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PARALLEL METHODS FOR SOLVING THE LINEAR ALGEBRAIC SYSTEMS

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Abstract

By using the domain decomposition methodology, we construct several algebraic domain decomposition methods for certain algebraic systems with sparse matrix. These methods are highly parallelizable. We show that these methods are convergent and we also discuss the eigenvalue distributions of the corresponding iterative matrices in order to analyse the convergenge factors of these methods.

1. Algebraic domain decomposition methods

Let us consider the linear algebraic system:

$$Au = f, \tag{1.1}$$

where matrix A is a block square matrix denoted by:

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,2p-1} \\ \vdots & & \\ A_{2p-1,1} & \cdots & A_{2p-1,2p-1} \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ \vdots \\ u_{2p-1} \end{bmatrix},\ \ f = \begin{bmatrix} f_1 \\ \vdots \\ f_{2p-1} \end{bmatrix}.$$

As in the domain decomposition methods, we partition the unknown vector u into p new subvectors $x_1, x_p, x_i,\ 2 \le i \le p-1$ ::

$$\tilde{u} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \text{ with } x_1 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ \ x_p = \begin{bmatrix} \tilde{u}_{2p-2} \\ u_{2p-1} \end{bmatrix},\ x_i = \begin{bmatrix} \tilde{u}_{2i-2} \\ u_{2i-1} \\ u_{2i} \end{bmatrix}. \tag{1.2}$$

Note that $\tilde{u}_{2i}$ are the unknown vectors associated with the overlapping to the subvector $\tilde{u}_{2i} = u_{2i},\ i = 1, ..., p$. In the same way, we can introduce a new

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vector $\tilde{f}$ from righthandside vector f. Thus, the matrix A is divided into $p \times p$ corresponding block submatrices with overlapping. Then, a corresponding new matrix $\tilde{A}$ can be defined by these $p \times p$ block submatrices:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{1,1} & \cdots & \tilde{A}_{1,p} \\ \vdots & & \\ \tilde{A}_{p,1} & \cdots & \tilde{A}_{p,p} \end{bmatrix}, \tag{1.3}$$

where

$$\tilde{A}_{1,1} = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} \end{bmatrix}, \quad \tilde{A}_{p,p} = \begin{bmatrix} A_{2p-2,2p-2} & A_{2p-2,2p-1} \\ A_{2p-1,2p-2} & A_{2p-1,2p-1} \end{bmatrix},$$

$$\tilde{A}_{1,p} = \begin{bmatrix} 0 & \tilde{A}_{1,2p-1} \\ 0 & \tilde{A}_{2,2p-1} \end{bmatrix}, \quad \tilde{A}_{1,i} = \begin{bmatrix} 0 & A_{1,2i-1} & \tilde{A}_{1,2i} \\ 0 & A_{2,2i-1} & \tilde{A}_{2,2i} \end{bmatrix}, \ 2 \le i \le p-1,$$

$$\tilde{A}_{p,1} = \begin{bmatrix} A_{2p-2,1} & 0 \\ A_{2p-1,1} & 0 \end{bmatrix}, \ \tilde{A}_{p,i} = \begin{bmatrix} A_{2p-2,2i-2} & A_{2p-2,2i-1} & 0 \\ A_{2p-1,2i-2} & A_{2p-1,2i-1} & 0 \end{bmatrix},$$

$2 \le i \le p-1$,

$$\tilde{A}_{i,1} = \begin{bmatrix} A_{2i-2,1} & 0 \\ A_{21-1,1} & 0 \\ A_{2i,1} & 0 \end{bmatrix}, \ \tilde{A}_{i,p} = \begin{bmatrix} 0 & A_{2i-2,2p-1} \\ 0 & A_{2i-1,p-1} \\ 0 & A_{2i,2p-1} \end{bmatrix}, \ 2 \le i \le p-1$$

$$\tilde{A}_{i,j} = \begin{bmatrix} 0 & A_{2i-2,2j-1} & A_{2i-2,2j} \\ 0 & A_{2i-1,2j-1} & A_{2i-1,2j} \\ 0 & A_{2i,,2j-1} & A_{2i,2j} \end{bmatrix}, \ 2 \le i < j \le p-1,$$

$$\tilde{A}_{j,i} = \begin{bmatrix} A_{2i-2,2j-2} & A_{2i-2,2j-1} & 0 \\ A_{2i-1,2j-2} & A_{2i-1,2j-1} & 0 \\ A_{2i,,2j-2} & A_{2i,2j-1} & 0 \end{bmatrix}, \ 2 \le i < j \le p-1,$$

$$\tilde{A}_{i,i} = \begin{bmatrix} A_{2i-2,2i-2} & A_{2i-2,2i-1} & A_{2i-2,2i} \\ A_{2i-1,2i-2} & A_{2i-1,2i-1} & A_{2i-1,2i} \\ A_{2i,,2i-1} & A_{2i,2i-1} & A_{2i,2i} \end{bmatrix}, \ 2 \le i \le p-1.$$

From these definitions, we obtain a new linear system

$$\widehat{A}\widehat{u} = \widehat{f}, \tag{1.4}$$

which is associated with the system (1.1). The following theorem gives the relation between the equation (1.1) and the system (1.4).

Theorem 1.1. *Suppose that $A_{2i,2i}$ are nonsingular for all i=1, ..., p-1. Then, the solution of the system (1.1) can be constructed from the solution of the problem (1.4) and vice versa.*

Proof. Let u be the solution of the system (1.1). We can construct a new vector $\tilde{u}$ as above from u, by letting the overlapping . Then this $\tilde{u}$ is the solution of the system (1.4).

Suppose that $\tilde{u}$ is the solution of the problem (1.4). Because $A_{2i,2i}$ are nonsingular for $i = 1, ..., p-1$, we have $u_{2i} = \tilde{u}_{2i},\ i = 1, ..., p-1$ from the equation $A_{2i,2i}(u_{2i} - \tilde{u}_{2i}) = 0$. A direct consequence is that the vector $u = \left(u_1^T, ..., u_{2p-1}^T\right)$ is the solution of the system (1.1).

