1 - \delta \bar{b}_0 \\ \bar{d}_1 + \rho \overline{B}_1^* \bar{b}_1 \\ \bar{b}_1 \\ ---------------- \\ \bar{c}_2 + \rho \overline{A}_2^* \bar{b}_2 - \delta \bar{b}_1 \\ \bar{d}_2 \rho \overline{B}_2^* \bar{b}_2 \\ \bar{b}_2 \\ ----------- \\ \vdots \\ ---------------- \\ \bar{c}_{N_t-1} + \rho \overline{A}_{N_t-1}^* \bar{b}_{N_t-1} - \delta \bar{b}_{N_t-2} \\ \bar{d}_{N_t-1} + \rho \overline{B}_{N_t-1} \bar{b}_{N_t-1} \\ \bar{b}_{N_t-1} \end{pmatrix}
$$

Fig. 1.1.

$$
\left(\begin{array}{cc|ccc|ccc|ccc|ccc|c}
-\rho\overline{B}_0^* & \overline{S}_0^\rho & 0 & 0 & \overline{B}_0 & & & & & & & & & & \\
-I & \overline{B}_0 & 0 & 0 & 0 & & & & & & & & & & \\
\hline
\overline{Q}_1^{\rho\delta} & 0 & -\rho\overline{A}_1^* & \overline{R}_1^\rho & -I & 0 & 0 & \overline{A}_1^* & & & & & & & \\
\left(\overline{R}_1^\rho\right)^* & 0 & -\rho\overline{B}_1^* & \overline{S}_1^\rho & 0 & 0 & 0 & -\overline{B}_1^* & & & & & & & \\
\overline{A}_1 & 0 & -I & \overline{B}_1 & 0 & 0 & 0 & 0 & & & & & & & \\
\hline
-\delta\overline{A}_1 & 0 & \overline{Q}_2^{\rho\delta} & -\delta\overline{B}_1 & 0 & -\rho\overline{A}_2^* & \overline{R}_2^\rho & -I & 0 & 0 & \overline{A}_2^* & & & & \\
0 & 0 & \left(\overline{R}_2^\rho\right)^* & 0 & 0 & -\rho\overline{B}_2^* & \overline{S}_2^\rho & 0 & 0 & 0 & \overline{B}_2^* & & & & \\
0 & 0 & \overline{A}_2 & 0 & 0 & -I & \overline{B}_2 & 0 & 0 & 0 & 0 & & & & \\
 & & & & & & & & \ddots & & & & & & \\
\hline
 & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & \ddots & & & & \\
\hline
 & & & & & -\delta\overline{A}_{N_t-1} & 0 & 0 & \overline{Q}_{N_t-1}^{\rho\delta} & -\delta\overline{B}_{N_t-1} & 0 & -\rho\overline{A}_{N_t-1}^* & \overline{R}_{N_t-1}^\rho & -I & \overline{A}_{N_t-1}^* \\
 & & & & & 0 & 0 & 0 & \left(\overline{R}_{N_t-1}^\rho\right)^* & 0 & 0 & -\rho\overline{A}_{N_t-1}^* & \overline{S}_{N_t-1}^\rho & 0 & \overline{B}_{N_t-1}^* \\
 & & & & & 0 & 0 & 0 & \overline{A_{N_t-1}} & 0 & 0 & -I & B_{N_t-1} & 0 & 0 \\
 & & & & & & & & -\delta\overline{A_{N_t-1}} & 0 & 0 & Q_{N_t}+\delta I & -\delta\overline{B}_{N_t-1} & 0 & -I
\end{array}\right)
$$

Fig. 1.2.

Since the operators $\overline{S}_i, i = 0, ..., N_t - 1$ are strictly positive (Theorem 1.3.1), the diagonal blocks of A in Fig. 1.4.2 are continuously invertible.

