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ON SOME ANALYTICAL MANIFOLDS OF CONSTANT SECTIONAL CURVATURE

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

This paper is a survey on some classes of n - dimensional differentiable manifolds with indefinite metric, of index $l(\leq n)$, and of constant sectional curvature. These manifolds, denoted by $\mathcal{V}_l^n(q)$, $(q \in \mathbb{K}^*,\ \mathbb{K} \leq \mathbb{R})$, comprise six types of non - Euclidean spaces.Two topologies, as well as a metric structure and an analytical manifold structure on the spaces $\mathcal{V}_l^n(q)$ are introduced. To make these, some isometries with specific quadrics in a pseudo - Euclidean space of dimension $(n+1)$ and the solutions of elliptic type and of hyperbolic type of a system of functional equations are used.

1. Introduction

In his book, [12], J.A.WOLF studied some analytical manifolds of constant sectional curvature $K(\neq 0)$, called *pseudo-spherical* and *pseudo-hyperbolical space forms,*

$$S_s^n := \{\mathbf{x} \in \mathbb{R}_s^{n+1} : b_s^{n+1}(\mathbf{x}, \mathbf{x}) = r^2\}$$
$$H_s^n := \{\mathbf{x} \in \mathbb{R}_{s+1}^{n+1} : b_{s+1}^{n+1}(\mathbf{x}, \mathbf{x}) = -r^2\}$$

where $r > 0$, and, for $\mathbf{x} = (x^i),\ \mathbf{y} = (y^i) \in \mathbb{R}_k^{n+1},\ (0 \leq k \leq n+1)$,

$$b_k^{n+1}(\mathbf{x}, \mathbf{y}) := -\sum_{i=1}^{k} x^i y^i + \sum_{j=k+1}^{n+1} x^j y^j .$$

The manifolds so obtained are Riemannian or pseudo-Riemannian real manifolds of signature $(s,\ n-s)$ and of constant curvatures $K = 1/r^2$ or $K = -1/r^2$.

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In our paper [3], we established isometries of the pseudo-spherical and pseudo- hyperbolic pseudo-Riemannian manifolds mentioned above and some types of non- Euclidean spaces, $\mathcal{V}_l^n(q)$, as were defined in [2]. $\mathcal{V}_l^n(q)$ are n – submanifolds, of (positive) index l, associated to a nonnul real number q, of which points are obtained by identification of all pairs of points that are diametrically opposite on the quadric:

$$\Sigma = \{\mathbf{x} \in \mathbb{R}_l^{n+1} | \ \sum_{i=1}^{n} \varepsilon_i (x^i)^2 - q(x^{n+1})^2 = \rho^2\}, \tag{1}$$

where $\varepsilon_i = +1$ for $i \leq l$, $\varepsilon_i = -1$ for $i > l$, and $\rho \in \mathbf{C}_\nu$; here $\mathbf{C}_\nu$ denotes a second order algebra with the minimal polynomial $\varphi(t) = t^2 - q \in \mathbb{R}$ and basis $\{1, \nu\}$.

Σ is a hypersphere of radius ρ in $\mathbb{R}_l^{n+1}$; as an element of $\mathbf{C}_\nu$, ρ can be taken as a real or an imaginary number: $\rho = \nu$, or $\rho = \nu'$, ($\nu' = \nu \ /\sqrt{-1}$); we have $\nu^2 - q = 0$.

So, the sectional curvature of $\mathcal{V}_l^n(q)$ is either $1/+q$, or $1/-q$, for some $l \in \overline{1,n}$ and $q > 0$ or $q < 0$.

Because $\mathbb{R}_s^{n+1}$ is linearly isometric with $\mathbb{R}_l^{n+1}$ for $l = n - s+1$, the $\mathcal{V}_l^n(q)$ are locally isometric with Σ.

2. Table of non-Euclidean spaces contained by $\mathcal{V}_l^n(q)$.

The non-Euclidean spaces $\mathcal{V}_l^n(q)$, and their "models" of type S or H in the pseudo- Euclidean spaces $\mathbb{R}_k^{n+1}$ as one or another of the quadrics Σ of which radii satisfy the equation $\rho^2 = \varepsilon r^2$, ($\varepsilon = \pm 1$), are presented in the following table:

Non-Euclidian space $\mathcal{V}_l^n(q)$	Sectional curvature		Type of manifold	Isometric quadric ,,The model"	Hypersphere	
	value	sign			of radius	of the space
$\mathcal{R}^n(q)$	$1/-q$	> 0	Riemannian	S_0^n	$\rho = r$	$\mathbb{R}^{n+1}$
$\mathcal{L}^n(q)$		< 0		H_0^n	$\rho = \sqrt{-1}r$	$\mathbb{R}_1^{n+1}$
$\mathcal{E}_l^n(q)_+$	$1/-q$	> 0	Pseudo-Riemannian	S_{s-1}^n	$q = r$	$\mathbb{R}_{s-1}^{n+1}$
$\mathcal{H}_l^n(q)_-$		< 0		H_{s-1}^n	$\rho = \sqrt{-1}r$	$\mathbb{R}_s^{n+1}$
$\mathcal{E}_l^n(q)_-$		< 0	Pseudo-Riemannian	S_s^n	$\rho = r$	
		> 0		H_s^n	$\rho = \sqrt{-1}r$	$\mathbb{R}_{s+1}^{n+1}$
$\mathcal{H}_l^n(q)_-$						$s+1$

