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# ON SOME ACTUARIAL MODELS INVOLVING SUMS OF DEPENDENT RISK

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**Abstract**

This paper presents some applications to the theoretical results obtained in [7], on the problem of approximating the tail probability of a randomly weighted sum of random variables. The results are supported by some simulation conclusions.

## 1 Introduction

In this paper we investigate the tail probabilities of the randomly weighted sums

$$S_n(\theta) = \sum_{k=1}^{n} \theta_k X_k, \qquad n \geq 1, \tag{1}$$

where $(X_k)_{k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) real-valued random variables (r.v.'s) with generic r.v. $X$, while $(\theta_k)_{k\geq 1}$ is another sequence of positive r.v.'s, independent of the sequence $(X_k)_{k\geq 1}$.

Such sums and their maxima are often encountered in actuarial and economical situations. For example, in an insurance context, the discounted sum of losses within a finite or infinite time period can be described as a randomly weighted sum of a sequence of independent r.v.'s. These independent r.v.'s $(X_k)_{k\geq 1}$ denote the amounts of losses in successive time periods (e.g. years), while the weights $(\theta_k)_{k\geq 1}$ denote the discount factors and are modelled by

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r.v.'s that can be independent or dependent. There is an increasing literature on the problem of approximating the tail probability of such weighted sums (see e.g. [3], [7], [8], [9], [10], etc.). Some advanced results can be found in [7], which considers the case when the losses $(X_k)_{k\geq 1}$ are Pareto-like distributed and the weights $(\theta_k)_{k\geq 1}$ are dependent r.v.'s. The present paper presents some applications to these results, supported by some simulation conclusions.

The structure of the paper is as follows: in section 2 we recall the results from [7], while in section 3 we present three applications. The first two applications compare the approximated and the exact (simulated) tail probabilities for some actuarial models, while the third application compares some upper and lower bounds for (1) from an asymptotic point of view.

## 2 Some theoretical results

We start by introducing some notations. For any real number $x$, we write its positive part by $x^+ = \max\{x, 0\}$. For two positive infinitesimals $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\limsup\limits_{x\to\infty} \dfrac{a(x)}{b(x)} = 1$.

The distribution function (d.f.) of the r.v. $X$ from (1) will be denoted by $F(x) = 1 - \overline{F}(x) = \Pr(X \leq x)$ for $x \in (-\infty, \infty)$. We assume that the right tail of $F$ is regularly varying in the sense that there exists some constant $0 < \alpha < \infty$ and a positive slowly varying function $L(\cdot)$ such that

$$\overline{F}(x) = x^{-\alpha} L(x), \qquad x > 0. \tag{2}$$

For simplicity we designate the fact (2) by $F \in \mathcal{R}_{-\alpha}$. This class contains the famous Pareto distributions, widely use in insurances to model the losses. For more details on this class see [1], [2] or [6].

The following results are from [7]. The first result deals with the case of randomly weighted sums of finite summands.

**Theorem 2.1** *Consider the randomly weighted sum (1) and let $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. We have*

$$\Pr\left(\max_{1\le m\le n}\sum_{k=1}^{m}\theta_k X_k > x\right) \sim \Pr\left(\sum_{k=1}^{n}\theta_k X_k > x\right) \sim \overline{F}(x)\sum_{k=1}^{n}\mathrm{E}\theta_k^{\alpha} \tag{3}$$

*if there exists some $\delta > 0$ such that*

*(1)* $\mathrm{E}\theta_k^{\alpha+\delta} < \infty$ *for each* $1 \le k \le n$.

The following result extends Theorem 2.1 to the case of infinite summands.

