An. Şt. Univ. Ovidius Constanţa Vol. 10(2), 2002, 79–86

ON SOME DIOPHANTINE EQUATIONS (II)

Diana Savin

Abstract

In [7] we have studied the equation $m^4 - n^4 = py^2$, where p is a prime natural number $p \geq 3$. Using the above result, in this paper, we study the equations $c_k(x^4 + 6px^2\ y^2\ + p^2y^4) + 4pd_k(x^3y + pxy^3) = 32z^2$ with $p \in \{5, 13, 29, 37\}$, where (c_k, d_k) is a solution of the Pell equation $\left|c^2 - pd^2\right| = 1$.

1. Preliminaries.

In order to solve our problems, we need some auxiliary results.

Proposition 1.1. ([3], pag.74) *The integer solutions of the Diophantine equation* $x_1^2 + x_2^2 + ... + x_k^2 = x_{k+1}^2$ *are the following ones:*

$$\begin{cases} x_1 = \pm(m_1^2 + m_2^2 + ... + m_{k-1}^2 - m_k^2) \\ x_2 = 2m_1 m_k \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x_k = 2m_{k-1} m_k \\ x_{k+1} = \pm(m_1^2 + m_2^2 + ... + m_{k-1}^2 + m_k^2), \end{cases},$$

with $m_1, ..., m_k$ *integer number. From the geometrical point of view, the elements* $x_1, x_2, ..., x_k$ *are the sizes of an orthogonal hyper-parallelipiped in the space* $\mathbf{R}^k$ *and* x_{k+1} *is the length of its diagonal.*

Proposition 1.2. ([1], pag.150) *For the quadratic field* $Q(\sqrt{d})$*, where* $d \in \mathbf{N}^*$*,* d *is square free, its ring of integers* A *is Euclidian with respect to the norm* N *, in the cases* $d \in \{2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}$.

Key Words: Diophantine equation; Pell equation.

Proposition 1.3.([1], pag141) *Let* K=$\boldsymbol{Q}(\sqrt{d})$ *be a quadratic field with* A *as its ring of integers . For* $a\in$A, $a\in$U(A) *if and only if* $N(a)=1$.

Proposition 1.4. ([7] ,Theorem 3.2.). *Let* p *be a natural prime number greater than* 3.*If the equation* $m^4 - n^4 = py^2$ *has a solution* $m, n, y \in \mathbf{Z}^*$, *then it has an infinity of integer solutions.*

2 Results.

Proposition 2.1. *The equation* $m^4 - n^4 = 5y^2$ *has an infinity of integer solutions.*

Proof. The equation $m^4 - n^4 = 5y^2$ has nontrivial integer solutions, for example $m = 245$, $n = 155$, $y = 24600$. Following Proposition 1.4., the equation $m^4 - n^4 = 5y^2$ has an infinity of integer solutions.

Proposition 2.2. *The equation* $m^4 - n^4 = 13y^2$ *has an infinity of integer solutions.*

Proof. It is sufficient to show that the equation $m^4 - n^4 = 13y^2$ has nontrivial integer solutions. In deed $m = 127729$, $n = 80929$, $y = 4144257960$ is such a solution.By Proposition 1.4., the equation $m^4 - n^4 = 13y^2$ has an infinity of integer solutions.

Now, we study our equations for $p \in \{5, 13\}$.

Proposition 2.3. *The equations*

$$c_k(x^4 + 6px^2\ y^2\ + p^2y^4) + 4pd_k(x^3y + pxy^3) = 32z^2,$$

with $p \in \{5, 13\}$, *where* (c_k, d_k) *is a solution of the Pell equation* $\left|c^2 - pd^2\right| =$ =1, *have an infinity of integer solutions.*

Proof. If $p \in \{5, 13\}$, then $p \equiv 5 \pmod 8$. By Proposition 1.2., the ring A of the integers of the quadratic field $\mathbf{Q}(\sqrt{p})$ is Euclidian with respect to the norm N. But $p \equiv 5 \pmod 8$ implies $p \equiv 1 \pmod 4$and A=$\mathbf{Z}\left[\frac{1+\sqrt{p}}{2}\right]$.

