An. Şt. Univ. Ovidius Constanţa Vol. 11(1), 2003, 93–100

# SOME PROPERTIES OF THE COMPOSITION ALGEBRAS

**Cristina Flaut**

*To Professor Silviu Sburlan, at his 60's anniversary*

**Abstract**

It is known that the composition algebras have many applications in physics. In this paper we give some properties of these algebras.

**Definition 1.** Let $A$ be an arbitrary algebra over the field $K$. It is called a **division algebra**, if for every $a, b \in A, a \neq 0$, each of the following equations:

$$ax = b,\ ya = b, a, b \in A, a \neq 0,$$

has a unique solution in $A$.

Every division algebra is a simple algebra. If it is an associative algebra, then $A$ is a unitary division algebra.

**Definition 2.** An algebra $A$ is called an **alternative algebra** if the following relations

$$x^2 y = x\,(xy)\, and yx^2 = (yx)\, x, \forall x, y \in A,$$

are satisfied.

**Definition 3.** An algebra $A$ is called a **power associative algebra** if every element of this algebra generated an associative subalgebra.

**Definition 4.** An algebra $A$ is called a **composition algebra** if a quadratic form $n : A \to K$ has been defined on the vector space $A$ such that:

$$n\,(xy) = n\,(x)\, n\,(y, )$$

for every $x, y \in A$, and the associated symmetric bilinear form $f : A \times A \to K,\ f\,(x, y) = \frac{1}{2}\,(n\,(x + y) - n\,(x) - n\,(y))\,,$ is non-degenerate. In this case we say that the form $n$ **permits composition** on $A$. The quadratic

Key Words: alternative algebra, composition algebra, division algebra, flexible algebra, Hurwitz algebra, power associative algebra.

Mathematical Reviews subject classification: 17D05; 17D99.

form $n$ is called the **norm** over $A$. The associated bilinear form $f$ is **associative** if and only if:

$$f(xy, z) = f(x, yz), \forall x, y, z \in A. \tag{1}$$

An unitary composition algebra is called a **Hurwitz** algebra. A Hurwitz algebra is an alternative algebra and a quadratic algebra, i.e. each element satisfies the relation:

$$x^2 - t(x)x + n(x) = 0, \text{with } t(x), n(x) \in K$$

and they are called the **trace** and the **norm** of the element $x$.

**Proposition 5.**[El, Pé; 96] (Hurwitz). *Every Hurwitz algebra is isomorphic with one of the following algebras:*

*i) The base field $K$, if char$K \neq 2$.*

*ii)* $\mathbb{K}(\alpha) = (K, \alpha), \alpha \neq 0$, *if the polynomial $X^2 + \alpha$ is irreducible over $K$, otherwise* $\mathbb{K}(\alpha) = K \oplus K$ .

*iii)* $\mathbb{H}(\alpha, \beta) = (\mathbb{K}(\alpha), \beta), \beta \neq 0$, *the algebra of generalized quaternions.*

*iv)* $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma), \gamma \neq 0$, *the algebra of generalized octonions or the Cayley-Dickson algebra.*□

**Proposition 6.**[Ko, Sh; 95] *Let $A$ be a Hurwitz algebra. The following sttements are equivalent:*

*i) There is $x \in A, x \neq 0$ such that $n(x) = 0$.*

*ii) There are $x, y \in A, x \neq 0, y \neq 0$ such that $xy = 0$.*

*iii) $A$ contains a non-trivial idempotent (i.e. $e \neq 0, 1$ such that $e^2 = e$ ).*□

**Definition 7.** If a Hurwitz algebra satisfies one of the equivalent conditions from Proposition 6, then it is called **a split Hurwitz algebra.**

Every Hurwitz algebra is split or it is a division algebra.( see [Ko; Sh, 90], p. 91). When the field $K$ is an algebraically closed field, then we find only the split algebras.

**Definition 8**. An algebra $A$ is called a **flexibile algebra** if the following relation:

$$x(yx) = (xy)x, \forall x, y \in A$$

is satisfied.

**Proposition 9.** ([El, Pé; 96], Lemma 2.4., and [My; 86], p. 68, Lemma 2.3.). *Let $A$ be a composition algebra, over the field $K$ with char$K \neq 2$. Then the associated nondegenerate symmetric bilinear form $f$ permits composition and it is associative if and only if the following relations:*

$$(xy)x = x(yx) = f(x, x)y, \tag{2}$$

*are true.*

**Proposition 10.**[Ok, Os; 81] *An algebra $A$, over the field $K$ with char$K \neq 2$, with nondegenerate associative symmetric bilinear form permitting composition, has dimension* $1, 2, 4, 8$.