**Theorem 2.2** *For the randomly weighted sum (1) with $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, we have*

$$\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \sim \Pr\left(\sum_{k=1}^{\infty}\theta_k X_k^{+} > x\right) \sim \overline{F}(x)\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha} \tag{4}$$

*if one of the following assumptions holds:*

*(2)* $0 < \alpha < 1$ *and*

$$\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha+\delta} < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha-\delta} < \infty \qquad \textit{for some } \delta > 0; \tag{5}$$

*(3)* $1 \le \alpha < \infty$ *and*

$$\sum_{k=1}^{\infty}\left(\mathrm{E}\theta_k^{\alpha+\delta}\right)^{\frac{1}{\alpha+\delta}} < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty}\left(\mathrm{E}\theta_k^{\alpha-\delta}\right)^{\frac{1}{\alpha+\delta}} < \infty \qquad \textit{for some } \delta > 0. \tag{6}$$

The following remarks hold:

**Remark 2.1.** Both Theorems 2.1 and 2.2 do not require any information about the dependence structure of the sequence $(\theta_k)_{k\ge1}$. □

**Remark 2.2.** Assume that the r.v. $X_k$ is the net payout during year $k$ and the random variables $\theta_k$ in (1) are interpreted as discount factors from time

$k$ to time 0. If $Y_n$ is the nonnegative r.v. discount factor from year $n$ to year $n-1$, $n=1,2,\ldots$, then $\theta_k$ can be expressed as

$$\theta_k = \prod_{j=1}^{k} Y_j, \qquad k=1,2,\ldots \tag{7}$$

As in the terminology of [10], we call $(X_k)_{k\geq 1}$ the insurance risks and $(Y_k)_{k\geq 1}$ the financial risks. If we also assume that $(Y_k)_{k\geq 1}$ are i.i.d., then clearly, in this standard case, assumption (1) of Theorem 2.1 is equivalent to

(4) $\mathrm{E}Y_1^{\alpha+\delta} < \infty$ for some $\delta > 0$,

and assumptions (2) and (3) of Theorem 2.2 are equivalent to

(5) $\mathrm{E}Y_1^{\alpha\pm\delta} < 1$ for some $\delta > 0$.

The following corollary will be useful in the first application.

**Corollary 2.1** *Under the assumptions of Theorem 2.2, if $M$ is a nonnegative, integer-valued and nondegenerate at 0 r.v., with $\mathrm{E}M < \infty$, independent of $(X_k)_{k\geq 1}$ and of $(\theta_k)_{k\geq 1}$, then*

$$\Pr\left(\sum_{k=1}^{M} \theta_k X_k^+ > x\right) \sim \overline{F}(x) \sum_{k=1}^{\infty} \mathrm{E}\theta_k^\alpha \Pr(M \geq k).$$

**Proof**. We rewrite

$$\sum_{k=1}^{M} \theta_k X_k^+ = \sum_{k=1}^{\infty} \tilde{\theta}_k X_k^+,$$

where $\tilde{\theta}_k = \theta_k I_{(M\geq k)}$. Applying now Theorem 2.2, we get

$$\Pr\left(\sum_{k=1}^{M} \theta_k X_k^+ > x\right) \sim \overline{F}(x) \sum_{k=1}^{\infty} \mathrm{E}\tilde{\theta}_k^\alpha = \overline{F}(x) \sum_{k=1}^{\infty} \mathrm{E}\theta_k^\alpha \Pr(M \geq k),$$

which completes the proof.

# 3 Applications

The results presented in the previous section are exemplified in [7] for different choices of a multivariate distribution for $(Y_k)_{k=1,\ldots,n}$ (e.g. lognormal and logelliptical). In the following two applications, we will consider more complex models than (1), models that are very common in insurances. The third application compares some upper and lower bounds derived in [8] for the sum $S_n(\theta)$ with the sum itself, from an asymptotic point of view.

## 3.1 A first application

We will now interpret

$$S_M(\theta) = \sum_{k=1}^{M} \theta_k X_k$$

as the total discounted claims of a policy that expires after $M$ years. A natural assumption is that the r.v. $M$ should be bounded above, so it can be written as $M\begin{pmatrix} 1 & 2 & \ldots m \\ p_1 & p_2 & \ldots p_m \end{pmatrix}$, with $0 \le p_i \le 1$ and $\sum_{i=1}^{m} p_i = 1$. Hence the Corollary 2.1 gives in this case

$$\Pr\left(\sum_{k=1}^{M} \theta_k X_k^+ > x\right) \sim \overline{F}(x) \sum_{k=1}^{m} \mathrm{E}\theta_k^\alpha \Pr(M \ge k) = \overline{F}(x) \sum_{k=1}^{m} \mathrm{E}\theta_k^\alpha \sum_{i=k}^{m} p_i. \quad (8)$$