We shall study the equation $m^4 - n^4 = py^2$, where p is a prime number, $p \equiv 5 \pmod 8$ and $(\ m, n\ ) = 1$, in the ring A. The equation $m^4 - n^4 = py^2$ is equivalent with $(m^2 - y\sqrt{p})(m^2 + y\sqrt{p}) = n^4$. Let $\alpha \in A$ be a common divisor of $m^2 - \sqrt{p}y$ and $m^2 + \sqrt{p}y$. As $\alpha \in A$, $\alpha = \frac{c}{2} + \frac{d}{2}\sqrt{p}$, $c, d \in \mathbf{Z}$, and c, d are simultaneously even or odd. As $\alpha/(m^2+y\sqrt{p})$ and $\alpha/(m^2-y\sqrt{p})$, we have $\alpha/2m^2$ and $\alpha/2y\sqrt{p}$, therefore $N(\alpha)/4m^4$ (in $\mathbf{Z}$) and $N(\alpha)/4py^2$(in $\mathbf{Z}$), hence $N(\alpha)/(4m^4,4py^2)$. $(m,n)=1$ implies $(m,y)=1$ (if $(m,y)=d>1$ then m and n would not be relatively prime). Analogously, $(m,p)=1$ implies in turn that $(4m^4,4py^2)=4$, hence $N(\alpha)\in\in\{1,2,4\}$.

If $N(\alpha)=2$, then $\left|\frac{c^2}{4}-p\frac{d^2}{4}\right|=2$. If $\frac{c^2}{4}-p\frac{d^2}{4}=2$, then $c^2-pd^2=8, c,d\in\mathbf{Z}$ and c, d are simultaneously even or odd. If c and d are odd numbers, then c^2, $d^2\equiv 1\ (\bmod 8)$. But $p\equiv 5\ (\bmod 8)$. Then $c^2-pd^2\equiv 4\ (\bmod 8)$, which implies that the equation $c^2-pd^2=8$ does not have integer solutions.

If c and d are even numbers, then let us take them $c=2c'$, $d=2d'$,with $c',d'\in\mathbf{Z}$. We get $c^2-pd^2=8$, then $(c')^2-p(d')^2=2$. But $p\equiv 5\ (\bmod 8)$ implies:

$(c')^2-p(d')^2\equiv 4\ (\bmod 8)$, if c',d' are odd numbers,

$(c')^2-p(d')^2\equiv 0$ or $4\ (\bmod 8)$, if c',d' are even numbers,

$(c')^2-p(d')^2$ = an odd number, if c',d' are one even and the other odd.

Therefore the equation $(c')^2-p(d')^2=2$ does not have integer solutions.

If $\frac{c^2}{4}-p\frac{d^2}{4}=-2$, that means $c^2-pd^2=-8$, with $c,d\in 2\mathbf{Z}+\mathbf{1}$ or $c,d\in 2\mathbf{Z}$.

If c and d are odd numbers, then $c^2,d^2\equiv 1\ (\bmod 8)$.

As $p\equiv 5\ (\bmod 8)$, this implies $c^2-pd^2\equiv 4\ (\bmod 8)$, which gives us that the equation $c^2-pd^2=-8$ does not have integer solutions. If c and d are even numbers, then $c=2c'$, $d=2d'$, $c',d'\in\mathbf{Z}$. We get $c^2-pd^2=-8$,which means that $(c')^2-p(d')^2=-2$. But, as above, $p\equiv 5\ (\bmod 8)$ implies that the equation $(c')^2-p(d')^2=-2$ does not have integer solutions.We get $N(\alpha)\neq 2$.

If $N(\alpha)=4$, then $\left|\frac{c^2}{4}-p\frac{d^2}{4}\right|=4$. If $\frac{c^2}{4}-p\frac{d^2}{4}=4$, then $c^2-pd^2=16$, where $c,d\in\mathbf{Z}$ and c,d are simultaneously either even or odd.

