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BIVARIATE OPERATORS OF SCHURER-STANCU TYPE

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

Considering two given non negative integers p, q and four real parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfying the conditions $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$, the Schurer-Stancu bivariate operator

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C([0,1+p] \times [0,1+q]) \to C([0,1] \times [0,1]),$$

defined by

$$\left(\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y) = \\ = \sum_{k=0}^{m} \sum_{j=0}^{n} \widetilde{p}_{m,k}(x)\widetilde{p}_{nj}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right)$$

is constructed and some approximation properties of the sequence $\left\{\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right\}_{m,n\in\mathbb{N}}$ are established.

1 Preliminaries

Let $p \geq 0$ be a given positive integer. In 1962, F. Schurer (see ([7])) constructed the sequence $\left(\widetilde{B}_{m,p}\right)_{m\in\mathbb{N}}$, $\widetilde{B}_{m,p} : C([0,1+p] \to C([0,1] \times [0,1])$, defined for any $f \in C([0,1+p])$ and any $m \in \mathbb{N}$ by

$$\left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f(k/m) \tag{1}$$

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where

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} \tag{2}$$

are the fundamental Schurer polynomials. Note that for $p = 0$ the operators (1.1) are reduced to the classical Bernstein operators.

Considering the real parameters α, β satisfying the conditions $0 \leq \alpha \leq \beta$, D.D. Stancu (see ([9])) introduced and studied the sequence of positive linear operators $\left(\widetilde{P}^{(\alpha,\beta)}\right)_{m\in\mathbb{N}}$, $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ by

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum\nolimits_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{3}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{4}$$

are the Bernstein fundamental polynomials.

In our recent paper ([6]) was constructed the Schurer-Stancu type operator $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$, defined for any $m \in \mathbb{N}$ and any $f \in C([0,1+p])$ by

$$\left(\widetilde{S}_{m,n}^{(\alpha,\beta)} f\right)(x) = \sum\nolimits_{k=0}^{m+p} \widetilde{p}_{mk}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{5}$$

and some approximation properties of this operator were studied.

Note that for $\alpha = \beta = 0$ the operator (1.4) reduces to Schurer operator (1.1) and for $p = 0$, (1.4) reduces to Stancu operator (1.5).

The purposes of the present paper are the following:

(i) to construct the bivariate Schurer-Stancu type operator

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C([0,1+p] \times [0,1+q]) \to C([0,1] \times [0,1]),$$

where $p, q \geq 0$ are given integers and $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$ are real parameters.

(ii) to establish approximation properties for the sequence $\left\{\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right\}_{m,n\in\mathbb{N}}$, under some assumptions on the approximated function f.

2 The construction of Schurer-Stancu bivariate operators

Let $p \geq 0, q \geq 0$ be given integers, $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$ and let

$$\widetilde{S}_{m,p}^{(\alpha_1,\beta_1)} : C([0,1+p]) \to C([0,1]),$$

$$\widetilde{S}_{n,q}^{(\alpha_2,\beta_2)} : C([0,1+q]) \to C([0,1])$$

be the univariate Schurer-Stancu operators defined by (1.4).
The parametric extensions (for this term see ([4])) of these operators are

$${}_x\widetilde{S}_{m,p}^{(\alpha_1,\beta_1)}, {}_y\widetilde{S}_{n,q}^{(\alpha_2,\beta_2)} : C([0,1+p] \times [0,1+q]),$$

defined respectively by

$$\left({}_x\widetilde{S}_{m,p}^{(\alpha_1,\beta_1)} f\right)(x,y) = \sum\nolimits_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f(k/m, y), \tag{6}$$

$$\left({}_y\widetilde{S}_{n,q}^{(\alpha_2,\beta_2)} f\right)(x,y) = \sum\nolimits_{j=0}^{n+q} \widetilde{p}_{n,j}(y) f(x, j/n). \tag{7}$$

Lemma 2.1 *The parametric extensions* (6) *and* (7) *are linear positive operators.*