**Proof.** Since the bilinear form $f$ is nondegenerate we can find $a \in A$ such that $f(a, a) \neq 0$,and let $x = f^{-1}(a, a)\, a^2$. Then $f(x, x) = f^{-2}(a, a)\, f\left(a^2, a^2\right) = 1$, since $f$ permits composition. We define a new multiplication on $A$ :

$$v * w = (xv)(wx). \tag{3}$$

It follows from the relation (2) that $x^2$ is the identity element for the algebra $(A, *)$. Indeed, we have:
$x^2 * y = \left(xx^2\right)(yx) = f(x, x)\, x\,(yx) = f(x, x)\, y = y$ and
$y * x^2 = (xy)\left(x^2 x\right) = f(x, x)(xy)\, x = x$.
Moreover, the bilinear form $f$ permits composition on the algebra $(A, *)$ :
$f(v * w, v * w) = f((xv)(wx), (xv)(wx)) = f(xv, xv)\, f(wx, wx) =$
$= f(x, x)\, f(v, v)\, f(w, w)\, f(x, x) = f(v, v)\, f(w, w)$. So that $(A, *)$ is a Hurwitz algebra and has the dimension $1, 2, 4, 8$.□

**Proposition 11.**[Fl; 01] *Let $A$ be a division composition algebra, $f : A \times A \to K$, $n : A \to K$, be the associated bilinear form and the norm form. Then, for every $v, w \in A$, with $v \neq 0, w \neq 0$, we have*

$$f^2(v, w) = f(v, v)\, f(w, w)$$

*if and only if $v = rw$ with $r \in K, r \neq 0$.*

**Proof.** If $v = rw, r \in K$, then the equality holds.

Reciprocally, if $f^2(v, w) = f(v, v)\, f(w, w)$, for $v \neq 0, w \neq 0$, it results $f^2(v, w) \neq 0$. We suppose that doesn't exist $r \in K$ such that $v = rw$. Then, for every two nonzero elements $a, b \in K$ we have $av + bw \neq 0$.(Indeed, if $av + bw = 0 \Rightarrow v = -\frac{b}{a} w$ with $-\frac{b}{a} \in K$, false). Hence we obtain

$$f(av + bw, av + bw) \neq 0.$$

We get then $a^2 f(v, v) + b^2 f(w, w) + 2ab f(v, w) \neq 0$. Let $a = f(w, w)$. We have $f(w, w)\, f(v, v) + b^2 + 2b f(v, w) \neq 0$ and for $b = -f(v, w)$ we obtain $f(w, w)\, f(v, v) + f^2(v, w) - 2f^2(v, w) \neq 0$. Hence we get the relation:

$$f(w, w)\, f(v, v) \neq f^2(v, w),$$

that it isn't true. Therefore $av + bw = 0 \Rightarrow v = rw$.□

**Proposition 12**. *If $A$ is a division Hurwitz algebra, and $a, b \in A$ with $t(a) = t(b) = 0$, then the equality*

$$n(a)\,n(b) = \frac{1}{4}(ab+ba)^2 \tag{4}$$

*is true if and only if $a = rb$, with $r \in K$. If $n(a) = n(b)$, then $r = 1$ or $r = -1$.*

**Proof**. Let $A$ be a Hurwitz algebra. By Proposition 5, we know that there are only four types of Hurwitz algebras. If $A = K$ or $A = \mathbb{K}(\alpha)$, these relations are true.

**Case I**. $A = \mathbb{H}(\alpha, \beta)$. The relation (4) is equivalent with $n(ab) = (\alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3)^2$. We get then

$$ab = -(\alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3) + (-a_3 b_2 \beta + a_2 b_3 \beta)\, e_1 +$$

$$+(-a_1 b_3 \alpha + a_3 b_1 \alpha)\, e_2 + (a_1 b_2 - a_2 b_1)\, e_3,$$

therefore $n(ab) = (\alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3)^2 + \alpha(-a_3 b_2 \beta + a_2 b_3 \beta)^2 +$

$$+\beta(-a_1 b_3 \alpha + a_3 b_1 \alpha)^2 + \alpha\beta(a_1 b_2 - a_2 b_1)^2.$$

Relation (4) is equivalent with

$$\alpha(-a_3 b_2 \beta + a_2 b_3 \beta)^2 + \beta(-a_1 b_3 \alpha + a_3 b_1 \alpha)^2 + \alpha\beta(a_1 b_2 - a_2 b_1)^2 = 0.$$

Then the element

$$(-a_3 b_2 \beta + a_2 b_3 \beta)\, e_1 + (-a_1 b_3 \alpha + a_3 b_1 \alpha)\, e_2 + (a_1 b_2 - a_2 b_1)\, e_3$$

has the norm zero and we obtain $-a_3 b_2 \beta + a_2 b_3 \beta = -a_1 b_3 \alpha + a_3 b_1 \alpha = a_1 b_2 - a_2 b_1 = 0$, since $\mathbb{H}(\alpha, \beta)$ is a division algebra.