3. Tangent hyperspaces and polar hyperplanes

Let us consider a numerical field $\mathbb{K}$ (that is a subfield of $\mathbb{C}$, as $\mathbb{Q}, \mathbb{Q}(\sqrt{2}$ $), \ldots, \mathbb{R})$. Thus, for $q \in \mathbb{K}$, $\mathbf{C}_\nu$ is isomorphic with a subalgebra of $\mathbb{C}$. Now $\mathbb{R}_l^{n+1}$ will be replaced by a pseudo-Euclidean vector space $\mathbf{V}_l^{n+1}(q) \doteq \mathbf{V}$ over the field $\mathbb{K}$ with the metric structure defined by the following bilinear form:

$$\langle X, Y \rangle_f := \sum_{i=1}^{n} \varepsilon_i x^i y^i - q x^{n+1} y^{n+1}, (X, Y \in \mathbf{V}),$$

where ε_i takes the same values as before.

A vector $X \in \mathbf{V}$ is said to be "a representative" of a point $\mathbf{x} \in \mathcal{V}_l^n(q)$ if $\langle X, X \rangle_f = \rho^2$; if X is a representative of a point in $\mathcal{V}_l^n(q)$, then also $-X$ will be a representative of the same point.

Let $\varphi_X \in \mathbf{V}^*$ be the linear form associated to X that sets in correspondence to $Y \longmapsto \langle X, Y \rangle_f \in \mathbb{K}$. Let us denote by $\mathbf{V}_1 (\subset \mathbf{V})$ the orthogonal complement of φ_X. This is both a proper maximal subspace of $\mathbf{V}$ and a normal divisor of the additive group $(\mathbf{V}, +)$.

In [3] it was shown that, for every $A \in \mathbf{V}$ there exists a canonical epimorphism $h : \mathbf{V} \longrightarrow \mathbf{V} \,/\, \mathbf{V}_1$such that when $\mathbf{V}_1 = \varphi_A^{-1}(0)$ and $\mathbf{V}_1 \oplus \mathbf{V}_2 = \mathbf{V}$, where $\mathbf{V}_2 = A\mathbb{K}$, the image $h\,(A) \doteq H_A$ is a hyperplane (orthogonal to A) and also has been put in evidence a family of hyperplanes $\{H_{\alpha A}\}_{(\alpha \in \mathbb{K})}$with the same n-dimensional direction as that of H_A and being in correspondence with the elements of the subspace of $\mathbf{V}^*$, $\Phi_1 = [\varphi_A]$,generated by φ_A.

Definition 1. Consider $\alpha \in \mathbb{K} \backslash \{-1, 0, 1\}$ and $A \in \mathbf{V}$, which is a representative of the point $\mathbf{a} \in \mathcal{V}_l^n(q)$.The intersection $\mathcal{V}_l^n(q) \cap H_{\alpha A} \doteq_\alpha S^{n-1}(\mathbf{a})$,when it is not empty, is called a *non-Euclidean hypersphere* of center $\mathbf{a}$.

Remarks 1. Let us fix $l = n$ and $\mathbb{K} = \mathbb{R}$. For $|\alpha| < 1$ and $q < 0$ the hypersphere ${}_\alpha S^{n-1}(\mathbf{a})$ is real, and for $q > 0$ it is imaginary. Conversely, for $|\alpha| > 1$.

2. We may consider only the case $l \geq (n+1)/2$, because the spaces $\mathcal{V}_l^n(q)_-$ and $\mathcal{V}_{n-l+1}^n(q)_+$ are isometric; the signs $\pm$ at lower position indicate the type of curvature.

Definition 2. The *tangent space* at $\mathbf{x} \in \mathcal{V}_l^n(q)$ is the set $T_\mathbf{x}(\mathcal{V})$ of all elements $Z \in \mathbf{V}_l^{n+1}(q)$ with the property $\langle X', Z \rangle_f = 0$, where $X' (= \pm X)$ is one of the representatives of the point $\mathbf{x}$.

Proposition 1. *If* $\mathbf{a} \in \mathcal{V}_l^n(q)$ *and* A *is its representative in* $\mathbf{V}_l^{n+1}(q)$ *then* $H_{\alpha A}$ *for* $\alpha = \pm 1$ *is the tangent space at* $\mathbf{a}$ *to* $\mathcal{V}_l^n(q)$.