Proof. The assertions follow by direct computation from their definitions. ■

Lemma 2.2 *The parametric extensions* ${}_x\widetilde{S}_{m,p}^{(\alpha_1,\beta_1)}, {}_y\widetilde{S}_{n,q}^{(\alpha_2,\beta_2)}$ *of Schurer-Stancu operator commute on* $C([0,1+p] \times [0,1+q])$. *Their product is the bivariate Schurer-Stancu type operator*

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C([0,1+p] \times [0,1+q]) \to C([0,1] \times [0,1])$$

defined for any $f \in C([0,1+p] \times [0,1+q])$ *and any* $m, n \in \mathbb{N}$ *by*

$$\left(\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y) = \\ = \sum\nolimits_{k=0}^{m+p} \sum\nolimits_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{nj}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right) \tag{8}$$

Proof. Taking into account of (6), (7) and Lemma 2.1, we get

$$\left({}_x\widetilde{S}^{(\alpha_1,\beta_1)}_{m,p,} \circ_y \widetilde{S}^{(\alpha_2,\beta_2)}_{n,q} f\right) =$$
$$=_x \widetilde{S}^{(\alpha_1,\beta_1)}_{m,p,} \left(\sum_{j=0}^{n+q} \widetilde{p}_{n,j}(y) f\left(x, \frac{j+\alpha_2}{n+\beta_2}\right)\right) =$$
$$= \sum_{j=0}^{n+q} \widetilde{p}_{n,j}(y) \left({}_x\widetilde{S}^{(\alpha_1,\beta_1)}_{m,p,} f\right)\left(x, \frac{j+\alpha_2}{n+\beta_2}\right) =$$
$$= \sum_{j=0}^{n+q} \widetilde{p}_{nj}(y) \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right) =$$
$$= \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right) =$$
$$= \left(\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q} f\right)(x,y).$$

In a similar way, it follows the equality ${}_y\widetilde{S}^{(\alpha_2,\beta_2)}_{n,q} \circ_x \widetilde{S}^{(\alpha_1,\beta_1)}_{m,p,} = S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q}$. ■

Lemma 2.3 *The bivariate Schurer-Stancu type operator* (2.3) *is linear and positive.*

Proof. The assertions follow from (8), taking into account the linearity of ${}_x\widetilde{S}^{(\alpha_1,\beta_1)}_{m,p}$ and ${}_y\widetilde{S}^{(\alpha_2,\beta_2)}_{n,q}$. ■

3 Approximations properties of the sequence $\left\{\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q} f\right\}$

First, the convergence of the sequence $\left\{\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q} f\right\}_{m,n\in\mathbb{N}}$ will be established.
Like usually, $e_{i,j}(s,t) = s^i t^j, (i,j) \in \mathbb{N} \times N$ denote the test functions.
We need the following Bohman-Korovkin Theorem for the approximation of bivariate functions (see ([11]])).

Theorem 3.1 *Let* $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of linear positive operators applying the space* $C([a,b]\times[c,d])$ *into it self and having the properties*

(i) $L_{m,n}(1;x,y) = 1 + u_{m,n}(x,y)$

(ii) $L_{m,n}((e_{10}-x)^2;x,y) = v_{m,n}(x,y)$

(iii) $L_{m,n}((e_{01}-y)^2;x,y) = w_{m,n}(x,y)$

(iv) $\lim\limits_{m,n\to\infty} u_{m,n}(x,y) = \lim\limits_{m,n\to\infty} v_{m,n}(x,y) = \lim\limits_{m,n\to\infty} w_{m,n}(x,y) = 0$, *uniformly on* $[a,b]\times[c,d]$.

Then, the sequence $(L_{m,n}f)_{m,n\in\mathbb{N}}$ *converges to* f, *uniformly on* $[a,b]\times[c,d]$, *for any* $f\in C([a,b]\times[c,d])$.