If c and d are odd numbers, then c^2, $d^2\equiv 1\ (\bmod 8)$, and, since $p\equiv 5\ (\bmod 8)$, $c^2-pd^2\equiv 4(\bmod 8)$,which implies that the equation $c^2-pd^2=16$ does not have integer solutions.

If c and d are even numbers, then $c=2c'$, $d=2d'$, with c', $d'\in\mathbf{Z}$, therefore $(c')^2-p(d')^2=4$. This equation may have integer solutions only if c',d' are simultaneously either even or odd. The equation $(c')^2-p(d')^2=4$ is equivalent with$\left(\frac{c'}{2}\right)^2-\left(\frac{d'}{2}\right)^2=1$. If we denote $\alpha'=\left(\frac{c'}{2}+\frac{d'}{2}\sqrt{p}\right)\in$ A, with $c',d'\in 2\mathbf{Z}+\mathbf{1}$ or $c',d'\in 2\mathbf{Z}$, we get $\alpha'\in\mathbf{U}(A)$.From $\alpha=\frac{c}{2}+\frac{d}{2}\sqrt{p}$, we obtain that $\alpha=2\alpha'$, $\alpha'\in\mathbf{U}(A)$. Supposing that 2 is reducible in A,hence there exist $\frac{a_1}{2}+\frac{b_1}{2}\sqrt{p}$, $\frac{a_2}{2}+\frac{b_2}{2}\sqrt{p}\in\mathbf{A}$ (a_1,a_2,b_1, $b_2\in\mathbf{Z}$, a_1,b_1, as well as, a_2,b_2 being simultaneounsly odd or even) such that $2=(\frac{a_1}{2}+.\frac{b_1}{2}\sqrt{p})(\frac{a_2}{2}+\frac{b_2}{2}\sqrt{p})$,hence $N(2)=N(\frac{a_1}{2}+\frac{b_1}{2}\sqrt{p})N(\frac{a_2}{2}+\frac{b_2}{2}\sqrt{p})$. This is equivalent with

$4 = N(\frac{a_1}{2} + \frac{b_1}{2}\sqrt{p})N(\frac{a_2}{2} + \frac{b_2}{2}\sqrt{p})$. But we have previously proved that there aren't elements in A having the norm equal with 2. We get $N(\frac{a_1}{2} + \frac{b_1}{2}\sqrt{p}) = 1$ or $N(\frac{a_2}{2} + \frac{b_2}{2}\sqrt{p}) = 1$, therefore $\frac{a_1}{2} + \frac{b_1}{2}\sqrt{p} \in \mathbf{U}(A)$or $\frac{a_2}{2} + \frac{b_2}{2}\sqrt{p} \in \mathbf{U}(A)$, hence 2 is irreducible in A. We come back to the fact that $\alpha/(m^2 + y\sqrt{p}$ and $\alpha/(m^2 - y\sqrt{p}$. This implis $2\alpha'/(m^2 + y\sqrt{p})$ and $2\alpha'/(m^2 - y\sqrt{p})$, then $2/(m^2 + y\sqrt{p})$ and $2/(m^2 - y\sqrt{p})$,therefore $4/(m^4 - py^2)$. This means $4/n^4$. As 2 is irreducible in A, we get $2/n$ (in A), hence $2^4/n^4$. This is equivalent with $2^4/(m^2 + y\sqrt{p}) \cdot (m^2 - y\sqrt{p})$ (in A), which implies $2^k/(m^2 + y\sqrt{p})$or $2^k/(m^2 - y\sqrt{p}), k \in \mathbf{N}, k \geq 2$. As $2^k/(m^2 + y\sqrt{p}), k \in \mathbf{N}, k \geq 2$, implies $2^2/(m^2 + y\sqrt{p})$, hence there exists $\left(\frac{a}{2} + \frac{b}{2}\sqrt{p}\right) \in A$ (either $a, b \in 2\mathbf{Z} + \mathbf{1}$ or $a, b \in 2\mathbf{Z}$) such that $m^2 + y\sqrt{p} = 2^2\left(\frac{a}{2} + \frac{b}{2}\sqrt{p}\right)$, hence $m^2 = 2a$ and $y = 2b$ (in $\mathbf{Z}$), then $2/m$ and $2/y$ (in $\mathbf{Z}$). As $m^4 - n^4 = py^2$, this implies $2/n$ (in $\mathbf{Z}$), in contradiction with the fact that $(m, n) = 1$. Analogously we get to contradiction in the case of the equation $\frac{c^2}{4} - p\frac{d^2}{4} = -4$. Therefore $N(\alpha) \neq 4$.