**Numerical results**

In order to illustrate the above result, we consider $(X_k)_{k\ge 1}$ to be i.i.d. $Pareto(\alpha, \beta)$, $\alpha > 1, \beta > 0$, with density

$$f_X(x) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}},\ x > \beta.$$

We also take $\mathbf{\Theta} = (\theta_n)_{n=1,\ldots,m}$ to be a sequence of lognormal dependent r.v.'s defined as $\ln \mathbf{\Theta} = (\ln\theta_1, \ldots, \ln\theta_m)$ to follow an $m-$dimensional Normal distribution $N_m(\mu, \mathbf{\Sigma})$, with parameters $\mu = (\mu_i)_{i=1,\ldots,m} \in \mathbb{R}^m$ and $\mathbf{\Sigma} = (\sigma_{ij})_{i,j=1,\ldots,m}$ being a positive defined matrix. Then in this particular case, from [7] we have

$$\mathrm{E}\theta_k^\alpha = e^{-\alpha(\mu_1+...+\mu_k)+\frac{\alpha^2}{2}\sum\limits_{1\le i,j\le k}\sigma_{ij}}, \tag{9}$$

so that formula (8) becomes

$$\Pr\left(\sum_{k=1}^{M}\theta_k X_k > x\right) \sim \left(1-\left(\frac{\beta}{x}\right)^\alpha\right)\sum_{k=1}^{m} e^{-\alpha(\mu_1+...+\mu_k)+\frac{\alpha^2}{2}\sum\limits_{1\le i,j\le k}\sigma_{ij}}\left(\sum_{i=k}^{m}p_i\right).$$

For simulation we considered $m = 10$ (i.e. ten years),

$$M\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{160} & \frac{1}{160} & \frac{1}{160} & \frac{1}{160} & \frac{1}{160} \end{pmatrix},$$

$\mu_1 = ... = \mu_{10} = 0.1$ and

$$\mathbf{\Sigma} = \begin{bmatrix} 0.05 & 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.1 & 0.01 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0.1 & 0.01 & 0.02 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0.01 & 0.05 & 0.05 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & 0.05 & 0.1 & 0.01 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0.01 & 0.1 & 0.02 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0.02 & 0.05 & 0.01 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 & 0.01 & 0.02 & 0.01 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0.01 & 0.1 & 0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0.05 & 0.05 \end{bmatrix}.$$

**Table 1.** Simulated versus asymptotic values of the tail probability for Pareto claim sizes with lognormal discounting factors

| | $\alpha = 1.2,$ | $\beta = 2$ | | | $\alpha = 1.5,$ | $\beta = 2$ | |
|---|---|---|---|---|---|---|---|
| $x$ | Simulated | Asymptotic | $1-\frac{asym.}{sim.}$ | $x$ | Simulated | Asymptotic | $1-\frac{asym.}{sim.}$ |
| 20 | 0.16612 | 0.11472 | 0.30939 | 50 | 0.02320 | 0.01477 | 0.36335 |
| 100 | 0.02047 | 0.01663 | 0.18766 | 100 | 0.00708 | 0.00522 | 0.26292 |
| 200 | 0.00818 | 0.00723 | 0.11540 | 200 | 0.00220 | 0.00184 | 0.16140 |
| 300 | 0.00489 | 0.00444 | 0.09037 | 300 | 0.00116 | 0.00100 | 0.13868 |
| 500 | 0.00252 | 0.00241 | 0.04454 | 400 | 0.00073 | 0.00065 | 0.11174 |
| 700 | 0.00172 | 0.00160 | 0.06406 | 500 | 0.00051 | 0.00046 | 0.09815 |
| 1000 | 0.00110 | 0.00104 | 0.05299 | 600 | 0.00037 | 0.00035 | 0.05985 |
| 1200 | 0.00088 | 0.00084 | 0.04193 | 700 | 0.00030 | 0.00028 | 0.06618 |
| 1400 | 0.00071 | 0.00070 | 0.01723 | 800 | 0.00024 | 0.00023 | 0.04222 |
| 1500 | 0.00064 | 0.00064 | 0.00149 | 1000 | 0.00017 | 0.00016 | 0.06156 |

Some results are given in Table 1. The number of simulation was 1,000,000. The considered values of $\alpha$, 1.2 and 1.5, are realistic in fire insurance, see [1].