Proof. Fixing $\alpha = 1$ we have $H_A \in \mathbf{V} \,/\, \mathbf{V}_1$,where $\mathbf{V}_1 = \varphi_A^{-1}(0)$, with $0 \in \mathbb{K}$. If $Z \in \mathbf{V}_1$ then as $\varphi_A(Z) = 0$ we have $\langle A, Z \rangle_f = 0$.But $\mathbf{V}_1$is maximal in $\mathbf{V}$ and $H_A = A + \mathbf{V}_1$. It results that $\mathbf{V}_1$ is the set of all vectors at $\mathbf{a}$ with the property in definition of the tangent space. Because the case $\alpha = -1$ does not change the previous assertions, $-A$ being the representative of the same point $\mathbf{a} \in \mathcal{V}_l^n(q)$, the proof is end.

Proposition 2. *The tangent space* $T_{\mathbf{x}}(\mathcal{V})$, *when* $\mathcal{V}_l^n(q)$ *is real, is:*
(i). an Euclidean space, $\mathbb{R}^n$,*at any point* $x \in \mathcal{R}^n(q)$ *or* $x \in \mathcal{L}^n(q)$,
(ii). a pseudo-Euclidean space, $\mathbb{R}_l^n$,*at every point* $x \in \mathcal{E}_l^n(q)_+$ *or* $x \in \mathcal{H}_l^n(q)_-$,
(iii).a pseudo-Euclidean space, $\mathbb{R}_{l+1}^n$ *or* $\mathbb{R}_{l-1}^n$,*at every point* $x \in \mathcal{E}_l^n(q)_-$ *or* $x \in \mathcal{H}_l^n(q)_+$, *respectively.*

Proof. It is enough to observe that any quadratic form $\langle X\,', X\,' \rangle_f$, when $X\,'$ are representatives of some points of $\mathcal{E}_l^n(q)_+$ or $\mathcal{E}_l^n(q)_-$, will contains $l+1$ positive terms, while for the points of $\mathcal{H}_l^n(q)_+$ or $\mathcal{H}_l^n(q)_-$ will contains only l positive terms. ·

Remark 3. The isotropic cone of $\mathbb{R}_l^{n+1}$, defined by $\langle X\,', X\,' \rangle_f = 0$, limits two regions of $\mathcal{V}_l^n(q)$, known as 'proper domain' and 'ideal domain', while the cone itself is the 'absolute domain' of the non-Euclidean space.

In the sequel by notation $\alpha \longrightarrow 0$ we mean that α runs through a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ which is convergent with limit 0.

Let $\mathbf{a}$ be an arbitrary point of one of the non-Euclidean spaces $\mathcal{V}_l^n(q)$. The set ${}_{[\mathbf{a}]}S^{n-1} := \lim_{\alpha \longrightarrow 0} (H_{\alpha A} \cap \mathcal{V}_l^n(q))$ is said to be the polar hyperplane of the point $\mathbf{a}$. This is the variety that we call a *non-Euclidean hyperplane.*

Remark 4. It results that the polar hyperplane of the point $\mathbf{a}$, ${}_{[\mathbf{a}]}S^{n-1}$, is the limit of the hyperspheres of center $\mathbf{a}$, ${}_{\alpha}S^{n-1}(\mathbf{a})$, when $\alpha \longrightarrow 0$.

Definition 3. For $r \in \overline{1, n-1}$ let us fix $m = n - r$. Then, if the intersection $\cap_{i=1}^r \{\mathcal{V}_l^n(q) \cap H_{\alpha_i A_i}\} \doteq S^m$ is not empty, S^m is called a *non-Euclidean* m*- sphere.*

Consequently, for $\alpha_i \longrightarrow 0, (i \in \overline{1, r}), S^m$ will define a *non-Euclidean* m*-plane.*

4. Topological structures on a non-Euclidean space $\mathcal{V}_l^n(q)$

Let us denote by $\mathcal{V}$ the connect component of $\mathcal{V}_l^n(q)$, or even this space if it is connected. Let $\mathbf{x} \in \mathcal{V}$ and $T_{\mathbf{x}}(\mathcal{V})$ be a point and the corresponding tangent space. We also consider the $\mathbb{K}$-vector space $\mathbf{V}_l^{n+1}(q) \doteq \mathbf{V}$ of the representatives of points of $\mathcal{V}_l^n(q)$ and denote by $X\,'$ one of the representatives $\pm X$ of the chosen point, $\mathbf{x}$, of $\mathcal{V}$.