Solving the problem with the Gauss-Seidel iterations

All block Gauss-Seidel methods require the solution of the systems (1.4.1a), (1.4.1b), (1.4.1c) for $(\bar{y}_1, \bar{u}_0), (\bar{y}_{i+1}, \bar{u}_i, \bar{p}_i)$ and p_{N_t} respectively. We use Remark 1.3.3. to rewrite these systems in the notation of the original problem and to discuss what is required to solve them. For $i \in \{0, ..., N_t - 1\}$, let y_i be the solution of:

$$\frac{\partial}{\partial t} y_i + A(t) y_i(t) = B(t) \bar{u}_i(t) + f(t),\ t \in (T_i, T_{i+1}), \tag{1.4.2a}$$

$$y_i(T_i) = \bar{y}_i. \tag{1.4.2b}$$

Remark 1.3.3 shows that:

$$\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i = y_i(T_{i+1}).$$

For $i \in \{0, ..., N_t - 1\}$, let p_i be the solution of the state equation:

$$-\frac{\partial}{\partial t} p_i(t) + A(t)^* p_i(t) = \alpha_1 C(t)^* \left(C(t) y_i(t) - z_1\right),\ t \in (T_i, T_{i+1}),$$

$$p_i(T_{i+1}) = \begin{cases} \bar{p}_{i+1} + \rho \left(y_i(T_{i+1}) - \bar{y}_{i+1}\right) & ,\ i = 0, ..., N_{t-2}, \\ \bar{p}_{i+1} + \rho \left(y_i(T_{i+1}) - \bar{y}_{i+1}\right) + \alpha_2 C_T^* \left(C_T y_i(T) - z_2\right) & ,\ i = N_{t-1}, \end{cases} \tag{1.4.2d}$$

which solves the system (1.4.2a), (1.4.2b).

Remark 1.3.3 shows that:

$$\overline{A}_i^* \bar{p}_{i+1} + \overline{Q}_i \bar{y}_i + \overline{R}_i \bar{u}_i + \bar{c}_i + \rho \overline{A}_i^* \left(\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i - \bar{y}_{i+1}\right) = p_i(T_i).$$

Finally, for $i \in \{0, ..., N_t - 1\}$, the equation

$$B(t)^* p_i(t) + \bar{u}_i(t) = 0,\quad t \in (T_i, T_{i+1}), \tag{1.4.2e}$$

where p_i solves (1.4.2c), (1.4.2d), is just the equation:

$$\overline{S}_i \bar{u}_i + \overline{R}_i^* \bar{y}_i + \overline{B}_i^* \bar{p}_{i+1} + \bar{d}_i + \rho \overline{B}_i^* \left(\overline{A}_i \bar{y}_i + \overline{B}_i \bar{u}_i + \bar{b}_i - \bar{y}_{i+1}\right) = 0.$$

Now we are able to discuss the solution of the block diagonal systems. Let $i = 0$. If $(y_0, \bar{u}_0, p_0)$ solves (1.4.2), then the solution $(\bar{y}_1, \bar{u}_0)$ of (1.4.1a) is given by:

$$\bar{y}_1 = y_0(T_1).$$

Let $i \in \{1, ..., N_t - 1\}$. If $(\bar{y}_i, \bar{u}_i, \bar{p}_i)$ solves (1.4.2) and y_{i-1} solves (1.4.2a) with $i = i - 1$, then the solution $(\bar{y}_{i+1}, \bar{u}_i, p_i)$of (1.4.1b) is given by:

$$\bar{y}_{i+1} = y_i(T_{i+1}), \bar{p}_i = p_i(T_i) + \delta(\bar{y}_i - y_{i-1}(T_i)).$$

Finally, since $\overline{Q}_{N_t} = 0,\ c_{N_t} = 0$, the solution of (1.4.1c) is given by:

$$\bar{p}_{N_t} = \delta\left(\bar{y}_{N_t} - y_{N_t - 1}(T_{N_t})\right),$$

where $y_{N_t} - 1$ solves (1.4.2a), (1.4.2b) with $i = N_t - 1$.

Notice that the system (1.4.2) for $i \in \{1, ..., N_t - 1\}$ is the optimality system for the quadratic optimization problem:

$$\min \frac{1}{2} \int\limits_{T_i}^{T_{i+1}} ||\bar{u}_i(t)||_u^2 dt + \frac{\alpha_1}{2} \int_{T_i}^{T_{i+1}} ||C(t) y_i(t) - z_1(t)||_Z^2 dt +$$

$$+ \langle y_i(T_{i+1}), \bar{p}_{i+1} + \rho(y_i(T_{i+1}) - \bar{y}_{i+1} \rangle_H ,$$

$$\frac{\partial}{\partial t} y_i(t) + A(t) y_i(t) = B(t) \bar{u}_i(t) + f(t),\ t \in (T_i, T_{i+1}),\ y_i(T_i) = \bar{y}_i, \quad (1.4.3)$$

where for $i = N_t - 1$, the term $\langle y_i(T_{i+1}), \bar{p}_{i+1} \rangle_H$ in the objective function has to be replaced by

$$\langle y_i(T_{i+1}), \bar{p}_{i+1} + \rho(y_i(T_{i+1}) - \bar{y}_{i+1}) + \alpha_2 C_T^* (C_T y_i(T) - z_2) \rangle_H .$$