Theorem 1.1. implies that if the solution of (1.1) is unique, the problem (1.4) has only one solution. □

Let us denote by $\lambda\{A\}$ the set of the eigenvalues of the matrix A.

Theorem 1.2. $\lambda\{\tilde{A}\} = \lambda\{A\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{A_{2i,2i}\}\right)$.

Proof. We first prove $\lambda\{\tilde{A}\} \subset \lambda\{A\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{A_{2i,2i}\}\right)$.
From $\widetilde{A}\tilde{u} = \widetilde{\lambda}\tilde{u}$ we have $A_{2i,2i}(u_{2i} - \tilde{u}_{2i}) = \widetilde{\lambda}(u_{2i} - \tilde{u}_{2i}),\ i = 1, ..., p-1$. If $u_{2i} \neq \tilde{u}_{2i}$ for some $i \in \{1, ..., p-1\}$, then $\tilde{\lambda} \in \lambda\{A_{2i,2i}\}$. Otherwise, if $u_{2i} = \tilde{u}_{2i} (\forall) i \in \{1, ..., p-1\}$, then $\widetilde{\lambda} \in \lambda\{A\}$.

Now we prove that $\lambda\{A\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{A_{2i,2i}\}\right) \subset \lambda\{\tilde{A}\}$. From $Au = \lambda u$, we have $\widetilde{A}\tilde{u} = \lambda\tilde{u}$ with $u_{2i} = u_{2i}$ as in (1.2). If $A_{2i,2i}v_{2i} = \lambda v_{2i}$, for some $i \in \{1, ..., p\}$, then we can construct a vector $\tilde{u}$ in the following way such that $\widetilde{A}\tilde{u} = \lambda\tilde{u}$. Let $\omega = -(0, ..., 0, v_{2i}^T A_{2i+1,2i}^T, ..., v_{2i}^T A_{2p-1,2i})$. If this $\lambda \notin \lambda\{A\}$ then the equation $(A - \lambda I)u = \omega$ has only one solution u. By setting

$$\tilde{u}_{2j} = \begin{cases} u_{2j} & j \neq i \\ u_{2i} + v_{2j} & j = i, \end{cases}$$

for this $\tilde{u}$, we can easily show that $\widetilde{A}\tilde{u} = \lambda\tilde{u}$. Hence,

$$\lambda\{A\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{A_{2i,2i}\}\right) \subset \lambda\{\tilde{A}\}. \ \square$$

Before giving a definition of an asynchronous Schwarz algorithms for the problem (1.1), we first decompose the linear system (1.1) into p subproblems: to find x_i^* such that

$$\tilde{A}_{i,i}x_i^* = \tilde{f}_i - \sum_{j \neq i} \tilde{A}_{i,j}x_j,\ 1 \leq i \leq p, \tag{1.5}$$

where $\tilde{u} = \left(x_1^T, ..., x_p^T\right)^T$ is regarded as a known vector and $\tilde{u}^* = \left(x_1^{*T}, ..., x_p^{*T}\right)$ as an unknown vector. Let φ_i denote a solver of (1.5). This φ_i is either a direct solver or an iterative solver. Now we list several possible choices φ_i as the examples:

$$a)\quad \tilde{u}^* = \varphi_i(\tilde{u}) \text{ and } \tilde{A}_{i,i}\tilde{u}^* = \tilde{f}_i - \sum_{j\neq i} \tilde{A}_{i,j}x_j;$$

$$b)\quad \tilde{u}^* = \varphi_i(\tilde{u}) \text{ is defined by } r_i = \tilde{f}_i - \sum_{1\leq j\leq p} \tilde{A}_{i,j}x_j,$$

$$\alpha_i = (r_i, r_i)/(\tilde{A}_{i,i}r_i, r_i),\quad x_i^* = x_i + \alpha_i r_i;$$

$c)$ Let $\tilde{A}_{i,i} = M_i - N_i$, where M_i is an invertible matrix. Then

$$\tilde{u}^* = \varphi_i(\tilde{u}) \text{ and } \; x_i^* = x_i + M_i^{-1}\left(\tilde{f}_i - \sum_{i\leq j\leq p} \tilde{A}_{i,j}x_j\right).$$

Let's distribute the computation on the machine with p processors. Each processor is assigned to solve one subproblem (1.5). Denote by N as the whole positive integer set. If we let all processors kept on calculating by using the most recent available data from neighbour processors, then we have the following asynchronous method:

Let $\tilde{u}^{(0)}$ be a given initial guess vector. The vector sequence $\tilde{u}(k)$ will be defined by the recursion:

$$\begin{cases} x_i^{k+1} = \phi_i\left(x_1^{(s_1(k)}, ..., x_p^{s_p(k)}\right), \\ x_i^{(k+1)} = x_i^{s_i(k)} + \omega_i\left(x_i^{(k+1)} - x_i^{s_i(k)}\right) \text{ if } i \in J(k), \\ x_i^{(k+1)} = x_i^{(k)} \text{ if } i \notin J(k), \\ u^{(k+1)} = \left(x_1^{(k+1)}, ..., x_p^{(k+1)}\right), \end{cases} \tag{1.6}$$

where $\{J(k)\}_{k\in N}$ is a sequence of nonempty subsets of the set $\{1, ..., p\}$. In fact, $J(k)$ is the set of subvectors to be updated at the step k. Here, $S = \{s_1(k), ..., s_p(k)\}$ is a sequence of elements of N^p with the following properties: $s_i(k) \leq k,\ \forall k \in N,\ (\forall)i \in \{1, ..., p\}$, and $\lim_{k\to\infty} s_i(k) = \infty,\ (\forall)i \in \{1, ..., p\}$.