If $\{E_i, E_{n+1}\}, (i = 1, 2, ..., n)$,is an 'orthonormal' basis of $\mathbf{V}$ in selected it in such a manner that E_{n+1} should have the direction of $X\,'$, its subsystem

$\{E_i\}, \left(i \in \overline{1,n}\right)$, will constitute an orthonormal basis for $T_{\mathbf{x}}(\mathcal{V})$, and we have (for $\varepsilon = \pm 1,\ \varepsilon q = \rho^2$ and $q \in \mathbb{K}$

$$\langle E_i, E_j \rangle_f = \varepsilon \delta_{ij},\ \ \langle E_i, E_{n+1} \rangle_f = 0, (i,j = 1,2,...,n), \tag{3}$$

where $\langle \cdot, \cdot \rangle_f$ is the inner product on $\mathbf{V}$ defined by the nondegenerate bilinear form f, whose image on the pair of repeated last vector of the basis is $f(E_{n+1}, E_{n+1}) = q$, that complete the list of conditions (3).

At each point $\mathbf{x}$ of $\mathcal{V}$ we consider the subspace of $T_{\mathbf{x}}(\mathcal{V})$, $U_{\mathbf{x}\ |\ r} := \langle X_\alpha \rangle_r$, generated by the finite system of vector fields $\{X_\alpha\}, (\alpha = 1,2,...,r\ ;\ r \leq n)$; $X_o = 0$ and we put $U_{\mathbf{x}\ |\ o} = \langle 0 \rangle_o$ for the null subspace.

Now we define

$$U^{\perp}_{\mathbf{x}\ |\ r} := \{X_{\mathbf{x}} \in T_{\mathbf{x}}(\mathcal{V}) :\ \langle X_{\mathbf{x}}, X_{\boldsymbol{\alpha}} \rangle_f = 0, (\forall)\, X_{\boldsymbol{\alpha}} \in U_{\mathbf{x}\ |\ r}\}. \tag{4}$$

As well as in the case of $U_{\mathbf{x}\ |\ o}$ the condition from (4) is fulfilled for every $X_{\mathbf{x}}$, such that we have $U^{\perp}_{\mathbf{x}\ |\ o} = T_{\mathbf{x}}(\mathcal{V})$.As for the rest, $U^{\perp}_{\mathbf{x}\ |\ r}$ being a proper linear subspace of $T_{\mathbf{x}}(\mathcal{V})$, we have $\dim U^{\perp}_{\mathbf{x}\ |\ r} + \dim U_{\mathbf{x}\ |\ r} = n$, and, so, $U^{\perp}_{\mathbf{x}\ |\ r}$ is the orthogonal complement of the subspace $U_{\mathbf{x}\ |\ r} \leq T_{\mathbf{x}}(\mathcal{V})$. It is a nondegenerate subspace because of the restriction $f\ |T_{\mathbf{x}}(\mathcal{V})$, which is a nondegenerate bilinear form. This tells us that $T_{\mathbf{x}}(\mathcal{V}) = U^{\perp}_{\mathbf{x}\ |\ r} \oplus U_{\mathbf{x}\ |\ r}$.

Concerning these elements the following result was established ([4]):

Theorem 3. *Fixing $\lambda_o > 0$, for every $\mathbf{x} \in \mathcal{V}^n_l(q)$ we define the set*

$$\mathrm{V}_{\mathbf{x}\ [\lambda_o,\ r]} = \lambda X\ ' + \mathrm{U}^*_{\mathbf{x}\ |\ r},$$

where λ crosses one of the intervals $(\lambda_o, 1] \doteq \mathbb{I}_E$ if $q < 0$ or $[1, \lambda_o) \doteq \mathbb{I}_H$ if $q > 0$ (with λ_o chosen such that this thing be possible), $X\ '$ is one of the representatives $+X$ or $-X$ of the point $\mathbf{x}$ in $\mathbf{V}^{n+1}_l(q)$, and

$$U^*_{\mathbf{x}\ |\ r} = \{X_{\mathbf{x}} \in U^{\perp}_{\mathbf{x}\ |\ r} : \langle X_{\mathbf{x}}, X_{\mathbf{x}} \rangle_f = \rho^2(1 - \lambda^2)\}.$$

Let $\mathcal{U}_{\mathbf{x}}$ be a part of $\mathcal{V}^n_l(q)$ with the property that any be $Y \in V_{\mathbf{x}\ [\lambda_o,\ r]}$ this is a representative of a point $\mathbf{y} \in \mathcal{U}_{\mathbf{x}}$. Let us now symbolize by $\mathcal{V}_{\mathbf{x}}$ the family of these sets when $\mathbf{x}$ crosses $\mathcal{V}^n_l(q)$ and for every $r \leq n$.

In these conditions $\mathcal{V}_{\mathbf{x}}$ is a fundamental system of neighborhoods for a topology $\tau_{\mathcal{V}}$ on $\mathcal{V}^n_l(q)$.

Remarks 5. The family $\mathcal{V}_{\mathbf{x}} \subset \mathcal{P}(\mathcal{V}^n_l(q))$ is a basis for the topology $\tau_{\mathcal{V}}$ because a sufficient condition for this to be true (acc. to [11], Theorem 7.3) is that for every $\mathcal{U}^1_{\mathbf{x}}, \mathcal{U}^2_{\mathbf{x}} \in \mathcal{V}_{\mathbf{x}}$ we have $\mathcal{U}^1_{\mathbf{x}} \cap \mathcal{U}^2_{\mathbf{x}} \in \mathcal{V}_{\mathbf{x}}$.