If $n(a) = n(b)$, by equality (4) it results that the relation $(n(a) + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3)(n(a) - \alpha a_1 b_1 - \beta a_2 b_2 - \alpha\beta a_3 b_3) = 0$ is true if and only if $a = rb, r \in K$. Replacing $a = rb$ in the last relation we get $(n(a) + rn(a))(n(a) - rn(a)) = 0$, therefore $n(a)^2(1+r)(1-r) = 0$. It results $r = -1$ or $r = 1$.

**Case II**. $A = \mathbb{O}(\alpha, \beta, \gamma)$. Indeed, we make the ccomputation like in the case I, the relation (4) is equivalent to:

$$n(a)\,n(b) = (\alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3 + \gamma a_4 b_4 + \alpha\gamma a_5 b_5 + \beta\gamma a_6 b_6 + \alpha\beta\gamma a_7 b_7)^2.$$

If $n(a)$=$n(b)$ , since $a$=$rb$, we obtain $(n(a)$-$rn(a))(n(a)+rn(a))$=0,therefore $r = 1$ or $r = -1$.□

**Proposition 13.** [Fl; 01] *Let $A$ be an unitary division power associative algebra (possibly infinite dimensional). Then every subalgebra of $A$ is unitary.*

**Proof.** Let $B$ be a subalgebra of $A$ and $b \in B, b \neq 0$ We denote with $B(b)$ the $B-$ subalgebra generated by $b$, which is an associative division algebra. We get that $B(b)$ is unitary, therefore $B$ is unitary.□

**Proposition 14.**[Fl; 02] *Let $(A, *)$ be a non-unitary algebra over the field $K$, with the symmetric nondegenerate bilinear form $S$ and the associated quadratic form, denoted in the same way, satisfyng the relations:*

$$(a * b) * a = a * (b * a) = S(a)\, b. \tag{5}$$

*If the elements $\{x, x * x\}$ are linearly dependent for all $x \in A$, then*

$$x * x = \alpha(x)\, x$$

*and $\alpha : A \to K$ is a $K$-algebra morphism.*

**Proof.** If in the relation (5) we take $a = b = x$ and we have: $(x * x) * x = S(x)\, x$. It results $(\alpha(x)\, x) * x = S(x)\, x$, therefore $\alpha^2(x)\, x = S(x)\, x$, and we obtain:

$$\alpha^2(x) = S(x) \tag{6}$$

Let the elements $x, y \in A$ be arbitrarily chosen . Then we get: $(x + y) * (x + y) = \alpha(x + y)(x + y)$ , therefore

$$x * x + x * y + y * x + y * y = \alpha(x + y)\, x + \alpha(x + y)\, y \Rightarrow$$
$$\Rightarrow \alpha(x)\, x + \alpha(y)\, y + x * y + y * x = \alpha(x + y)\, x + \alpha(x + y)\, y,$$

and we obtain:

$$(\alpha(x+y) - \alpha(x))\, x + (\alpha(x+y) - \alpha(y))\, y = x * y + y * x, \forall x, y \in A. \tag{7}$$

Since

$$S(x, y) = \frac{1}{2}(S(x+y) - S(x) - S(y)) = \frac{1}{2}\left(\alpha^2(x+y) - \alpha^2(x) - \alpha^2(y)\right),$$

we have:

$$\alpha^2(x + y) = \alpha^2(x) + \alpha^2(y) + 2S(x, y), \forall x, y \in A. \tag{8}$$

We apply $S\left( \ ,x\right)$ to the relation (7) and we obtain:

$$\left( \alpha\left( x+y\right) -\alpha\left( x\right) \right) S\left( x,x\right) +\left( \alpha\left( x+y\right) -\alpha\left( y\right) \right) S\left( x,y\right) =$$

$$=S\left( x\ast y,x\right) +S\left( y\ast x,x\right) .$$

By hypothesis we get then $S\left( \ ,\right)$ is associative and permits composition, no matter if $\omega$ is in $K$ or not. Therefore $S\left( x\ast y,x\right) =S\left( x,x\ast y\right) =S\left( x\ast x,y\right) =\alpha\left( x\right) S\left( x,y\right)$and $S\left( y\ast x,x\right) =S\left( y,x\ast x\right) =\alpha\left( x\right) S\left( x,y\right) .$