Such a procedure is called the **chaotic relaxation Schwarz (CRS) algorithm** and is identified by $\left(\phi_i, \tilde{u}^{(0),J,S}\right)$. Selecting the setting $J(k)$ and set $S = \{s_1(k), ..., s_p(k)\}$, we give several special cases of CRS algorithm:

a) **Algebraic Multiplicative Schwarz (AMS) Algorithm** with:

$$s_i(k) = k,\ J(k) \equiv (1+k) mod p,\ \forall k.$$

b) **Algebraic Additive Schwarz (AAS) Algorithm** with:

$$S_i(k) = k,\ J(k) = \{1, ..., p\}\ \ \forall k.$$

2. Direct Subsolver for All Sub-problems

We use a direct solver for the subproblems in our CRS method. In order to analyse the convergence factor, we discuss the eigenvalue distribution of the iterative matrix of AMS and AAS methods. Let the matrix be

$$\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U},$$

where $\tilde{D}$ is a block diagonal matrix, $\tilde{L}$ is a block lower triangular matrix, and $\tilde{U}$ is a block upper triangular matrix

$$\tilde{D} = \begin{bmatrix} \tilde{A}_{1,1} & & 0 \\ & \ddots & \\ 0 & & \tilde{A}_{p,p} \end{bmatrix},\quad \tilde{L} = \begin{bmatrix} 0 & & & \\ \tilde{A}_{2,1} & 0 & & \\ \vdots & \ddots & \ddots & \\ \tilde{A}_{p,1} & \cdots & \tilde{A}_{p,p-1} & 0 \end{bmatrix},$$

$$\widetilde{U} = \begin{bmatrix} 0 & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,p} \\ & \ddots & \ddots & \vdots \\ & & 0 & \tilde{A}_{p-1,p} \\ & & & 0 \end{bmatrix}.$$

Assume that, for $k = 1, 2, ...$, we define a new sequence $y^{(k)}$ by:

$$y^{(k)} = \begin{bmatrix} x_1^{(k-1)p+1} \\ \vdots \\ x_p^{kp} \end{bmatrix},\ \text{with } y^{(0)} = \tilde{u}^{(0)}.$$

If p subproblems are all solved by the direct solver φ_i with $\omega_i = \omega$, the AMS method can be described in one simple form

$$\widetilde{H}y^{(k+1)} = \widetilde{B}y^{(k)} + \omega\widetilde{f},$$

where $\widetilde{H} = \widetilde{D} + \omega\widetilde{L}$ and $\widetilde{B} = (1-\omega)\widetilde{D} - \omega\widetilde{U}$ and the AAS method can be rewritten as

$$\widetilde{D}\tilde{u}^{(k+1)} = \left(\widetilde{D} - \omega\widetilde{A}\right)\tilde{u}^{(k)} + \omega\widetilde{f}.$$

Then the iterative matrices of the AMS method and AAS method satisfy the following:

Theorem 2.1. *If* $A_{i,2j}=0$ *for* $|i-2j|\geq 2$, *then*
i) The iterative matrix of the AMS method satisfies:

$$\lambda\{\widetilde{H}^{-1}\widetilde{B}\}\subseteq\lambda\left\{\left(\ddot{D}+\omega\dot{L}\right)^{-1}(1-\omega)\,\ddot{D}-\omega\ddot{L}\right\}$$

and

$$\lambda\left\{\left(\ddot{D}+\omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D}-\omega\ddot{L}\right)\right\}\setminus\{1-\omega\}\subseteq\lambda\{\widetilde{H}^{-1}\widetilde{B}\}.$$

ii) The iterative matrix of the AAS method satisfies:

$$\lambda\left\{\widetilde{D}^{-1}\left(\widetilde{D}-\omega\widetilde{A}\right)\right\}\subseteq\lambda\left\{\ddot{D}^{-1}\left(\ddot{D}-\omega\ddot{A}\right)\right\}$$

and

$$\lambda\left\{\ddot{D}^{-1}\left(\ddot{D}-\omega\ddot{A}\right)\right\}\setminus\{1-\omega\}\subseteq\lambda\left\{\widetilde{D}^{-1}\left(\widetilde{D}-\omega\widetilde{A}\right)\right\}.$$

Here the block diagonal matrix $\ddot{D}$, *the lower triangular matrix* $\ddot{L}$ *and the upper triangular matrix* $\ddot{U}$ *in the sum expression* $\dot{A}=\ddot{D}+\ddot{L}+\ddot{U}$ *have the forms:*

$$\ddot{D}=\begin{bmatrix}\ddot{A}_{1,1} & & & \\ & \ddot{A}_{3.3} & 0 & \\ & 0 & \ddots & \\ & & & \ddot{A}_{2p\text{-}1,2p\text{-}1}\end{bmatrix},\ \ddot{L}=\begin{bmatrix}0 & & & \\ \ddot{A}_{3,1} & 0 & & \\ \ddot{A}_{5,1} & \ddot{A}_{5,3} & & \\ \vdots & \ddots & \ddots & \\ \ddot{A}_{2p\text{-}1,1} & & \ddot{A}_{2p\text{-}1,2p\text{-}3} & 0\end{bmatrix},$$

$$\ddot{U}=\begin{bmatrix}0 & & \ddot{A}_{1,3\ldots} & \ddot{A}_{1,2p-1}\\ & \ddots & \cdots & \vdots\\ & & A_{2p-5,2p-3} & A_{2p-5,2p-1}\\ & & 0 & A_{2p-3,2p-1}\\ & & & 0\end{bmatrix},$$

where

$$\ddot{A}_{1,2j-1}=A_{1,2j-1}-A_{1,2}A_{2,2}^{-1}A_{2,2j-1},\quad \textit{for}\ \ 1\leq j\leq p,$$