Gauss-Seidel Iterations

The matrix A in the Gauss-Seidel iterations is given by:

$$A = D - L - U,$$

where $D = diag(A)$ is the block diagonal part, $-L$ is the strictly lower block triangular part, and $-U$ is the strictly upper block triangular part of A. Then, a Gauss-Seidel iteration is given by:

$$x_{k+1} = (D - L)^{-1}(b + U x_k).$$

For the system (1.4.1) a Gauss-Seidel iteration is given by the computation of $x_{k+1} = (D - L)^{-1}(b + U x_k)$:

$a)$ Solve (1.4.1a) for $(\bar{y}_1, \bar{u}_0)$.

$b)$ For $i = 1, ..., N_t - 1$ (1.4.4),solve (1.4.1b) for $(\bar{y}_{i+1}, \bar{u}_i, \bar{p}_i)$.

$c)$ Compute p_{N_t} from (1.4.1c).

We notice that, because of the third equation in (1.4.1b), all terms in (1.4.1b) involving ρ will be zero, all terms in (1.4.1b), (1.4.1c) involving ρ will be zero. Hence, Gauss-Seidel is independent of ρ and δ and we may consider $\rho = \delta = 0$..

Using our discussions we can formulate the Gauss-Seidel method as the computation of $x_{k+1} = (D - L)^{-1}(b + Ux_k)$:

$$\begin{array}{l} \text{For } i = 0, ..., N_t - 1 : \\ \text{Solve } (1.4.2a - 1.4.2e \text{) } (or\ 1.4.3) \\ \text{ Set } \bar{y}_{i+1} = y_i(T_{i+1}),\ \bar{p}_i = p_i(T_i). \\ \text{(If } i = 0, \text{ only } y_1, u_0 \text{ are computed).} \\ \text{ Set } p_{N_t} = 0. \end{array} \tag{1.4.5}$$

In the Gauss-Seidel method, the states computed as the solutions of (1.4.2a), (1.4.2b) are continuous in time in the sense that:

$$y_i(T_{i+1}) = \bar{y}_{i+1} = y_{i+1}(T_i),\ i = 0, ..., N_t - 1.$$

If one step of the Gauss-Seidel method (1.4.5) with starting value $\bar{y}_i = 0$, $u_i = 0, \bar{p}_i = 0, i = 1, ..., N_t$, is applied, then the problem (1.4.3) which will be solved in the ith substep of the Gauss-Seidel method is identical to the original optimal control problem (1.1.1), (1.1.2) with initial conditions $y(T_i) = \bar{y}_i$.

In the Gauss-Seidel method, the solution $(\bar{y}_{i+1}, \bar{u}_i, \bar{p}_i)$ of the system (1.4.1b) depends only on (y_{j+1}, u_j, p_j), with $j = i \pm 1$. Thus, we can solve in parallel the diagonal block systems (1.4.2) corresponding to even indices i and then we can solve in parallel the diagonal block systems (1.4.2) corresponding to odd indices i, analogous to the red-black-ordering.

1.5 Numerical experiment

It is considered the problem of heat distribution in a homogeneous finite bar due to some internal sources and having sources and heat leaks at the borders. It is required to perform the control of temperature, meaning to determine the control u and the temperature y in a point situated at distance x towards the end of the bar with the minimalization of the following performance criteria :

$$\min \frac{1}{2}\int_0^T u^2(t)dt + \frac{\alpha_1}{2}\int_0^T\int_a^b (y(t,x) - z_1(t,x))^2 dxdt + \frac{\alpha_2}{2}\int_0^1 (y(T,x) - z_2(x))^2 dx \tag{1.5.1}$$

with:

$$\frac{\partial}{\partial t}y(t,x) - \frac{\partial^2}{\partial x^2}y(t,x) = f(t,x),\ \frac{\partial}{\partial x}y(t,0) = u(t),\ t \in (0,T), \tag{1.5.2}$$

$$\frac{\partial}{\partial x}y(t,1) = r(t),\ t \in (0,T), y(0,x) = y_0(x),\ \ x \in [0,1].$$

$y(t,0)$ and $y(t,1)$ represent the temperature regime imposed at the ends of the bar, and $y(0,x) = y_0(x)$ represents the temperature distribution from the bar at the beginning of the experiment ($t = 0$). The solving is being done progressively on intervals. From the initial data (at $t = 0$), one can determine the temperature distribution at n the next time, from which the distribution over a longer period is determined and so on.