Apart from the values of $x$, of the simulated and asymptotic tail probabilities, we also display the values of $1-\frac{\text{asymptotic}}{\text{simulated}}$. Theoretically, this values must tend to 0 when $x\to\infty$, which seems to be the case from Table 1.

We can conclude that when $\alpha$ decreases, i.e. the distribution of $X$ becomes more heavy-tailed, the asymptotic results perform better. This is a reasonable conclusion, since the theoretical results are established for heavy-tailed distributions.

### 3.2 A second application: the compound model

We will now assume that every loss $X_k$ results from a compound process, i.e.

$$X_k=\sum_{i=0}^{N_k}C_{ki},$$

where $N_k$ is the r.v. number of claims for year $k$ and $C_{k1},C_{k2},...$ are i.i.d. claim amounts, independent of $N_k$. We take $C_{k0}=0$. We will also assume that $C_{ki}$ are i.i.d. for any $k$ and $i$, with d.f. $F_C\in\mathcal{R}_{-\alpha}$, while $N_1,...,N_n$ are independent, but not necessarily identically distributed. Then using first a step from the proof of Theorem 2.1 (see [7]) and secondly Theorem 2 in [3], it holds that

$$\Pr\left[\sum_{k=1}^{n}X_k\theta_k>x\right]\sim\sum_{k=1}^{n}\mathrm{E}\left(\theta_k^{\alpha}\right)\overline{F}_{X_k}\left(x\right)\sim\overline{F}_C\left(x\right)\sum_{k=1}^{n}\mathrm{E}\left(N_k\right)\mathrm{E}\left(\theta_k^{\alpha}\right). \quad (10)$$

On the other hand, we can rewrite

$$S_n\left(\theta\right)=\sum_{k=1}^{n}\theta_kX_k=\sum_{k=1}^{M}\tilde{\theta}_k\tilde{C}_k,$$

where $M=\sum_{i=1}^{n}N_i$, the sequence $\left(\tilde{\theta}_k\right)_{k\geq1}$ is defined as $\tilde{\theta}_1=...=\tilde{\theta}_{N_1}=\theta_1,...,\tilde{\theta}_{M-N_n+1}=...=\tilde{\theta}_M=\theta_n$ and the r.v.'s $\left(\tilde{C}_k\right)_{k\geq1}$ are i.i.d., with the same d.f. $F_C$. Then from Corollary 2.1 we have

$$\Pr\left(\sum_{k=1}^{n}X_k\theta_k>x\right)=\Pr\left(\sum_{k=1}^{M}\tilde{\theta}_k\tilde{C}_k>x\right)\sim\overline{F}_C\left(x\right)\sum_{k=1}^{\infty}\mathrm{E}\tilde{\theta}_k^{\alpha}\Pr\left(M\geq k\right),$$

and we are back again in the context of the previous application.

**Numerical results**

We will illustrate the result (10) considering as before $(X_k)_{k=1,\ldots,n}$ i.i.d. $Pareto\,(\alpha,\beta)$, $\alpha > 1, \beta > 0$, $\mathbf{\Theta} = (\theta_k)_{k=1,\ldots,n}$ lognormal dependent r.v.'s, and $(N_k)_{k=1,\ldots,n}$ i.i.d. Poisson$(\lambda)$, $\lambda > 0$. Then using again (9), the formula becomes

$$\Pr\left(\sum_{k=1}^{M} \theta_k X_k > x\right) \sim \left(1 - \left(\frac{\beta}{x}\right)^{\alpha}\right) \lambda \sum_{k=1}^{m} e^{-\alpha(\mu_1+\ldots+\mu_k)+\frac{\alpha^2}{2}\sum\limits_{1\leq i,j\leq k}\sigma_{ij}}.$$

Using 1,000,000 simulations, we obtained the results in Tables 2 and 3.