6. For $r = 0$, $U^{\perp}_{\mathbf{x}\ |\ o}$ is a hyperplane of $\mathcal{V}^n_l(q)$. Because of this fact the topology $\tau_{\mathcal{V}}$ on $\mathcal{V}^n_l(q)$, defined by the fundamental system of neighborhoods $\mathcal{V}_{\mathbf{x}}((\forall)\, \mathbf{x} \in \mathcal{V}^n_l(q))$, is said to be a "topology of hyperplanes".

7. The neighborhoods of the form $\mathcal{U}_{\mathbf{x}}$ of a point $\mathbf{x} \in \mathcal{V}_l^n(q)$ can be reduced to open neighborhoods of that point if any be a point $\mathbf{y} \in \mathcal{U}_{\mathbf{x}}$ there exists $\lambda \in \mathbb{I}_E$(or, respectively, $\mathbb{I}_H$) such that its representative in $\mathbf{V}_l^{n+1}(q)$ can be set under the form $Y = \lambda X\ ' + X_{\mathbf{x}}$, and the following condition $\langle X_{\mathbf{x}}, X_{\mathbf{x}} \rangle_f = \rho^2(1-\lambda^2)$ holds.

Now we also have in view the 'natural topology' $\mathcal{T}_{\ \mathcal{V}}$ on $\mathcal{V}_l^n(q)$.It can be defined with the help of the family of open sets on some hyperquadrics Σ in $\mathbb{R}_l^{n+1}$, 'the models' of the corresponding non-Euclidean spaces $\mathcal{V}_l^n(q)$, as were put in evidence in the section **1.**

Thus, we can establish the following result:

Theorem 4. *Let us consider the space $\mathbb{R}_l^{n+1}$ endowed with the natural topology $\mathcal{T}$. If $\mathcal{V}_l^n(q)$ is one of the non-Euclidean space stated above and Σ is its model in $\mathbb{R}_l^{n+1}$, then to the intersection of the open sets belonging to $\mathcal{T}$ with Σ will correspond open sets on $\mathcal{V}_l^n(q)$ by the mapping which attaches to every point of the model the corresponding point of the non-Euclidean space represented.*

Proof. We consider the topological space $(\Sigma, \mathcal{T}_{\ \Sigma})$, whose topology is consisting in the family of sets $\mathcal{T}_{\ \Sigma} := \{G_\alpha \cap \Sigma\}_{\alpha \in A}$, where G_α is an open set of the natural topology $\mathcal{T}$ of $\mathbb{R}_l^{n+1}$.Let us denote by U the intersection of Σ with an open set of $\mathcal{T}$ and let G be that set of the family $\{G_\alpha\}_{\alpha \in A}$ whose intersection with Σ is U. Then $U \in \mathcal{T}_{\ \Sigma}$, hence it is open in Σ.

Thus $\mathcal{T}_{\ \Sigma}$ is an induced topology on Σ by the natural topology $\mathcal{T}$ on $\mathbb{R}_l^{n+1}$, the environmental space of the manifold consisting in all points of the hyperquadric.

Let us now consider the mapping $\Im$ defined on the topological space $(\Sigma, \mathcal{T}_{\ \Sigma})$ into $\mathcal{V}_l^n(q)$, which attaches to every point $\mathbf{x}\ ' = (x'^i)_{n+1} \in \Sigma$ the corresponding point $\mathbf{x}\ = (x^i)_{n+1}$ in the non-Euclidean space whose model is Σ. This mapping is an isometry. Together with the point $\mathbf{x}\ '$ having as image the point $\mathbf{x} \in \mathcal{V}_l^n(q)$ will have the same image $-\mathbf{x}\ '$ as well, whose coordinates differ by sign from those of the point $\mathbf{x}\ '$. Let $U_+ \in \mathcal{T}_{\ \Sigma}$ be an arbitrary open set containing the point $\mathbf{x}\ '$.If we put $\Im(U_+) = \mathcal{U}$, then from the definition of the mapping $\Im$, we also have $\Im(U_-) = \mathcal{U}$, where U_- denotes the part of $\mathbb{R}_l^{n+1}$ containing the points $-\mathbf{x}\ '$ when $\mathbf{x}\ '$ crosses U_+ and which is, evidently, a part of Σ, $\mathcal{T}_{\ \Sigma}$- open. Since $U_- \in \mathcal{T}_{\ \Sigma}$, the pre-image $\Im^{-1}(\mathcal{U})$ of the set $\mathcal{U} \subset \mathcal{V}_l^n(q)$ will be $\mathcal{T}_{\ \Sigma}$- open, that is an open set on Σ, because $U_+ \cup U_- \in \mathcal{T}_{\ \Sigma}$ $\cdot$