The spaces corresponding to the general case are: $H = L^2(0,1)$,
$V = H^1(0,1)$ and $U = L^2(0,T)$, $Z = Z_T = L^2(a,b)$. The following specifications are considered for the respective problem:

$$\alpha_1 = \alpha_2 = 10^3,\ [a,b] = [0,1],\ f(t,x) = (4\pi^2(1-e^{-t}) + e^{-t})\sin(2\pi x)$$
$$r(t) = 2\pi(1-e^{-t}),\ \ y_0(x) = 0,\ \ z_1 = z_2 = 1.$$

The above data have been chosen so that, if $u = r$, then $y(t,x) = \sin(2\pi x)(1-e^{-t})$ is a solution of the problem.

To obtain the equation with rank two differences it is easy to prolong the solution outside the definition domain ($0 < x < 1$) with an interval h to the left and to the right. This means that we are going to have a reticular domain :

$$x_k = kh,\ k = -1, 0, 1, ..., N.N+1,\ h = 1/N.$$

On this network, we define the approximate of the state equation and of the limit conditions under the form

$$\frac{dy_k}{dt} + \frac{-y_{k-1} + 2y_k - y_{k+1}}{h^2} = f_k(t),\ k = 0, 1, ..., N,$$
$$\frac{y_i - y_{-1}}{2h} = u(t), \frac{y_{N+1} - y_{N-1}}{2h} = r(t),\ y_0 = 0. \tag{1.5.3}$$

To solve the problem (1.5.3) one has to eliminate the limit conditions, through the elimination of the two new unknown y_{-1} and y_{N+1}. Through this, the limit conditions are being solved in relation to y_{-1} and y_{N+1} one obtains:

$$y_{-1} = y_1 - 2hu(t),\ \ y_{N+1} = y_{N-1} + 2hr(t).$$

The values obtained are replaced in (1.5.3):

$$\frac{dy_0}{dt} + 2\frac{y_0 - y_1}{h^2} = f_0 - \frac{2}{h}u_0(t),$$
$$\frac{dy_k}{dt} + \frac{-y_{k-1} + 2y_k - y_{k+1}}{h^2} = f_k(t),\ \ k = 1, ..., N-1, \tag{1.5.4}$$

$$\frac{dy_N}{dt} + 2\frac{-y_{N-1} + y_N}{h^2} = f_N - \frac{2r_N(t)}{h}, \ \ y_0 = 0.$$

The solving of the system (1.5.4) over the interval $t \in [t_j, t_{j+1}]$ leads to the system:

$$\begin{aligned}
\frac{y_0^{j+1} - y_0^j}{\tau} &= 2\frac{\bar{y}_1^j - \bar{y}_0^j}{h^2} + \bar{f}_0^j - \frac{2}{h}\bar{u}_0^j, \\
\frac{y_k^{j+1} - y_k^j}{\tau} &= \frac{\bar{y}_{k-1}^j - 2\bar{y}_k^j + \bar{y}_{k+1}^j}{h^2} + \bar{f}_k^j, \ k = 1, ..., N-1, \\
\frac{y_N^{j+1} - y_N^j}{\tau} &= 2\frac{\bar{y}_N^j - \bar{y}_{N-1}^j}{h^2} + \bar{f}_N^j - \frac{2r_N(t)}{h},
\end{aligned}$$

where $y_k^j = y_k(t_j), \ \tau = \Delta t$,

$$\bar{y}_k^j = \frac{1}{\Delta t}\int_{t_j}^{t_j+1} y_k dt, \ \bar{f}_k^j = \frac{1}{\Delta t}\int_{t_j}^{t_j+1} f_k dt, \ \ \bar{u}_k^j = \frac{1}{\Delta t} = \frac{1}{\Delta t}u_k dt.$$

If the explicit approximation scheme of the triangle is obtained $\bar{y}_k^j \approx y_k^j, \ \bar{f}_k^j \approx f_k^j, \ \bar{u}_k^j \approx u_k^j$, then the system is:

$$\begin{aligned}
y_0^{j+1} &= y_0^j + 2\mu(y_1^j - y_0^j) + \tau(f_0^j - \frac{2}{h}u_0^j) \\
y_k^{j+1} &= y_k^j + \mu(y_{k-1}^j - 2y_k^j + y_{k+1}^j) + \tau f_k^j, \ k = 1, ..., N-1, \\
y_N^{j+1} &= y_N^j + 2\mu(y_N^j - y_{N-1}^j) + \tau\left(f_N^j - \frac{2r_N^j(t)}{h}\right),
\end{aligned} \tag{1.5.5}$$

where $\mu = \frac{\tau}{h^2}$ and to which the initial conditions are added: $y_0^j = 0$.