We have :

$$(\alpha(x+y)\text{-}\alpha(x))\alpha^{2}(x)\text{+}(\alpha(x+y)\text{-}\alpha(y))S(x,y)=2\alpha(x)S(x,y),\forall x,y\in A. \tag{9}$$

We denote $\alpha\left( x+y\right) =z,\alpha\left( x\right) =a,\alpha\left( y\right) =b$ and (8) and (9) becomes:

$$z^{2}=a^{2}+b^{2}+2S\left( x,y\right) , \tag{10}$$

$$\left( z-a\right) a^{2}=\left( b+2a-z\right) S\left( x,y\right) . \tag{11}$$

In the next we suppose $\alpha\left( x\right) \neq0.$ If $b+2a-z=0,$ it results $\alpha\left( y\right) +2\alpha\left( x\right) =\alpha\left( x+y\right) ,$for every $x,y\in A.$ If we let $y=0,$ we obtain $2\alpha\left( x\right) =\alpha\left( x\right) ,$ therefore $\alpha\left( x\right) =0,$ false. Therefore $b+2a-z\neq0.$ By relation $\left( 11\right) ,$ we have $S\left( x,y\right) =\frac{\left( z-a\right) a^{2}}{b+2a-z}$ and replace this in the relation $\left( 10\right) .$ It results $z^{2}=a^{2}+b^{2}+2\frac{\left( z-a\right) a^{2}}{b+2a-z},$ therefore $\left( b\text{+}2a\text{-}z\right) z^{2}\text{=}\left( a^{2}+b^{2}\right) \left( b+2a-z\right) +2a^{2}\left( z-a\right) .$ We obtain:

$$-z^{3}+\left( b+2a\right) z^{2}+\left( b^{2}-a^{2}\right) z-b\left( a+b\right) ^{2}=0.$$

The polynomial $P\left( Z\right) \text{=-}Z^{3}\text{+}\left( b\text{+}2a\right) Z^{2}\text{+}\left( b^{2}\text{-}a^{2}\right) Z\text{-}b\left( a\text{+}b\right) ^{2}$has the next decomposition in to irreducible factors:

$$P\left( Z\right) =-\left( Z-a-b\right) ^{2}\left( Z+b\right) ,$$

therefore $z_{1}=a+b$ and $z_{2}=-b$ are the roots of the polynomial $P.$ If $z=a+b,$ it results that $\alpha\left( x+y\right) =\alpha\left( x\right) +\alpha\left( y\right) .$ If $z\text{=}-b,$ we obtain $\alpha\left( x+y\right) =-\alpha\left( y\right)$ and for $y=0,$ we have $\alpha\left( x\right) =0,$ contradiction. Therefore $\alpha\left( x+y\right) =\alpha\left( x\right) +\alpha\left( y\right) ,\forall x,y\in A,$with $\alpha\left( x\right) \neq0.$ If $\alpha\left( x\right) =0,$ by relation $\left( 9\right) ,$ we have $\left( z-b\right) S\left( x,y\right) =0$ for every $y\in A$ and $x\in A$ with $\alpha\left( x\right) =0.$ Let $z=b.$ We get then:

$$\alpha\left( x+y\right) =\alpha\left( y\right) =\alpha\left( y\right) +0=\alpha\left( y\right) +\alpha\left( x\right) .$$

If $S\left( x,y\right) =0,\forall y\in A,$we obtain $x=0\Rightarrow\alpha\left( 0\right) =0,$therefore:

$$\alpha\left( x+y\right) =\alpha\left( x\right) +\alpha\left( y\right) ,\forall x,y\in A. \tag{12}$$

In the relation (7) let $y = ax$, with $a \in K$ and we have:

$$\alpha\left(\left(a+1\right)x\right)x - \alpha\left(x\right)x + a\alpha\left(\left(a+1\right)x\right)x - a\alpha\left(ax\right)x = 2a\alpha\left(x\right)x \Rightarrow$$

$$\Rightarrow \alpha\left(ax\right)x + \alpha\left(x\right)x - \alpha\left(x\right)x + a\alpha\left(ax\right)x + a\alpha\left(x\right)x - a\alpha\left(ax\right)x = 2a\alpha\left(x\right)x,$$

therefore:

$$\alpha\left(ax\right) = a\alpha\left(x\right), \forall x \in A. \tag{13}$$

Since $S\left(\,,\right)$ permits composition, we obtain that

$$S\left(x*y, x*y\right) = S\left(x,x\right)S\left(y,y\right) \Rightarrow S\left(x*y\right) = S\left(x,x\right)S\left(y,y\right)$$