From the previously proved, $N(\alpha) \neq 2$ and $N(\alpha) \neq 4$, hence $N(\alpha) = 1$ and $\alpha \in \mathbf{U}(A)$. We obtained that $(m^2 + y\sqrt{p})$ and $(m^2 - y\sqrt{p})$ are relatively prime elements in A, but $(m^2 - y\sqrt{p})(m^2 + y\sqrt{p}) = n^4$, therefore there exists $\left(\frac{f}{2} + \frac{g}{2}\sqrt{p}\right) \in A$ with the property: $m^2 + y\sqrt{p} = (\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p})\left(\frac{f}{2} + \frac{g}{2}\sqrt{p}\right)^4$, $(\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p}) \in \mathbf{U}(A)$ (here $c_k, d_k \in \mathbf{Z}$, c_k, d_k are simultaneously odd or even, $N(\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p}) = 1$).This is equivalent to $m^2 + \sqrt{p}y =$
$= \left(\frac{c_k}{2} + \frac{d_k}{2}\right)\left(\frac{f^4}{16} + \frac{f^3 g\sqrt{p}}{4} + \frac{3f^2g^2p}{8} + \frac{fg^3p\sqrt{p}}{4} + \frac{g^4p^2}{16}\right)$, which is equivalent to

$$32(m^2 + y\sqrt{p}) = (c_k + d_k\sqrt{p})(f^4 + 4f^3g\sqrt{p} + 6f^2g^2p + 4fg^3p\sqrt{p} + g^4p^2),$$

implying the system:

$$\begin{cases} 32m^2 = c_k f^4 + 6pc_k f^2 g^2 + p^2 c_k g^4 + 4pf^3 g d_k + 4p^2 f g^3 d_k \\ 32y = 4c_k f^3 g + 4pc_k f g^3 + d_k f^4 + 6pd_k f^2 g^2 + p^2 d_k g^4, \end{cases}$$

equivalently

$$\begin{cases} 32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3) \\ 32y = d_k(f^4 + 6pf^2g^2 + p^2g^4) + 4c_k(f^3g + pfg^3). \end{cases}$$

We have already proved that the equation $m^4 - n^4 = py^2$, where $p \in \{5, 13\}$ has an infinity of integer solutions. Therefore, if the system:

$$\begin{cases} 32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3) \\ 32y = d_k(f^4 + 6pf^2g^2 + p^2g^4) + 4c_k(f^3g + pfg^3) \end{cases}$$

has an infinity of integer solutions and hence the equation

$$32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3)$$
has an infinity of integer solutions.

We do not know if the Diophantine equation $m^4 - n^4 = py^2$ has nontrivial solutions for $p \in \{29, 37\}$, but we may prove the following result.