$$\ddot{A}_{2p-1,2j-1}=A_{2p-1,2j-1}-A_{2p-1,2p-2}A_{2p-2,2p-2}^{-1}A_{2p-2,2j-1},\quad \textit{for}\ \ 1\leq j\leq p,$$

$$\ddot{A}_{2i-1,2j-1}=A_{2i-1,2j-1}-A_{2i-1,2i-2}A_{2i-2,2i-2}^{-1}A_{2i-2,2j-1}-A_{2i-1,2i}A_{2i,2i}^{-1}A_{2i,2j-1},$$

$$\textit{for}\ 1<i<p\ \textit{and}\ 1\leq j\leq p.$$

Proof. Suppose that $\lambda \in \lambda\{\tilde{H}^{-1}\tilde{B}\}$ and $\tilde{u}$ is the corresponding eigenvector, i.e. $\widetilde{H}^{-1}\widetilde{B}\tilde{u} = \lambda\tilde{u}$. So $\widetilde{B}\tilde{u} = \lambda\widetilde{H}\tilde{u}$. Let $\dot{u}= \left(u_1^T, ..., u_{2p-1}^T\right)^T$ be the vector defined by the subvectors of $\tilde{u}$. Then we have

$$\lambda\left(\ddot{D} + \omega\ddot{L}\right)\dot{u}= \left((1-\omega)\ddot{D} - \omega\ddot{U}\right)\dot{u}\ .$$

Hence, $\lambda \in \lambda\left(\ddot{D} + \omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D} - \omega\ddot{U}\right)$. Thus,

$$\lambda\left\{\widetilde{H}^{-1}\widetilde{B}\right\} \subseteq \lambda\left\{\left(\ddot{D} + \omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D} - \omega\ddot{U}\right)\right\}.$$

Let us consider $\lambda\left\{\left(\ddot{D} + \omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D} - \omega\ddot{U}\right)\right\}$ and $\dot{u}$ be the corresponding eigenvector, i.e. $\lambda\left(\ddot{D} + \omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D} - \omega\ddot{U}\right)\dot{u}$. Now we construct an eigenvector $\tilde{u}$ of $\widetilde{H}^{-1}\widetilde{B}$ from this eigenvector $\dot{u}$. Let the subvectors of $\tilde{u}$ be defined by the corresponding subvectors of $\dot{u}$. The other subvectors of $\tilde{u}$ are uniquely determined by the equation $\lambda\widetilde{H}\tilde{u} = \widetilde{B}\tilde{u}$ if $\lambda \neq 1-\omega$. It follows that $\lambda\left\{\widetilde{H}^{-1}\widetilde{B}\right\}$. Thus,

$$\lambda\left\{\left(\ddot{D} + \omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D} - \omega\ddot{U}\right)\right\}\backslash\{1-\omega\} \subseteq \lambda\{\widetilde{H}^{-1}\widetilde{B}\}.\ \square$$

We rewrite the block submatrices of A as follows:

$$A_{1,1} = \begin{bmatrix} \overline{A}_{0,0} & \overline{A}_{0,1} \\ \overline{A}_{1,0} & \overline{A}_{1,1} \end{bmatrix},\ A_{2p-1,2p-1} = \begin{bmatrix} \overline{A}_{2p-1,2p-1} & \overline{A}_{2p-1,2p} \\ \overline{A}_{2p,2p-1} & \overline{A}_{2p,2p} \end{bmatrix},$$

$$A_{2p-1,1} = \begin{bmatrix} \overline{A}_{2p-1,0} & \overline{A}_{2p-1,1} \\ \overline{A}_{2p,0} & \overline{A}_{2p,1} \end{bmatrix},\ A_{1,2p-1} = \begin{bmatrix} \overline{A}_{0,2p-1} & \overline{A}_{0,2p} \\ \overline{A}_{1,2p-1} & \overline{A}_{1,2p} \end{bmatrix},$$

$$A_{1,i} = \begin{bmatrix} \overline{A}_{0,i} \\ \overline{A}_{1,i} \end{bmatrix},\quad A_{i,1} = \left(\overline{A}_{i,0}, \overline{A}_{i,1}\right),\ A_{2p-1,i} = \begin{bmatrix} \overline{A}_{2p-1,i} \\ A_{2p,i} \end{bmatrix},$$

for $1 < i < 2p-1$,

$$A_{i,2p-1} = \left(\overline{A}_{i,2p-1}, \overline{A}_{i,2p}\right),\ u_1 = \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix},\ u_{2p-1} = \begin{bmatrix} \bar{u}_{2p-1} \\ \bar{u}_{2p} \end{bmatrix},\ 1 < i < 2p-1.$$

After these rewritings of the sub-matrices along the boundary of the matrix A, we can obtain the following theorem, which requires less zero sub-matrices in its assumption. The proof of this theorem is very similar to that of Theorem 2.1. and we omit it

Theorem 2.2 *If* $A_{i,2j} = 0$ *for* $|i-2j| \geq 2$, $\overline{A}_{i,0} = 0$, *for* $2 \leq i \leq 2p$, *and* $\overline{A}_{i,2p} = 0$, *for* $1 \leq i \leq 2p-2$, *then:*

i) The iterative matrix of the AMS method satisfies

$$\lambda\left\{\widetilde{H}^{-1}\widetilde{B}\right\} \subseteq \lambda\left\{\left(\ddot{D}+\omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D}-\omega\ddot{U}\right)\right\}$$

and

$$\lambda\left\{\left(\ddot{D}+\omega\ddot{L}\right)^{-1}\left((1-\omega)\ddot{D}-\omega\ddot{U}\right)\right\}\backslash\{1-\omega\} \subseteq \lambda\left\{\widetilde{H}^{-1}\widetilde{B}\right\}.$$

ii) The iterative matrix of the AAS method satisfies

$$\lambda\left\{\widetilde{D}^{-1}\left(\widetilde{D}-\omega\widetilde{A}\right)\right\} \subseteq \left\{\ddot{D}^{-1}\left(\ddot{D}-\omega\ddot{A}\right)\right\}$$

