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VARIANT OF LAX-MILGRAM LEMMA FOR BANACH SPACES

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In the paper, it is shown that, by taking a condition on the difference $| a(u,v) - a(v,u) |$ in the Lax-Milgram Lemma, we may apply it to the same problem in a Banach space.

We recall the **Lax-Milgram Lemma**:

Let X be a real Hilbert space, let $a(\cdot,\cdot)$ be a continuous, coercitive, bilinear form defined on X, and let f be a continuous linear form on X. Then there exists one and only one element $u \in X$ which satisfies

$$(\forall) v \in X,\ a(u,v) = f(v).$$

We establish the conditions of our work.

Let X be a Banach space, $a : X \times X \to \mathbb{R}$ be a bilinear form and $f : X \to \mathbb{R}$ be a linear continuous function.

Assume that:

1) $\exists\, M > 0$ a.i. $| a(u,v) | \le M \| u \| \cdot \| v \|$ (boundness);
2) $\exists\, K^2 > 0$ a.i. $a(u,u) \ge K^2 \| u \|^2$ (coercitivity).

Let

$$\alpha = \sup_{u,v \neq 0} \frac{| a(u,v) - a(v,u) |}{\| u \| \| v \|}$$

From $| a(u,v) | \le M \| u \| \cdot \| v \|$ we get $| a(u,v) - a(v,u) | \le 2M \| u \| \| v \|$ and therefore there exists α and $\alpha \le 2M$.

We denote $b(u,v) = a(u,v) + a(v,u)$ and for all $x \in X$ we denote $g_x(v) = f(v) + a(v,x)$

Obviously b becomes a bilinear form which is symmetric, continuous, and coercitive on X and g_x is linear and continuous.

Denoting

$$J_x(u) = \frac{1}{2} b(u,u) - g_x(u) = a(u,u) - g_x(u),$$

and using Ritz method, there exists a unique $u_x \in X$ which satisfies $J_x(u_x) = \inf_{u \in X} J_x(u)$ and in addition satisfies a variational equation

$$b(u_x v) = g_x(v),\ (\forall)\ v \in X\ i.e.$$

$$a(u_x, v) + a(v, u_x) = f(v) + a(v, x),\ (\forall)\ v \in X$$

By starting with $u_0 \in X$, we define the sequence $(u_n)_{n \in N}$ given by

$$a(u_n, v) + a(v, u_n) = f(v) + a(v, u_{n-1}).$$

We prove that:

$$\|\, u_n - u_{n-1} \,\| \ \le \left(\tfrac{\alpha}{4K^2} + \tfrac{1}{2}\right) \|\, u_{n-1} - u_{n-2} \,\|,\ n \ge 2,$$

where α and K are the nonsymmetry constant and, respectively the coercitivity constant for a.

Remark 1. The following inequality holds:

$$a\left(u_n - u_{n-1} - \frac{1}{2}(u_{n-1} - u_{n-2}),\ u_n - u_{n-1} - \frac{1}{2}(u_{n-1} - u_{n-2})\right) \ge$$

$$\ge K^2 \left\| u_n - u_{n-1} - \frac{1}{2}(u_{n-1} - u_{n-2}) \right\|^2 .$$

From $a(u_n, v) + a(v, u_n) - f(v) - a(v, u_{n-1}) = 0$, and
$a(u_{n-1}, v) + a(v, u_{n-1}) - f(v) - a(v, u_{n-2}) = 0$,

we get

$$a(u_n - u_{n-1}, v) + a(v, u_n - u_{n-1}) = a(v, u_{n-1} - u_{n-2}).$$

Denoting $v_n = u_n - u_{n-1},\ n \ge 1$, we have

$$a(v_n, v) + a(v, v_n) = a(v, v_{n-1}) = \frac{1}{2}(a(v, v_{n-1}) + a(v_{n-1}, v)) +$$

$$+\frac{1}{2}(a(v, v_{n-1}) - a(v_{n-1}, v)).$$

We deduce immediately:

$$a\left(v_n - \frac{1}{2}v_{n-1}, v\right) + a\left(v, v_n - \frac{1}{2}v_{n-1}\right) = \frac{1}{2}\left[a(v, v_{n-1}) - a(v_{n-1}, v)\right],$$

$$a\left(v_n - \frac{1}{2}v_{n-1}, v_n - \frac{1}{2}v_{n-1}\right) =$$

$$= \frac{1}{4}\left[a\left(v_n - \frac{1}{2}v_{n-1}, v_{n-1}\right) - a\left(v_{n-1}, v_n - \frac{1}{2}v_{n-1}\right)\right] \leq$$

$$\leq \frac{\alpha}{4}\ \|v_{n-1}\| \cdot \left\|v_n - \frac{1}{2}v_{n-1}\right\|$$

and, from Remark 1, we get the inequalities:

$$K^2\left\|v_n - \frac{1}{2}v_{n-1}\right\| \leq \frac{\alpha}{4}\|v_{n-1}\| \cdot \left\|v_n - \frac{1}{2}v_{n-1}\right\|,$$

$$\left\|v_n - \frac{1}{2}v_{n-1}\right\| \leq \frac{\alpha}{4K^2}\ \|v_{n-1}\|,$$

$$\|v_n\| \leq \left\|v_n - \frac{1}{2}v_{n-1}\right\| + \frac{1}{2}\|v_{n-1}\| \leq \left(\frac{\alpha}{4K^2} + \frac{1}{2}\right)\|v_{n-1}\|.$$

Proposition 2. *If* $\Lambda = \frac{\alpha}{4K^2} + \frac{1}{2} < 1$, for $\alpha < 2K^2$, *the sequence* $(X_n)_n$ *is convergent.*

Proof. We give the calculation for showing that $(u_n)_{n\in\mathbb{N}}$ is a fundamental sequence in the Banach space. First, we have:

$$||u_n - u_{n-1}|| \leq \Lambda||u_{n-1} - u_{n-2}|| \leq ... \leq \Lambda^{n-1}||u_1 - u_0||;$$

therefore

$$||u_{n+p} - u_n|| \leq ||u_{n+p} - u_{n+p-1}|| + ... + ||u_{n+1} - u_n|| \leq$$
$$\leq (\Lambda^{n+p-1} + ... + \Lambda^n)||u_1 - u_0|| \leq \frac{\Lambda^n}{1-\Lambda}||u_1 - u_0||,\ \Lambda \in (0,1)$$

and $(u_n)_n$ is fundamental

As X is complete, it folows that $(u_n)_n$ is convergent.

Let $u^* = \lim u_n$. As $a(u_n, v)+a(v, u_n) = f(v)+a(v, u_{n-1})$ we have $a(u^*, v) = f(v),\ (\forall)\ v \in V$, therefore, by weakining the nonsymmetry of the bilinear form, we may extend the Lax-Milgram Lemma to a Banach space. In addition, the solution is the limit of a recurrence sequence obtained by minimizing the energy functional J.

References

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