**Table 2.** Simulated versus asymptotic values of the tail probability for the compound Poisson-Pareto model with lognormal discounting factors $(\beta = 2,\ \lambda = 5)$

| | $\alpha = 1.2$ | | | | $\alpha = 1.5$ | | |
|---|---|---|---|---|---|---|---|
| $x$ | Simulated | Asymptotic | $1 - \frac{asym.}{sim.}$ | $x$ | Simulated | Asymptotic | $1 - \frac{asym.}{sim.}$ |
| 50 | 0.99115 | 0.88038 | 0.11175 | 100 | 0.79004 | 0.13152 | 0.83351 |
| 100 | 0.90197 | 0.38320 | 0.57514 | 500 | 0.08014 | 0.01176 | 0.85321 |
| 500 | 0.20007 | 0.05554 | 0.72236 | 800 | 0.02697 | 0.00581 | 0.78448 |
| 800 | 0.09098 | 0.03160 | 0.65264 | 1000 | 0.01572 | 0.00415 | 0.73546 |
| 1000 | 0.06114 | 0.02417 | 0.60455 | 1200 | 0.01001 | 0.00316 | 0.68412 |
| 1500 | 0.02963 | 0.01486 | 0.49850 | 1500 | 0.00579 | 0.00226 | 0.60910 |
| 2000 | 0.01788 | 0.01052 | 0.41141 | 2000 | 0.00299 | 0.00147 | 0.50867 |
| 2500 | 0.01234 | 0.00805 | 0.34764 | 2500 | 0.00187 | 0.00105 | 0.44000 |

**Table 3.** Simulated versus asymptotic values of the tail probability for the compound Poisson-Pareto model with lognormal discounting factors $(\alpha = 1.5,\ \beta = 2)$

| | $\lambda = 2$ | | | | $\lambda = 10$ | | |
|---|---|---|---|---|---|---|---|
| $x$ | Simulated | Asymptotic | $1 - \frac{asym.}{sim.}$ | $x$ | Simulated | Asymptotic | $1 - \frac{asym.}{sim.}$ |
| 100 | 0.28449 | 0.05261 | 0.81507 | 500 | 0.30576 | 0.02352 | 0.92305 |
| 500 | 0.01153 | 0.00470 | 0.59211 | 1000 | 0.07819 | 0.00831 | 0.89361 |
| 1000 | 0.00263 | 0.00166 | 0.36788 | 2000 | 0.01396 | 0.00294 | 0.73157 |
| 1500 | 0.00117 | 0.00090 | 0.22596 | 2500 | 0.00787 | 0.00210 | 0.73259 |
| 2000 | 0.00070 | 0.00058 | 0.16801 | 3000 | 0.00482 | 0.00160 | 0.66834 |
| 2500 | 0.00045 | 0.00042 | 0.07292 | 3500 | 0.00322 | 0.00127 | 0.60631 |
| 3000 | 0.00033 | 0.00032 | 0.04422 | 4000 | 0.00232 | 0.00103 | 0.55333 |
| 3200 | 0.00030 | 0.00029 | 0.03761 | 4500 | 0.00174 | 0.00087 | 0.49946 |
| 3500 | 0.00026 | 0.00025 | 0.03389 | 5500 | 0.00109 | 0.00064 | 0.41210 |

We can conclude that:

- For fixed $\beta, \lambda$, when $\alpha$ decreases, the asymptotic results perform better again. It is not recommended to consider $\alpha > 2$, i.e. the claim distribution must not be too light-tailed.

- For fixed $\beta, \alpha$, when $\lambda$ decreases, the asymptotic results perform better.

- When using these asymptotic results, one should be very careful with the choice of $x$. We can see that if $x$ is too small, then the differences between the simulated reality and asymptotics can be very important (see e.g. $x = 100$). This is also reasonable since an asymptotic result involves the limit for $x \to \infty$.