and

$$\lambda\left\{\ddot{D}^{-1}\left(\ddot{D}-\omega\ddot{A}\right)\right\}\backslash\{1-\omega\} \subseteq \lambda\left\{\widetilde{D}^{-1}\left(\widetilde{D}-\omega\widetilde{A}\right)\right\},$$

where

$$\ddot{A} = \ddot{D}+\ddot{L}+\ddot{U},$$

$$\ddot{D} = \begin{bmatrix} \ddot{A}_{1,1} & & & \\ & \ddot{A}_{3,3} & & \\ & & \ddots & \\ & & & \ddot{A}_{2P-1,2P-1} \end{bmatrix},$$

$$\ddot{L} = \begin{bmatrix} 0 & & & & \\ \ddot{A}_{3,1} & 0 & & & \\ \ddot{A}_{3,1} & \ddot{A}_{3,3} & 0 & & \\ \vdots & & \ddots & \ddots & \\ \ddot{A}_{2P-1,1} & & & \ddot{A}_{2P-1,2P-3} & 0 \end{bmatrix},$$

$$\ddot{U} = \begin{bmatrix} 0 & \ddot{A}_{1,3} & \cdots & \ddot{A}_{1,2P-1} \\ & \ddots & \cdots & \vdots \\ & & \ddot{A}_{2P-5,2P-3} & \ddot{A}_{2P-5,2P-3} \\ & & 0 & \ddot{A}_{2P-3,2P-1} \\ & & & 0 \end{bmatrix},$$

where, for $1 \leq i \leq p$, $\ddot{A}_{1,2i-1} = \overline{A}_{1,2i-1} - \overline{A}_{1,0}\overline{A}_{0,0}^{-1}\overline{A}_{0,2i-1} - \overline{A}_{1,2}\overline{A}_{2,2}^{-1}A_{2,2i-1}$

$\ddot{A}_{2p-1,2i-1} = \overline{A}_{2p-1,2i-1} - \overline{A}_{2p-1,2p}\overline{A}_{2p,2p}^{-1}\overline{A}_{2p,2i-1} - \overline{A}_{2p-1,2p-2}\overline{A}_{2p-2,2p-2}^{-1}A_{2p-2,2i-1}$

and, for $2 \leq i \leq p-1, m = 1$ *or* $2p-1$,

$$\ddot{A}_{2i-1,m} = \overline{A}_{2i-1,m} - A_{2i-1,2i-2}\overline{A}_{2i-2,2i-2}^{-1}\overline{A}_{2i-2,m} - A_{2i-1,2i}\overline{A}_{2i,2i}^{-1}A_{2i,m}.$$

From Theorems 2.1 and 2.2, we establish the relation between eingenvalues of the the matrices A and $\dot{A}$ or $\ddot{A}$. In order to obtain the convergence factors of the AMS method, we further discuss the relation between A and $\dot{A}$, as well as A and $\ddot{A}$ in following lemma.

Lemma 2.1. *Suppose that* $A_{i,2j} = 0$ *and* $A_{2j,i} = 0$ *for* $|i-2j| \geq 2$; *then:*

i) If A *is symmetric, then* $\dot{A}$ *is symmetric. If* A *is positive definite, then* $\dot{A}$ *is positive definite.*

ii) Assume $\overline{A}_{i,0}, \overline{A}_{0,i} = 0$, *for* $2 \leq i \leq 2p$ *and* $\overline{A}_{i,2p} = 0$, $\overline{A}_{2p,i} = 0$, *for* $1 \leq i \leq 2p-2$.

Then, if A *is symmetric, then* $\ddot{A}$ *is symmetric. If* A *is positive definite, then* $\ddot{A}$ *is positive definite.*

iii) If A *is an* M *- matrix, then* $\dot{A}$ *is an* M *matrix.*

iv) Assume $\overline{A}_{i,0} = 0, \overline{A}_{0,i} = 0$ *for* $2 \leq i \leq 2p$ *and* $\overline{A}_{i,2p} = 0$, $\overline{A}_{2p,i} = 0$, *for* $1 \leq i \leq 2p-2$.*If* A *is an* M *- matrix, then* $\ddot{A}$ *is an* M *- matrix.*

Proof. Since the proof of *(ii)* and *(iv)* is similar to that of *(i)* and *(iii)*, we only prove *(i)* and *(ii)* here.

i) Because $A_{i,2j} = 0$ for $|i-2j| \geq 2$, it is obvious that $\dot{A}$ is symmetric.

From $\dot{u}^T\dot{A}\dot{u} = \sum\limits_{i,j=1}^{T} \dot{u}^T_{2i-1\dot{A}2i-1,2j-1}\dot{u}_{2j-1}$, we let

$$u_{2i} = -A^{-1}_{2i,2i}\left(A_{2i,2i-1}\,\dot{u}_{2i-1} + A_{2i,2i+1}\dot{u}_{2i+1}\right)$$

and construct a vector u by putting these subvectors $\dot{u}_{2i-1}$ and u_{2i}. Then

$$\dot{u}^T\dot{A}\dot{u} = \sum_{i,j=12i-1}^{p} \dot{u}^T_{2i-1}\dot{A}_{2i-1,2j-1}\dot{u}_{2j-1} = \sum_{i,j=1}^{2p-1} u_i^T A_{i,j}u_j.$$

Hence, if A is positive definite then $\dot{\text{A}}$ is positive definite.

iii) For any $\dot{b} = \left(\dot{b}_1^T, \dot{b}_3^T, ..., \dot{b}_{2p-1}^T\right)^T \geq 0$, there exists a vector $\dot{u}$ such that $\dot{A}\dot{u} = \dot{b}$. Let u be a vector whose subvectors are defined from the vector $\dot{u}$ and the solutions u_{2i} of $A_{2i,2i}u_{2i} = -\left(A_{2i,2i}\dot{u}_{2i-1} + A_{2i,2i+1}\dot{u}_{2i+1}\right)$.