Lemma 3.1 *The bivariate Schurer-Stancu type operators verify*

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(1;x,y)=1 \tag{9}$$

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((e_{10}-x)^2;x,y\right)=\frac{(p-\beta_1)^2}{(m+\beta_1)^2}+\frac{m+p}{(m+\beta_1)^2}x(1-x)+ \\ +\frac{2\alpha_1(mp-2m\beta_1-\beta_1^2)}{(m+\beta_1)^3}x+\frac{\alpha_1^2(3m+p)}{(m+\beta_1)^3} \tag{10}$$

$$\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((e_{01}-y)^2;x,y\right)=\frac{(q-\beta_2)^2}{(n+\beta_2)^2}+\frac{n+q}{(n+\beta_1)^2}y(1-y)+ \\ +\frac{2\alpha_2(nq-2n\beta_2-\beta_2^2)}{(n+\beta_2)^3}y+\frac{\alpha_2^2(3n+p)}{(n+\beta_2)^3} \tag{11}$$

Proof. We make use of the linearity of $\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ and of the properties of univariate operator Schurer-Stancu (see ([6])). ∎

Theorem 3.2 *The sequence* $\left(\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}f\right)_{m,n\in\mathbb{N}}$ *converges to* f, *uniformly on* $[0,1]\times[0,1]$, *for any* $f\in C([0,1+p]\times[0,1+q])$

Proof. We apply Theorem 2.1, with $u_{m,n}(x,y)=0$ and

$$v_{m,n}(x,y)=\frac{(p-\beta_1)^2}{(m+\beta_1)^2}+\frac{m+p}{(m+\beta_1)^2}x(1-x)+ \\ +\frac{2\alpha_1(mp-2m\beta_1-\beta_1^2)}{(m+\beta_1)^2}x+\frac{\alpha^2(3m+p)}{(m+\beta_1)^3}$$

$$w_{m,n}(x,y)=\frac{(q-\beta_2)^2}{(n+\beta_2)^2}+\frac{n+q}{(n+\beta_2)}y(1-y)+ \\ +\frac{2\alpha_2(nq-2n\beta_2-\beta_2^2)}{(n+\beta_2)^2}y+\frac{\alpha_2^2(3n+q)}{(n+\beta_2)^3}$$

■

A quantitative estimation of approximation order of a continuous bivariate function using a sequence of linear positive bivariate operators is given in the following theorem, due to O. Shisha and B. Mond (see ([11])).

Theorem 3.3 *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of bivariate linear positive operators,*

$$L_{m,n} : C([a,b]\times[c,d]) \to C([a,b]\times[c,d]),$$

reproducing the constant functions. For any $f \in C([a,b]\times[c,d])$, any $(x,y)\in [a,b]\times[c,d]$ and any $(\delta_1,\delta_2)\in[0,b-a]\times[0,d-c]$, the following

$$\begin{aligned}|(L_{m,n}f)(x,y)-f(x,y)| \le \Big(1+\delta_1^{-1}\sqrt{L_{m,n}((\cdot-x)^2;x,y)}+\\ +\delta_2^{-1}\sqrt{L_{m,n}((*-y)^2;x,y)}+\\ +\delta_1^{-1}\delta_2^{-1}\sqrt{L_{m,n}((\cdot-x)^2;x,y)\cdot L_{m,n}((*-y)^2;x,y)}\Big)\times\\ \times\,\omega_1(\delta_1,\delta_2) \quad (12)\end{aligned}$$

holds.

Note that in (12), ω_1 denotes the first order modulus of smoothness for bivariate functions (see ([11])).

Theorem 3.4 *For any $f \in C([0,1+p]\times[0,1+q])$, any $(x,y)\in[0,1]\times[0,1]$, the bivariate Schurer-Stancu type operators verify*

$$\left|\left(\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}f\right)(x,y)-f(x,y)\right| \le 4\omega_1\sqrt{(\delta_{m,p,\alpha_1,\beta_1,x}\cdot\delta_{n,q,\alpha_2,\beta_2,y})} \quad (13)$$