### 3.3 Upper and lower bounds for discounted amounts of claims

Consider two r.v.'s $X$ and $Y$. Then $X$ is said to precede $Y$ in the convex order sense, denoted $X \leq_{cx} Y$, if we have

$$\mathrm{E}\left[v\left(X\right)\right] \leq \mathrm{E}\left[v\left(Y\right)\right]$$

for all convex real functions $v$ such that the expectations exists.

Kaas et al. [8] derived upper and lower bounds in the convex order for the sum $S_n\left(\theta\right)$, when $X_1, ..., X_n$ are deterministic values of arbitrary sign,

$\theta_k = e^{-(Z_1+...+Z_k)}$ for $k = 1, ..., n$, and $(Z_1, ..., Z_n)$ has a multivariate normal distribution (see also the reviews [4], [5]). Assuming that $X_k = x_k, k = 1, ..., n$, denoting by $Z(k) = Z_1 + ... + Z_k$ and by $\Lambda = \sum_{k=1}^{n} \beta_k Z_k$ a conditioning r.v., then in [8] it was proved that

$$S_d^l \leq_{cx} S_n(\theta) \leq_{cx} S_d^u \leq_{cx} S_d^c.$$

The above bounds are defined as follows

$$\begin{aligned}
S_d^l &= \mathrm{E}\left[S_n(\theta)|\Lambda\right] = \sum_{k=1}^{n} x_k \exp\left\{-\mathrm{E}\left[Z(k)\right] - r_k \sigma_{Z(k)} W + \frac{1}{2}\left(1 - r_k^2\right)\sigma_{Z(k)}^2\right\}, \\
S_d^u &= \sum_{k=1}^{n} x_k \exp\left\{-\mathrm{E}\left[Z(k)\right] - r_k \sigma_{Z(k)} W + sign(x_k)\sqrt{1 - r_k^2}\sigma_{Z(k)} V\right\}, \\
S_d^c &= \sum_{k=1}^{n} x_k \exp\left\{-\mathrm{E}\left[Z(k)\right] + sign(x_k)\sigma_{Z(k)} V\right\},
\end{aligned}$$

where $W$ and $V$ are independent $N(0,1)$ distributed r.v.'s and $r_k = \dfrac{cov\left[Z(k), \Lambda\right]}{\sigma_{Z(k)}\sigma_\Lambda}$.

This result can be extended to $X_1, ..., X_n$ non-negative r.v.'s as follows:

**Proposition 3.1** *If $X_1, ..., X_n$ are non-negative r.v.'s and $(Z_1, ..., Z_n)$ has a multivariate normal distribution, then the following order hold*

$$S^l \leq_{cx} S_n(\theta) \leq_{cx} S^u \leq_{cx} S^c,$$

*where*

$$S^l = \sum_{k=1}^{n} X_k \theta_k^l = \sum_{k=1}^{n} X_k \exp\left\{-\mathrm{E}\left[Z(k)\right] - r_k \sigma_{Z(k)} W + \frac{1}{2}\left(1 - r_k^2\right)\sigma_{Z(k)}^2\right\}, \tag{11}$$

$$S^u = \sum_{k=1}^{n} X_k \theta_k^u = \sum_{k=1}^{n} X_k \exp\left\{-\mathrm{E}\left[Z(k)\right] - r_k \sigma_{Z(k)} W + \sqrt{1 - r_k^2}\sigma_{Z(k)} V\right\}, \tag{12}$$

$$S^c = \sum_{k=1}^{n} X_k \theta_k^c = \sum_{k=1}^{n} X_k \exp\left\{-\mathrm{E}\left[Z(k)\right] + \sigma_{Z(k)} V\right\}, \tag{13}$$

*with $W$ and $V$ independent $N(0,1)$ distributed r.v.'s and $r_k$ defined above.*