Proposition 2.4. *If the equations* $c_k(x^4 + 6px^2\ y^2\ + p^2y^4) + 4pd_k(x^3y + pxy^3) = 32z^2$,*with* $p \in \{29, 37\}$, *where* (c_k, d_k) *is a solution of the Pell equation* $\left|c^2 - pd^2\right| = 1$,*have a solution* $x, y, z \in Z^*$, *then they have an infinity of integer solutions.*

Proof. In our case again $p \equiv 5 \pmod 8$ and the ring A of the integers of the quadratic field $\mathbf{Q}(\sqrt{p})$ is Euclidian with respect to the norm N, A being $\mathbf{Z}\left[\frac{1+\sqrt{p}}{2}\right]$. We study the equation $m^4 - n^4 = py^2$, where p is prime number, $p \equiv 5 \pmod 8$ in the ring A.The given equation is equivalent to $(m^2 - y\sqrt{p})(m^2 + y\sqrt{p}) = n^4$. Let $\alpha \in A$ be a common divisor of $m^2 - y\sqrt{p}$ and $m^2 + y\sqrt{p}$. Then $\alpha = \frac{c}{2} + \frac{d}{2}\sqrt{p}$, with $c, d \in 2\mathbf{Z} + \mathbf{1}$ or $c, d \in 2\mathbf{Z}$.

As $\alpha/(m^2 + y\sqrt{p})$ and $\alpha/(m^2 - y\sqrt{p})$, we have $\alpha/2m^2$ and $\alpha/2y\sqrt{p}$, so $N(\alpha)/4m^4$ and $N(\alpha)/4py^2$ (in $\mathbf{Z}$), hence $N(\alpha)/(4m^4, 4py^2)$.$(m, n) = 1$ implies $(m, y) = 1$ (if $(m, y) = d > 1$, then m and n are not relatively prime).

Analogously, $(m, p) = 1$ implies in turn that $(m^4, py^2) = 1$, $(4m^4, 4py^2) = 4$, hence $N(\alpha) \in \{1, 2, 4\}$. If $N(\alpha) = 2$, we have $\left|\frac{c^2}{4} - p\frac{d^2}{4}\right| = 2$. If $\frac{c^2}{4} - p\frac{d^2}{4} = 2$, that means $c^2 - pd^2 = 8, c, d \in \mathbf{Z}$ and c, d are simultaneously even or odd.

If c and d are odd numbers, then $c^2, d^2 \equiv 1 \pmod 8$. As $p \equiv 5 \pmod 8$,then $c^2 - pd^2 \equiv 4 \pmod 8$, which implies that the equation $c^2 - pd^2 = 8$ does not have integer solutions. If c and d are even numbers then $c = 2c'$, $d = 2d'$, $c', d' \in \mathbf{Z}$. We get $c^2 - pd^2 = 8$, therefore $(c')^2 - p(d')^2 = 2$. But $p \equiv 5 \pmod 8$ implies:$(c')^2 - p(d')^2 \equiv 4 \pmod 8$, if c', d' are odd numbers,$(c')^2 - p(d')^2 \equiv 0$ or $4 \pmod 8$, if c', d' are even numbers, $(c')^2 - p(d')^2 =$ an odd number, if c', d' are one even and another odd, and the equation $(c')^2 - p(d')^2 = 2$ does not have integer solutions. If $\frac{c^2}{4} - p\frac{d^2}{4} = -2$, then $c^2 - pd^2 = -8, c, d \in 2\mathbf{Z}$ or $c, d \in 2\mathbf{Z} + \mathbf{1}$. If c and d are odd numbers, then $c^2, d^2 \equiv 1 \pmod 8$. But $p \equiv 5 \pmod 8$ implies$c^2 - pd^2 \equiv 4 \pmod 8$, which gives us that the equation $c^2 - pd^2 = -8$ does not haveinteger solutions. If c and d are even numbers, then $c = 2c'$, $d = 2d'$, $c', d' \in \mathbf{Z}$. We get the equation $(c')^2 - p(d')^2 = -2$, which does not have integer solutions. We get $N(\alpha) \neq 2$. In the same way as above, we prove that $N(\alpha) \neq 4$. It remains only $\alpha \in \mathbf{U}(A)$ and $m^2 + y\sqrt{p}$, $m^2 - y\sqrt{p}$ are relatively prime elements in A.