where

$$\begin{aligned}\delta_{m,p,\alpha_1,\beta_1,x} &= \frac{(p-\beta_1)^2}{(m+\beta_1)^2}+\frac{m+p}{(m+\beta_1)^2}x(1-x)+\\ &+ \frac{2\alpha_1(mp-2m\beta_1-\beta_1^2)}{(m+\beta_1)^3}x+\frac{\alpha_1^2(3m+\beta_1)}{(m+\beta_1)^3}\end{aligned}$$

and

$$\begin{aligned}\delta_{n,q,\alpha_2,\beta_2,y} &= \frac{(q-\beta_2)^2}{(n+\beta_2)^2}+\frac{n+q}{(n+\beta_2)^2}y(1-y)+\\ &+ \frac{2\alpha_2(nq-2n\beta_2-\beta_2^2)}{(m+\beta_2)^2}y+\frac{\alpha_2^2(3n+\beta_2)}{(n+\beta_2)^3}\end{aligned}$$

Proof. Taking into account of Lemma 2.2 and applying Theorem 2.3, we get

$$\left|\left(S_{m,n,p,q}^{(\alpha,\beta,p,q)}f\right)(x,y)-f(x,y)\right|\leq$$
$$\leq\left(1+\delta_1^{-1}\cdot\sqrt{\delta_{m,p,\alpha_1,\beta_1,x}}+\delta_2^{-1}\sqrt{\delta_{n,q,\alpha_2,\beta_2,y}}+\right.$$
$$\left.+\delta_1^{-1}\delta_2^{-1}\sqrt{\delta_{m,p,\alpha_1,\beta_1,x}\cdot\delta_{n,q,\alpha_2,\beta_2,y}}\right)\cdot\omega_1(\delta_1,\delta_2)$$

Choosing $\delta_1=\sqrt{\delta_{m,p,\alpha_1,\beta_1,x}}$, $\delta_2=\sqrt{\delta_{n,q,\alpha_2,\beta_2,y}}$ in the above inequality, we arrive to the desired inequality (13), and the proof ends. ■

Remark 3.1 Theorem 2.4 contains the approximation order of local approximation of $f\in C([0,1+p]\times[0,1+q])$ by $S_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}f$ (i.e. the approximation order in each point (x,y)).

The order of global approximation is expressed in

Theorem 3.5 *For any $f\in C([0,1+p]\times[0,1+q])$ and any $(x,y)\in[0,1]\times[0,1]$ the operators $S_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ satisfy*

$$\left|\left(S_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}f\right)(x,y)-f(x,y)\right|\leq 4\omega_1(\delta_{m,p,\alpha_1,\beta_1},\delta_{n,q,\alpha_2,\beta_2}) \tag{14}$$

where

$$\begin{aligned}\delta_{m,p,\alpha_1,\beta_1}&=\max_{x\in[0,1]}\delta_{m,p,\alpha_1,\beta_1,x}\\ \delta_{n,q,\alpha_2,\beta_2}&=\max_{y\in[0,1]}\delta_{n,q,\alpha_2,\beta_2,y}\end{aligned}$$

and $\delta_{m,p,\alpha_1,\beta_1,x}$ respectively $\delta_{n,q,\alpha_2,\beta_2,y}$ are given in Theorem 2.4.

Proof. Taking into account that the first order modulus of smoothness is monotonous increasing with respect the natural order of $\mathbb{R}^2$, i.e.

$$\delta_1\leq\delta_1',\ \delta_2\leq\delta_2'\Rightarrow\omega_1(\delta_1,\delta_2)\leq\omega_1(\delta_1',\delta_2')$$

the desired inequality (3.6) follows from Theorem 2.4. ■

Remark 3.2 Clearly the expressions of $\delta_{m,p,\alpha_1,\beta_1}$ and respectively $\delta_{n,q,\alpha_2,\beta_2}$ depend of the relations between the parameters p,α_1,β_1 and q,α_2,β_2 and the inequality (14) can be rafinated.

Finally, we mention that the paper is devoted to all our dear colleagues from the Faculty of Mathematics from "Ovidius" University, and especially to Professor Silviu Sburlan, the Dean of this Faculty.

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