If the matrix is introduced:

$$A = \begin{bmatrix} -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

corresponding to the second equation from the system (1.5.4), the adjacent equation corresponding to the state equation becomes:

$$\frac{dp_k}{dt} - \frac{-p_{k-1} + p_k - p_{k+1}}{h^2} = \alpha_1(y_k - 1), \ k = 1, ..., N-1,$$

with the final condition: $p_N = \alpha_2(y_N - 1)$,or:

$$\frac{dp_k}{dt} = \frac{-p_{k-1} + p_k - p_{k+1}}{h^2} + \alpha_1(y_k - 1),\ k = 1, ..., N-1, \qquad (1.5.6)$$

$$p_N = \alpha_2(y_N - 1).$$

Solving the system (1.5.6) over the interval $t \in [t_j, t_{j+1}]$ and using the explicit scheme of approximation of the triangle, we get the system:

$$\frac{p_k^{j+1} - p_k^j}{\tau} = \frac{-p_{k-1}^j + p_k^j + p_{k+1}^j}{h^2} + \alpha_1(y_k^j - 1),\ k = 1, ..., N-1, \qquad (1.5.7)$$

$$p_N^j = \alpha_2(y_N^j - 1).$$

In the end, the vector $B = \left[-\frac{2}{h}, 0, 0, ..., 0\right]$ is introduced, the gradient equation corresponding to the problem $(u(t) + B^* p(t) = 0,\ t \in (0, T))$ becomes:

$$u_k^j = -\frac{2}{h} p_k^j,\quad k = 1, ..., N. \qquad (1.5.8)$$

The Gauss-Seidel algorithm will solve the problem (1.5.1),(1.5.2) having the following structure:
for $i = 0, ..., N-1$
solve the problem (1.5.5) and obtain the solution y_i,
using y_i solve the system (1.5.7) and obtain the solution p_i,
using p_i solve the system (1.5.8) and obtain the solution u_i,
set $y_{i+1} = y_i(t_{i+1}),\ p_i = p_i(t_i)$, as initial values for the following interval (t_{i+1}, t_{i+2})

repeat
$p_N = 0$.

Some numerical results for the temperature distribution are presented in Table 1.5.1

j t \ k / y	0 0	1 0.1	2 0.2	3 0.3	4 0.4	5 0.5
0 0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000
10.001	0	0.2000	0.4000	0.6000	0.8000	0.9600
2 0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280
3 0.003	0	0.2000	0.4000	0.5986	0.7986	0.9016
4 0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792
5 0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597
...	...	...	...	...	...	...
10 0.010	0	0.2000	0.3968	0.3968	0.7281	0.7867

Table 1.5.1

It has been noticed though that the method Gauss-Seidel converges slowly, the spectral ray of the iteration matrix $M^{-1}N = (D - L)^{-1}U$, tending to 1. This is why it is recommendable that the method Gauss-Seidel is used for another iterative method, for example, the conjugated gradient method.

References

[1] Berggren M., *Parallel solution of optimal-control problems by time-domain decomposition*, Computational Science for the 21st Century, 102-112,Chichester, 1997.

[2] Chmielewski D., Manousiouthakis V., *On constrained infinite-time linear quadratic optimal control*, System & Control Letters, **29**: 121-129, 1996.

[3] Popoviciu I., *Algoritmi de paralelizare a metodei iterative a relaxărilor succesive*, Ses. de comunicări ştiinţifice cu participare internaţională, Academia Navală "Mircea cel Bătrân", Constanţa, mai, 2001.

[4] Scokaert P.O., Rawlings J.B., *Infinite horizon linear quadratic control with constrains*, Proceedings of the 13th IFAC World Congress, San Francisco, 1996.

[5] Scokaert P.O., Rawlings J.B., *Constrained linear quadratic regulation*, IEEE Transactions on Automatic Control, December, 1995.

Department of Mathematics and Informatics
Naval Academy "Mircea cel Bãtrân"
Fulgerului 1, Constanta, ROMANIA, 900218
e-mail: bhbro@yahoo.com