Then (according to [11] ,Theorems 10,11) the family $\mathcal{T}_{\ \mathcal{V}} \subset \mathcal{P}\ (\mathcal{V}_l^n(q))$, that consists in all the sets $\mathcal{U} \subset \mathcal{V}_l^n(q)$ of which pre-images by $\Im^{-1}$ belong to $\mathcal{T}_{\ \Sigma}$, is a topology on $\mathcal{V}_l^n(q)$. By this, the set $\mathcal{U}$ is $\mathcal{T}_{\ \mathcal{V}}-$open. Taking now $U = U_+$ or $U = U_-$, the assertion is proved.$\cdot$

Between the two topologies $\mathcal{T}_{\mathcal{V}}$ and $\tau_{\mathcal{V}}$ defined on $\mathcal{V}_l^n(q)$ by the previous two theorems there exists a certain relationship that will be emphasized below:

Theorem 5. *The topologies $\mathcal{T}_{\mathcal{V}}$ and $\tau_{\mathcal{V}}$ satisfy the relation of partial order $\mathcal{T}_{\mathcal{V}} < \tau_{\mathcal{V}}$, that is $\tau_{\mathcal{V}}$ is a finer topology on $\mathcal{V}_l^n(q)$ than $\mathcal{T}_{\mathcal{V}}$.*

Proof. Indeed, we observe that for every set $\mathcal{U}$ which is $\mathcal{T}_{\mathcal{V}}-$open a point $\mathbf{x} \in \mathcal{U}$ and a number λ_o can be found such that its corresponding neighborhood in $\mathcal{V}_{\mathbf{x}}$ for $r = 0$, $\mathcal{U}_{\mathbf{x}}$, to coincide with $\mathcal{U}$. It results that $\mathcal{U}$ is $\tau_{\mathcal{V}}$ -open, which ends the proof.·

Theorem 6. *The $\mathcal{V}_{\mathbf{x}}^o$ subfamily of $\tau_{\mathcal{V}}$ made up of all the $\mathcal{U}_{\mathbf{x}}$ neighborhoods (for $r = 0$) of the point $\mathbf{x} \in \mathcal{V}_l^n(q)$ and of $\mathcal{V}_l^n(q)$ itself generates properly a topology on the space $\mathcal{V}_l^n(q)$ which is exactly $\mathcal{T}_{\mathcal{V}}$.*

Proof. Let us consider the family $\mathcal{B}(\mathcal{V}_{\mathbf{x}}^o)$ containing all the finite intersections of elements from $\mathcal{V}_{\mathbf{x}}^o$.This is a basis because the intersections of two arbitrary elements from $\mathcal{B}(\mathcal{V}_{\mathbf{x}}^o)$ is the intersection of a finite number of elements from $\mathcal{V}_{\mathbf{x}}^o$ and, consequently, it can be found in $\mathcal{B}(\mathcal{V}_{\mathbf{x}}^o)$.Then, according to the Remark **5**. this is a basis for a topology on $\mathcal{V}_l^n(q)$. It results that $\mathcal{V}_{\mathbf{x}}^o$ is a subbasis of the same topology on $\mathcal{V}_l^n(q)$. Let us denote by $\tau_{\mathcal{V}}^o$ this topology. But, since a family of sets determines unically a topology for which it is subbasis and this one is the less finer topology containing the given family, it follows, according to Theorem **5**., that we have $\tau_{\mathcal{V}}^o = \mathcal{T}_{\mathcal{V}}$.

This ends the proof.

5. The metric structure on $\mathcal{V}_l^n(q)$

The metric structure of a non-Euclidean space $\mathcal{V}_l^n(q)$ follows from the formulas of angle between two non-Euclidean straight-lines at a point $\mathbf{x}$, defined as an angle between the tangent vectors in $T_{\mathbf{x}}(\mathcal{V})$ to the considered above lines. The original formulas (for pseudo-Euclidean spaces) can be found in [6] , ($pp.$49, 525), and may be applied in our case because the tangent space to $\mathcal{V}_l^n(q)$ at every $\mathbf{x}$ is one or another of the pseudo-Euclidean n - spaces $\mathbb{R}^n$, $\mathbb{R}^n_s$, $\mathbb{R}^n_{s-1}$, $\mathbb{R}^n_{s+1}$. In [7] , ($pp.$51, 127, 210, 211),B.A. ROZENFELD established the appropriate relations for the analyzed cases, separately.

In this section we want to give for all the cases presented in the first section a single formula for the distance between two points in anyone of the spaces contained in $\mathcal{V}_l^n(q)$.

To make it, the solutions of a system of two functional equations are used.