Therefore

$$\alpha\left(x*y\right) = \alpha\left(x\right)\alpha\left(y\right). \tag{14}$$

By the relations $(12), (13), (14)$ it results that $\alpha$ is a $K$-algebra morphism.□

**Proposition 15**. *Let $A$ be a composition algebra satisfying the conditions of the Proposition 14. If, in addition, $A$ is an alternative algebra and for every two elements $x, y \in A$ we have that $x*y = \alpha\left(x,y\right)y*x$, $\forall x,y \in A$, with $\alpha\left(x,y\right) \in K$, then $dim_K A \leq 2$.*

**Proof**. We know $\left(x*y\right)*x = x*\left(y*x\right) = S\left(x\right)y$. We get then $\alpha\left(x,y\right)\left(y*x\right)*x = \alpha\left(y,x\right)x*\left(x*y\right)$. Since $A$ is an alternative algebra, we obtain:

$$\begin{aligned}\alpha\left(x,y\right)\alpha\left(x\right)y*x &= \alpha\left(y,x\right)\alpha\left(x\right)x*y \Rightarrow \alpha\left(x,y\right)\alpha\left(x\right)y*x = \\ &= \alpha\left(y,x\right)\alpha\left(x\right)\alpha\left(x,y\right)y*x,\end{aligned}$$

therefore $\alpha\left(x,y\right)\alpha\left(x\right)=\alpha\left(y,x\right)\alpha\left(x\right)\alpha\left(x,y\right)$ and we have $\alpha\left(y,x\right)=1$. It results that $A$ is a commutative algebra. We apply the Propositions 9 and 10, and we find the element $a \in A$ such that $S\left(a,a\right) \neq 0$. We denote

$$\theta = S^{-1}\left(a,a\right)a*a = S^{-1}\left(a\right)a*a = \alpha^{-2}\left(a\right)\alpha\left(a\right)a = \alpha^{-1}\left(a\right)a.$$

Then $S\left(\theta,\theta\right)=S^{-2}\left(a,a\right)S\left(a*a,a*a\right)=\alpha^{-4}\left(a\right)\alpha^{2}\left(a\right)\alpha^{2}\left(a\right)=1$, since $S$ permits composition. But $1 = S\left(\theta,\theta\right) = S\left(\theta\right) = \alpha^{2}\left(\theta\right)$, and we obtain $\alpha^{2}\left(\theta\right) = 1$. We define a new multiplication on $A$ :

$$v\nabla w = \left(\theta*v\right)*\left(w*\theta\right),$$

and by calculus we obtain that $\theta^2 = \theta*\theta = \theta$ is the unity of the algebra $\left(A,\nabla\right)$. With this multiplication the algebra $A$ is a Hurwitz algebra. Since the multiplication "$*$" is commutative the multiplication "$\nabla$" is commutative, therefore $\dim_K A \leq 2.\square$

## References

[El, Pé; 96] Elduque, A., Pérez, J. M., *Composition algebras with associative bilinear form,* Comm. Algebra **24,** No 3 (1966) ,1091-1116.

[Fl; 01] Flaut, C., *Some equations in algebras obtained by the Cayley-Dickson process,* Analele Universitaţii " Ovidius " Constanţa, **9**, fasc. 2 (2001) .

[Fl; 02] Flaut, C., *Associativity condition for some alternative algebras of degree three*, Proceedings of the PAMM - Conference PC-139 / 2002, Constantza, 65-73.

[Ko, Sh; 90] Kostrikin, A. I., Shafarevich, I.R. (Eds), *Algebra I,* Springer-Verlag, 1990.

[Ko, Sh; 95] Kostrikin, A. I., Shafarevich, I.R. (Eds), *Algebra VI,* Springer-Verlag, 1995.

[Ok,Os; 81] Okubo, S., Osborne, J.M., *Algebras with nondegenerate associative symmetric bilinear forms permitting composition, I,* Comm. Algebra **9,** No 12,1981, 1233-1261.

[Sc; 66] Schafer, R. D., *An Introduction to Nonassociative Algebras,* Academic Press, New-York, 1966.

"Ovidius" University,
Departament of Mathematics,
Bd. Mamaia 124,
8700 Constantza,
Romania
e-mail: cflaut@univ-ovidius.ro