As $(m^2 - y\sqrt{p})(m^2 + y\sqrt{p}) = n^4$, there exists $\left(\frac{f}{2} + \frac{g}{2}\sqrt{p}\right) \in A$ such that: $m^2 + y\sqrt{p} = (\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p})\left(\frac{f}{2} + \frac{g}{2}\sqrt{p}\right)^4$, $(\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p}) \in \mathbf{U}(A)$ ($c_k, d_k \in \mathbf{Z}$, c_k, d_k are simultaneously even or odd, $N(\frac{c_k}{2} + \frac{d_k}{2}\sqrt{p}) = 1$), which is equivalent to$m^2 + y\sqrt{p} = (\frac{c_k}{2} + \frac{d_k}{2})\left(\frac{f^4}{16} + \frac{f^3 g\sqrt{p}}{4} + \frac{3f^2g^2p}{8} + \frac{fg^3p\sqrt{p}}{4} + \frac{g^4p^2}{16}\right)$,which is equivalent to $32(m^2 + y\sqrt{p}) = (c_k + d_k\sqrt{p})(f^4 + 4f^3g\sqrt{p} + 6f^2g^2p + 4fg^3p\sqrt{p} + g^4p^2)$. This implies the system:

$$\begin{cases} 32m^2 = c_k f^4 + 6pc_k f^2 g^2 + p^2 c_k g^4 + 4pf^3 g d_k + 4p^2 f g^3 d_k \\ 32y = 4c_k f^3 g + 4pc_k f g^3 + d_k f^4 + 6pd_k f^2 g^2 + p^2 d_k g^4, \end{cases}$$

which implies the system:

$$\begin{cases} 32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3) \\ 32y = d_k(f^4 + 6pf^2g^2 + p^2g^4) + 4c_k(f^3g + pfg^3). \end{cases}$$

We have already proved that, if the equation $m^4 - n^4 = py^2$ has a nontrivial solution in$\mathbf{Z}$, it has an infinity of integer solutions. Therefore, if the system

$$\begin{cases} 32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3) \\ 32y = d_k(f^4 + 6pf^2g^2 + p^2g^4) + 4c_k(f^3g + pfg^3), \end{cases}$$

has a nontrivial solution in $\mathbf{Z}$, it has an infinity of integer solutions. Therefore, if the equation $32\ m^2 = c_k(f^4 + 6pf^2g^2 + p^2g^4) + 4pd_k(f^3g + pfg^3)$ has a nontrivial solution in $\mathbf{Z}$, it has an infinity of integer solutions.

References

[1] T. Albu, I.D.Ion, *Chapters of the algebraic theory of numbers (* in Romanian*)*, Ed. Academiei, Bucureşti, 1984.

[2] V. Alexandru, N.M. Goşoniu, *The elements of the theory of numbers (* in Romanian), Ed.Universităţii Bucureşti, 1999.

[3] T. Andreescu, D. Andrica, *An Introduction in the Study of Diophantine Equations* (in Romanian), Ed. Gil, Zalău, 2002.

[4] L.J. Mordell, *Diophantine Equations,* Academic Press, New York,1969.

[5] L. Panaitopol, A. Gica, *An introduction in arithmetic and the theory of numbers*(in Romanian),Ed. Universităţii Bucureşti, 2001.

[6] Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory,*Springer-Verlag, 1992.

[7] D. Savin, *On Some Diophantine Equations* (I)Analele Universităţii Övidius",Constanţa, ser. Matematica, **11** (2002), fasc.1, pag. 121-134.

[8] W.Sierpinski, *What we know and what we do not know about prime numbers*(in Romanian), (transl. from the Polish edition,1964), Bucureşti.

[9] C. Vraciu, M. Vraciu, *The elements of Arithmetic,* Ed. All, Bucureşti, 1998.

"Ovidius" University of Constanta
Department of Mathematics and Informatics,
900527 Constanta, Bd. Mamaia, 124
Romania
e-mail: Savin.Diana@univ-ovidius.ro