So we consider the following system of functional equations

$$C^2(\varphi) - q\ S^2(\varphi) = 1 \tag{5}$$

$$C(\varphi - \psi) = C(\varphi)C(\psi) - q\ S(\varphi)S(\psi), \tag{5'}$$

where $q \in \mathbb{R}$, and $C, S : \mathbb{R} \to \mathbb{R}$ are continuous unknown functions. We observe that (5,5') generalize the system of trigonometric equations that define the usual functions $\{\cos\varphi, \sin\varphi\}$, as well as the system defining the hyperbolic functions $\{\cosh\varphi, \sinh\varphi\}$.

If $\{C(\varphi), S(\varphi)\}$ denotes a solution of the system (5,5'), we can prove that the following pairs of functions are solutions of this system with respect to the chosen q :

$$C(\varphi) = \cos\ q\varphi,\quad S(\varphi) = \frac{1}{\sqrt{-q}}\sin\ q\varphi,\ (q < 0) \tag{6}$$

called 'elliptical functions',

$$C(\varphi) = 1,\quad S(\varphi) = \varphi,\ (q = 0), \tag{7}$$

called 'parabolic functions', and

$$C(\varphi) = \frac{1}{2}(e^{q\varphi} + e^{-q\varphi}),\quad S(\varphi) = \frac{1}{2\sqrt{q}}(e^{q\varphi} - e^{-q\varphi}), (q > 0), \tag{8}$$

called 'hyperbolic functions'.

Now we define the number $q \in \mathbb{K}(\leq \mathbb{R})$ by means of the equation $\varepsilon q = \rho^2$, where ρ denotes the radius of the hyperquadric Σ, the 'model' of $\mathcal{V}_l^n(q)$ in $\mathbb{R}_l^{n+1}$, and $\varepsilon = \pm 1$.

Theorem 7. *Let $\mathcal{V}$ be a connected component of a non-Euclidean space of index l and dimension n. The the distance d between two points x_1 and x_2 of $\mathcal{V}$ is given by*

$$C(\frac{d}{\rho}) = \frac{\langle X_1, X_2 \rangle_f}{\rho^2}, \tag{9}$$

where X_1 and X_2 are the representatives of the considered above points in the associated $\mathbb{K}$-space $\mathbf{V}_l^{n+1}(q)$, and f is the corresponding bilinear form.

Proof. It results immediately by comprising the elliptic and hyperbolic cases.$\cdot$

In (9) the $C(.)$ is one or another of the first components of the solutions (6) or (8) of the system (5,5'). The specific choice is made with respect to the type of non-Euclidean space we have in view, as will be mentioned below

6. The analytical manifold structure on $\mathcal{V}_l^n(q)$

Using the previous elements one can introduces a real analytical manifold structure on $\mathcal{V}_l^n(q)$ by means of an anlytical mapping $f : U \longrightarrow \mathbb{R}^n$, where

U is an ope set in the natural topology of the pseudo-Euclidean space, of dimension $n+1$ an index l. Moreover, we need of an appropriate frame on $\mathcal{V}_l^n(q)$ to express the local coordinates of the points; this one is defined as follows.

A *selfpolar frame* on $\mathcal{V}_l^n(q)$ is a system of $n+1$ points, $\mathbf{e}_i$, of the space such that for every $j \neq i, (i,j = 1,2,...,n+1)$, to have $\mathbf{e}_j \in {}_{[\mathbf{e}_i]}S^{n-1}$, where ${}_{[\mathbf{e}_i]}S^{n-1}$ is the polar hyperplane of the point $\mathbf{e}_i$ (see section **3.**). This frame will be denoted by $\mathbf{R}_a = \{\mathbf{e}_i\}_{n+1}$.

Now, we can formulate the following result:

Theorem 8. *On the non-Euclidean spaces $\mathcal{V}_l^n(q)$ one can introduces a differentiable real manifold structure, of class C^∞ and of dimension n.*

The proof actually consists in the construction of such a structure on $\mathcal{V}_l^n(q)$, defined simultaneously for all the spaces contained in it. The manifolds so defined will be pseudo-Riemannian manifolds of constant sectional curvature (in the sense of [12]).

With respect to $\mathbf{R}_a$ the Cartesian coordinates $u^k, (k = 1,2,...,n)$, of a point $\mathbf{x} \in \mathcal{V}_l^n(q)$ by the following relations are defined:

$$u^{n-p} := d(\mathbf{x}^{(p)}, {}_{[\mathbf{e}_{n-p},...,\mathbf{e}_n]}S^{n-p-1}),\ (p = 0,1,...,n-1), \tag{10}$$

where $\mathbf{x}^{(p+1)}$ denotes the projection of the point $\mathbf{x}^{(p)}, (\mathbf{x}^{(0)} = \mathbf{x})$,on the $(n-p-1)$ - planes $[\mathbf{e}_{n+1}, \mathbf{e}_1, ..., \mathbf{e}_{n-p-1}]$, and the function d is a distance on $\mathcal{V}_l^n(q)$, given by the length of the metric segment that connects the points $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(p+1)}$ and is entirely enclosed in the $\tau_{\mathcal{V}}-$open set $\mathcal{U}_{\mathbf{x}}^{(p)}$ for $r = n-1$, (see section **4.**).