Then this vector u satisfies $Au = b$, where $b = \left(\dot{b}_1^T, 0, \dot{b}_3^T, 0, ..., 0, \dot{b}_{2p-1}^T\right)^T$.

Since $A^{-1} \geq 0$ and $b \geq 0$, it follows that $u = A^{-1}b \geq 0$. Thus $\dot{A}^{-1} \geq 0$. Note that $\dot{A}$ is positive definite. So the diagonal elements of $\dot{A}$ must be positive. Now we prove that the off-diagonal elements of $\dot{A}$ are negative. Let $\dot{u}$ be the vector such that only one component of $\dot{u}$ is one and the other components are zero. Let

$$\dot{A}\dot{u} = \dot{b} \text{ where } \dot{b} = \left(\dot{b}_1^T, 0, \dot{b}_3^T, 0, ..., 0, \dot{b}_{2p-1}^T\right) \text{ and}$$

$$A_{2i,2i}u_{2i} = -\left(A_{2i,2i-1}\dot{u}_{2i-1} + A_{2i,2i+1}\dot{u}_{2i+1}\right),\ 1 \le i \le p-1.$$

Because $A_{2i,2i-1}\dot{u}_{2i-1} + A_{2i,2i+1}\dot{u}_{2i+1} \le 0$ and $A_{2i,2i}^{-1} \ge 0$, we have $u_{2i} \ge 0, 1 \le i \le p-1$.

Then a vector u is defined and $Au = b$, with $b = \left(\dot{b}_1^T, 0, \dot{b}_3^T, 0, ..., \dot{b}_{2p-1}^T\right)$.

From this equation, we can obtain that the component of $\dot{b}$ corresponding to the nonzero components of $\dot{u}$ must be strictly positive and other components of $\dot{b}$ are negative. Then $\dot{A}$ is an M - matrix.

Remarks. We can use [3] one the spectrum δ of the block Jacobi iterative matrices of $\dot{A}$ and $\ddot{A}$ to get the optimal ω:

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \delta^2}}.$$

This choice makes the convergence factor $\lambda = \omega - 1$ minimum. Since $\delta < 1$, we prefer to choose $1 < \omega < 2$ in AMS and AAS methods.

3. Iterative Subsolver for All Sub-problems

We write the matrix A as the sum of a diagonal matrix D, a lower triangular matrix L and a upper triangular matrix U, $A = D + L + U = D + C$, where

$$D = \begin{bmatrix} D_{1,1} & & \\ & \ddots & \\ & & D_{2p-1,2p-1} \end{bmatrix},$$

$$L = \begin{bmatrix} L_{1,1} & & & \\ A_{2,1} & & L_{2,2} & \\ \vdots & & \ddots\ \ddots & \\ A_{2p-1,1} & \cdots & A_{2p-1,2p-2} & L_{2p-1,2p-1} \end{bmatrix}$$

$$U = \begin{bmatrix} U_{1,1} & A_{1,2} & \cdots & A_{1,2p-1} \\ & U_{2,2} & & \vdots \\ & & \ddots\ \ddots & A_{2p-2,2p-1} \\ & & & U_{2p-1,2p-1} \end{bmatrix}.$$

Then, the matrix $\tilde{A}$ has a corresponding decomposition denoted as

$$\tilde{A} = \widehat{D} + \widehat{L} + \widehat{U} = \widehat{D} + \widehat{C},$$

where

$$\widehat{D} = \begin{bmatrix} \widehat{D}_{1,1} & & \\ & \ddots & \\ & & \widehat{D}_{p,p} \end{bmatrix},\ L = \begin{bmatrix} \widehat{L}_{1,1} & & & \\ \widehat{A}_{2,1} & & \widehat{L}_{2,2} & \\ \vdots & & \ddots & \ddots \\ \widehat{A}_{p,1} & \cdots & \widehat{A}_{p,p-1} & \widehat{L}_{p,p} \end{bmatrix},$$

$$\widehat{U} = \begin{bmatrix} \widehat{U}_{1,1} & \widehat{A}_{1,2} & \cdots & & \widehat{A}_{1,p} \\ & \widehat{U}_{2,2} & & & \vdots \\ & & \ddots & \ddots & \widehat{A}_{p-1,p} \\ & & & & \widehat{U}_{p,p} \end{bmatrix}.$$

Here we let:

$$\widehat{D}_{1,1} = \begin{bmatrix} D_{1,1} & \\ & D_{2,2} \end{bmatrix},\ \widehat{D}_{p,p} = \begin{bmatrix} D_{2p-2,2p-2} & \\ & D_{2p-1,2p-1} \end{bmatrix},$$

$$\widehat{L}_{1,1} = \begin{bmatrix} L_{1,1} & \\ A_{2,1} & L_{2,2} \end{bmatrix},\ \widehat{L}_{p,p} = \begin{bmatrix} L_{2p-2,2p-2} & \\ A_{2p-1,2p-2} & L_{2p-1,2p-1} \end{bmatrix},$$

$$\widehat{U}_1 = \begin{bmatrix} U_{1,1} & A_{1,2} \\ & U_{2,2} \end{bmatrix},\ \widehat{U}_p = \begin{bmatrix} U_{2p-2,2p-2} & A_{2p-2,2p-1} \\ & U_{2p-1,2p-1} \end{bmatrix}.$$

$$\widehat{D}_i = \begin{bmatrix} D_{2i-2,2i-2} & & \\ & D_{2i-1,2i-1} & \\ & & D_{2i,2i} \end{bmatrix},$$

$$\widehat{L}_i = \begin{bmatrix} L_{2i-2,2i-2} & & \\ A_{2i-1,2i-2} & L_{2i-1,2i-1} & \\ A_{2i,2i-2} & A_{2i,2i-1} & L_{2i,2i} \end{bmatrix}$$

$$\widehat{U}_i = \begin{bmatrix} U_{2i-2,2i-2} & A_{2i-2,2i-1} & A_{2i-2,2i} \\ & U_{2i-1,2i-1} & A_{2i-1,2i} \\ & & U_{2i,2i} \end{bmatrix}.$$