**Proof**. For example, we will prove the middle inequality, the others resulting using similar arguments. Let $\phi$ be any convex function such that the following expectations exists. Then we have

$$\begin{aligned}
\mathrm{E}\left[\phi\left(S_n\left(\theta\right)\right)\right] &= \mathrm{E}\left\{\mathrm{E}\left[\phi\left(\sum_{k=1}^{n} X_k\theta_k\right)|X_1,...,X_n\right]\right\} \\
&\leq \mathrm{E}\left\{\mathrm{E}\left[\phi\left(\sum_{k=1}^{n} X_k\theta_k^u\right)|X_1,...,X_n\right]\right\} \\
&= \mathrm{E}\left[\phi\left(S^u\right)\right].
\end{aligned}$$

In order to derive the above inequality we used the order relation known for $X_k$ deterministic. It follows that the same order holds when $X_k$ are r.v.'s.

We will now apply Theorem 2.1 to the bounds (11), (12) and (13). We see that the values $\mathrm{E}\left[Z\left(k\right)\right]$ and $\sigma^2_{Z(k)}$ are given by

$$\mathrm{E}\left(Z_1+...+Z_k\right) = \mu_1+...+\mu_k, \qquad (14)$$

$$Var\left(Z_1+...+Z_k\right) = \sum_{1\leq i,j\leq k} \sigma_{ij}. \qquad (15)$$

From the fact that $\theta_k, \theta_k^u$ and $\theta_k^c$ have the same marginal distributions (see [8]) and from (9), we have that

$$\mathrm{E}\left[\left(\theta_k\right)^{\alpha}\right] = \mathrm{E}\left[\left(\theta_k^u\right)^{\alpha}\right] = \mathrm{E}\left[\left(\theta_k^c\right)^{\alpha}\right] = \exp\left\{-\alpha\left(\mu_1+...+\mu_k\right)+\frac{\alpha^2}{2}\sum_{1\leq i,j\leq k}\sigma_{ij}\right\},$$

so that in this case Theorem 2.1 gives the same result: the asymptotic tail probabilities are the same for $S_n\left(\theta\right), S^u$ and $S^c$, given by

$$\Pr\left(S_n\left(\theta\right)>x\right) \sim \overline{F}\left(x\right)\sum_{k=1}^{n} e^{-\alpha\left(\mu_1+...+\mu_k\right)+\frac{\alpha^2}{2}\sum_{1\leq i,j\leq k}\sigma_{ij}}.$$

This is not the case for $S^l$. Here we have

$$\begin{aligned}
\mathrm{E}\left[\left(\theta_k^l\right)^\alpha\right] &= \exp\left\{\alpha\left[-\mathrm{E}\left[Z\left(k\right)\right]+\frac{1}{2}\left(1-r_k^2\right)\sigma_{Z(k)}^2\right]\right\}\mathrm{E}\left[e^{-\alpha r_k\sigma_{Z(k)}W}\right] \\
&= \exp\left\{-\alpha\mathrm{E}\left[Z\left(k\right)\right]+\frac{\alpha}{2}\left(1-r_k^2\right)\sigma_{Z(k)}^2\right\}\exp\left\{\frac{\alpha^2r_k^2\sigma_{Z(k)}^2}{2}\right\} \\
&= \exp\left\{-\alpha\mathrm{E}\left[Z\left(k\right)\right]+\frac{\alpha}{2}\left[1+\left(\alpha-1\right)r_k^2\right]\sigma_{Z(k)}^2\right\}.
\end{aligned}$$

Since $r_k^2\leq 1$, we see that $\frac{\alpha}{2}\left[1+\left(\alpha-1\right)r_k^2\right]\leq\frac{\alpha^2}{2}$, so that $\mathrm{E}\left[\left(\theta_k^l\right)^\alpha\right]\leq\mathrm{E}\left[\left(\theta_k\right)^\alpha\right]$ which is reasonable.

Theorem 2.1 gives

$$\Pr\left(S^l>x\right)\sim\overline{F}\left(x\right)\sum_{k=1}^n e^{-\alpha(\mu_1+\ldots+\mu_k)+\frac{\alpha}{2}\left[1+(\alpha-1)r_k^2\right]\sum\limits_{1\leq i,j\leq k}\sigma_{ij}}.$$

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