From here it results that, as a function of the domain of parameter variation, $\mathbb{I}_E$ or $\mathbb{I}_H$, we have the following intervals of variation for the coordinates

$$-\pi\rho \leq u^1 \leq \pi\rho,\ -\frac{\pi}{2}\rho \leq u^k \leq \frac{\pi}{2}\rho, (k = 2,3,...,n),$$

whenever it is possible that $\lambda_o \longrightarrow 0$, and

$$-\infty \leq u^k \leq +\infty, (k = 1,2,...,n),$$

whenever it is possible that $\lambda_o \longrightarrow \infty$.

The u^k coordinates are connected with the corresponding angles in $\mathbf{V}_l^{n+1}(q)$ between the representatives X_p and X_{p+1} of the points $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(p+1)}$, for each $k = n-p$, by the following relations

$$\varphi^k = \frac{u^k}{\rho}(\equiv \frac{u^k}{\nu}\sqrt{-1}\ or\ \equiv \frac{u^k}{\nu}), \tag{11}$$

where $\rho \in \mathbf{C}_\nu(\cong \mathbb{K} + \nu\mathbb{K})$ is the radius of the model Σ of $\mathcal{V}_l^n(q)$ in the corresponding space $\mathbb{R}_l^{n+1}$,and$\{1,\nu\}$is the basis of the second order division algebra $\mathbf{C}_\nu$, defined in **1.**

Now we consider an open set $\mathcal{U} \in \tau_\mathcal{V}$ such that $\mathcal{U} \ni \mathbf{x}$ and also contains all its neighborhoods $\mathcal{U}_\mathbf{x}$ for every $r > 0$. Let χ be a homeomorphism of $\mathcal{U}$ into the arithmetic space $\mathbb{R}^n$. The coordinates of the point $\mathbf{x}$ in the local chart $(\mathcal{U},\chi)$ will be

$$u^k = (\xi^k \circ \chi)(\mathbf{x}), (k = 1, 2, ..., n), \tag{12}$$

where $\xi^k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the well known coordinate functions. The mapping χ can be analytically obtained by solving the equations which define its inverse mapping, χ^{-1},

$$x^k = q \prod_{\alpha=k+1}^{n} C(\frac{u^\alpha}{\rho})S(\frac{u^k}{\rho}), \tag{13}$$

$$x^{n+1} = \prod_{h=1}^{n} C(\frac{u^h}{\rho}), \quad (h, k = 1, 2, ..., n), \tag{13'}$$

where $(x^1, ..., x^{n+1}) = \eta(X)$ are the Weierstrass' coordinates of the representative X of $\mathbf{x}$ in the chart $(\mathbb{R}^{n+1}, \eta)$, $\mathbf{V} \cong \mathbb{R}^{n+1}$.

Here $\left\{C(\varphi^k), S(\varphi^k)\right\}$ are solutions of elliptic type of the system (5,5') in the case of the space $\mathcal{R}^n(q)$, and of hyperbolic type in the case of the space $\mathcal{L}^n(q)$. For the spaces $\mathcal{E}_l^n(q)_+$ and $\mathcal{H}_l^n(q)_-$ the first l functions are of elliptic type, while the remaining $n - l$ functions are of hyperbolic type; for the spaces $\mathcal{E}_l^n(q)_-$ and $\mathcal{H}_l^n(q)_+$, conversely.

According to the expressions (6-8) of the functions C and S, we observe these admit continuous derivatives of any order with respect to the variables u^k.

Besides of this, the choice of the charts whose geometrical domains are the sets $\mathcal{U}$ defined before to constitute a covering of $\mathcal{V}_l^n(q)$, as well as the chaange of the charts can be made such that to obtain an atlas of class C^∞ on the manifold $\mathcal{V}$.

Proposition 9. *The real non-Euclidean spaces $\mathcal{V}_l^n(q)$ are separable locally compact n - manifolds.*

Proof. Indeed, $\mathcal{V}_l^n(q)$ are real analytical manifolds which satisfy the condition: $\mathcal{V}_l^n(q)$ has dimension n at any point and, as a topological space, it is separable and locally compact. This results from the fact that the associate vector space $\mathbf{V}_l^{n+1}(q)$ is isomorphic with $\mathbb{R}_s^{n+1}$, for $s = n-l+1$, which has the mentioned above property because the field $\mathbb{R}$ itself is a nondiscrete normed field, complete with respect to the norm, and locally compact. $\cdot$

The metric characterization of the non-Euclidean spaces can be obtained from now by using the general characterization of the Riemannian or pseudo-Riemannian manifolds. For the symmetric Riemannian manifolds this is made by I. SZENTHE in [9] .

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