Suppose that p subproblems are solved using point Jacobi iterative method, with $\omega_i = \omega, i = 1, ..., p$. Then, the AAS algorithm can be written in the form:

$$\widehat{D}\widetilde{u}^{(k+1)} = \left(\widehat{D} - \omega\widehat{A}\right)\widetilde{u}^{(k)} + \omega f,$$

and the AMS method can be represented by

$$\left(\widehat{D} + \omega\widehat{L}\right) y^{(k+1)} = \left(\widehat{D} - \omega\left(\widehat{D} + \widehat{U}\right)\right) y^{(k)} + \omega f.$$

Theorem 3.1. *Suppose that $D_{i,i}$ are invertible, for $i = 1, ..., 2p-1$. Then we have*

$$\lambda\left\{\widehat{D}^{-1}\left(\widehat{D} - \omega\widehat{A}\right)\right\} = \lambda\left\{D^{-1}\left(D - \omega A\right)\right\} \cup \left(\bigcup_{i=1}^{p} \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}\right).$$

Proof. Suppose that $\lambda\left\{\widehat{D}^{-1}\left(\widehat{D} - \omega\widehat{A}\right)\right\}$ and $\hat{u}$ is the corresponding eigenvector, i.e. $\lambda\widehat{D}\hat{u} = \left(\widehat{D} - \omega\widehat{A}\right)\hat{u}$.

If $u_{2i} \neq \hat{u}_{2i}$ for $1 \leq i \leq p-1$, then we have$\lambda \in \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}$.

If $u_{2i} = \hat{u}_{2i}$ for $i = 1, ..., p-1$, then we have $\lambda \in \lambda\left\{D^{-1}\left(D - \omega A\right)\right\}$ and we already show that

$$\lambda\left\{\widehat{D}^{-1}\left(\widehat{D} - \omega\widehat{A}\right)\right\} \subseteq \lambda\left\{D^{-1}\left(D - \omega A\right)\right\} \cup \left(\bigcup_{i=1}^{p} \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}\right).$$

Now we prove that

$$\lambda\left\{D^{-1}\left(D - \omega A\right)\right\} \cup \left(\bigcup_{i=1}^{p} \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}\right) \subseteq \lambda\left\{\widehat{D}^{-1}\left(\widehat{D} - \omega\widehat{A}\right)\right\}.$$

Assume that

$$\lambda \in \lambda\left\{D^{-1}\left(D - \omega A\right)\right\} \cup \left(\bigcup_{i=1}^{p} \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}\right).$$

If $\lambda \in \lambda\left\{D^{-1}\left(D - \omega A\right)\right\}$ and u is the associated eigenvector, i.e., $-\lambda Du + (D - \omega A)u = 0$, then we construct an eigenvector $\hat{u}$ of $\widehat{D}^{-1}\left(\widehat{D} - \omega\widehat{A}\right)$, from the eigenvector $\hat{u}$, *by* letting $\hat{u}_{2i} = u_{2i}$, for $i = 1, ..., p-1$. If $\lambda = \left(\bigcup_{i=1}^{p} \lambda\left\{D_{2i}^{-1}\left(D_{2i} - \omega A_{2i,2i}\right)\right\}\right)$ and $\lambda \notin \lambda\left\{D^{-1}\left(D - \omega A\right)\right\}$, then an eigenvector $\hat{u}$ of the matrix $\widehat{D}^{-1}\left(\widehat{D}-\omega\widehat{A}\right)$is constructed by the following procedure. Assume that v_{2i} is the solution of the equation:

$$-\lambda D_{2i}v_{2i}+\left(D_{2i}-\omega A_{2i,2i}\right)v_{2i}=0,$$

i.e. v_{2i} is the eigenvector of $D_{2i}^{-1}\left(D_{2i}-\omega A_{2i,2i}\right)$. By solving the equation $-\lambda Du+(D-\omega A)u=w$, where $w=\left(0^T,...,0^T,\omega v_{2i}^T A_{2i+1,2i},...,\omega v_{2i}^T A_{2p-1,2i}^T\right)$ we obtain a vector u. Define $\hat{u}_{2i}=v_{2i}+u_{2i}$. The other subvectors of $\hat{u}$ are defined by $\hat{u}_{2j}=u_{2j}, j=1,...,i-1,i+1,...,p-1$. This $\hat{u}$ satisfies

$$-\lambda\widehat{D}\hat{u}+\left(\widehat{D}-\omega\widehat{A}\right)\hat{u}=0,\ \text{and then}\ \lambda\in\lambda\left\{\widehat{D}^{-1}\left(\widehat{D}-\omega\widehat{A}\right)\right\}.$$

Theorem 3.2. *Suppose that A is an M - matrix, the matrix*

$$\left(D+\omega\begin{bmatrix} 0 & & & \\ A_{2,1} & & & \\ \vdots & & & \\ A_{2p-1,1} & \cdots & A_{2p-1,2p-2} & 0 \end{bmatrix}\right)$$

exists and is nonnegative, and the matrix

$$D-\omega\begin{bmatrix} A_{1,1} & \cdots & A_{1,2p-1} \\ & \ddots & \vdots \\ & & A_{2p-1,2p-1} \end{bmatrix}$$

is nonnegative. Then $\widehat{A}^{-1}$ and $\left(\widehat{D}+\omega\widehat{L}\right)^{-1}$ exist and are nonnegative, and $\widehat{D}-\omega\left(\widehat{D}+\widehat{U}\right)$is nonnegative. So the spectrum of the AMS iterative matrix AMS $\left(\widehat{D}+\omega\widehat{L}\right)^{-1}\left(\widehat{D}-\omega\left(\widehat{D}-\widehat{U}\right)\right)$ is less than 1.

The proof of this theorem is similar to that of Lemma 2.1.

Let all subproblems be solved by the SOR method with $\omega_i=1$ and $i=1,...,p$.

Then, the AAS method can be expressed by

$$\left(\widehat{D}+\omega\begin{bmatrix} \widehat{L}_{1,1} & & \\ & \ddots & \\ & & \widehat{L}_{p,p} \end{bmatrix}\right)u^{(k+1)}=\left((1-\omega)\widehat{D}-\omega\left(\widehat{L}+\widehat{U}\right)\right)u^{(k)}+\omega f.$$

Hence, the corresponding iterative matrix is

$$\widehat{J}=\left(\widehat{D}+\omega\begin{bmatrix}\widehat{L}_{1,1} & & \\ & \ddots & \\ & & \widehat{L}_{p,p}\end{bmatrix}\right)^{-1}\left((1-\omega)\widehat{D}-\omega\left(\widehat{L}+\widehat{U}\right)\right).$$

The AMS method can also be written in the simple form:

$$\left(\widehat{D}+\omega\widehat{L}\right)y^{(k+1)}=\left((1-\omega)\widehat{D}-\omega\widehat{U}\right)y^{(k)}+\omega f,$$

with the iterative matrix:

$$\widehat{S}=\left(\widehat{D}+\omega\widehat{L}\right)^{-1}\left((1-\omega)\widehat{D}-\omega\widehat{U}\right).$$

Theorem 3.3. *Assume that A is an M - matrix,*

$$\left(D+\omega\begin{bmatrix}L_{1,1} & & \\ & \ddots & \\ & & L_{2p-1,2p-1}\end{bmatrix}\right)^{-1}$$ *exists and is nonnegative, and*

$$(1-\omega)D-\omega\begin{bmatrix}0 & & \\ A_{2,1} & & \\ \vdots & & \\ A_{2p-1,1} & \cdots & A_{2p-1,2p-2}\end{bmatrix}-\omega U$$ *is nonnegative.*

Then A^{-1} *and* $\left(\widehat{D}+\omega\begin{bmatrix}\widehat{L}_{1,1} & & \\ & \ddots & \\ & & \widehat{L}_{p,p}\end{bmatrix}\right)$ *exist and are nonnegative. So, the spectrum of the AAS iterative matrix $\widehat{J}$ is strictly less than 1.*

The proof is similar to that of Lemma 2.1.

Denote

$$M=(D+\omega L)^{-1}((1-\omega)D-\omega U)$$

and

$$M_{2i,2i}=\left(D_{2i,2i}+\omega L_{2i,2i}\right)^{-1}((1-\omega)D_{2i,2i}-\omega U_{2i,2i}.$$

Theorem 3.4. *Assume that all $D_{i,i}$ are nonsingular. Then, we have*

$$\lambda\{\widehat{S}\} = \lambda\{M\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{M_{2i,2i}\}\right).$$

Proof. Assume that $\lambda = \lambda\{\widehat{S}\}$ and $\hat{u}$ is the corresponding eigenvector, i.e.

$$\lambda\widehat{u} = \widehat{S}\hat{u},\ \lambda\left(\widehat{D} + \omega\widehat{L}\right)\hat{u} = \left((1-\omega)\widehat{D} - \omega\widehat{U}\right)\hat{u}.$$

If $u_{2i} \neq \hat{u}_{2i}$ for some $1 \leq i \leq p-1$ then $\lambda \in \lambda\{M_{2i,2i}\}$. If $u_{2i} = \hat{u}_{2i}$ for all $1 \leq i \leq p-1$ then $\lambda \in \lambda\{M\}$. Hence, $\lambda\{\widehat{S}\} \subseteq \lambda\{M\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{M_{2i,2i}\}\right)$.

Now we show that $\lambda\{M\} \cup \left(\bigcup_{i=1}^{p-1} \lambda\{M_{2i,2i}\}\right) \subseteq \lambda\{\widehat{S}\}$.

Let $\lambda \in \lambda\{M\}$ and u be the associated eigenvector, i.e.

$$\lambda u = Mu \text{ and } \lambda(D + \omega L)u = ((1-\omega)D - \omega U)u.$$

Define $\hat{u}$ by letting $\hat{u}_{2i} = u_{2i}$. Then, this $\hat{u}$ is the eigenvector of $\widehat{S}$ and $\lambda\widehat{u} = \widehat{S}\hat{u}$. Thus, $\lambda\{M\} \subseteq \lambda\{\widehat{S}\}$.

Assume $\lambda \in \lambda\{M_{2i,2i}\},\ \lambda \notin \lambda\{M\}$. Denote v_{2i} to be the corresponding eigenvector, i.e. $\lambda v_{2i} = M_{2i,2i}$. We solve the following equation and get a solution: $\left(\lambda(D + \omega L) + \omega U + (\omega - 1)D\right)u = w$,where
$w = -\lambda\omega\left(0^T, ..., 0^T, v_{2i}^T A_{2i+2i}, ..., v_{2i}A_{2p-1,2i}\right)^T$. Since $\lambda \notin \lambda\{M\}$, this problem has only one solution. We define a new vector by letting

$$\hat{u}_{2j} = \begin{cases} v_{2j} + u_{2i} & , j = i, \\ u_{2j} & , j \neq i. \end{cases}.$$

This $\hat{u}$ satisfies $\lambda\widehat{u} = \widehat{S}\hat{u}$. Hence, $\hat{u}$ is the eigenvector of $\widehat{S}$.

From above theorems and lemmas, we conclude that the convergence factors of the AAS method and AMS method are almost the same as the block Jacobi method and the SOR method.

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