An. Şt. Univ. Ovidius Constanţa Vol. 11(1), 2003, 105–112

THE USE OF LAGUERRE-SONINE POLYNOMIALS IN SOLVING BOLTZMANN'S EQUATION (II)

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

Generally, one can solve Botzmann's equation only by using approximation methods, and the expansion of the distribution function in spherical harmonics. In this paper, we shall study Boltzmann's equation for a fully ionised inhomogeneous plasma with Laguerre-Sonine polynomials as coefficients of the spherical harmonics expansion. We establish also the cross-coupling relations between Laguerre-Sonine polynomials of different orders, useful relations in order to obtain the approximative solutions of Boltzmann's equation.

1. Introduction

We base our considerations on the following assumptions:
(i) The Laguerre-Sonine polynomials are of the form:

$$L_r^{l+\frac{1}{2}}\left(p^2 v^2\right).$$

(ii) The weight function coincides with the equilibrium distribution function of electrons.

For Boltzmann's equation for electrons, we take the following form :

$$\frac{\partial f}{\partial t} + \overrightarrow{v}\nabla_r f + \frac{e}{m_e}\left[\overrightarrow{E} + \frac{1}{c}\left(\overrightarrow{v}\times\overrightarrow{B}\right)\right]\nabla_v f =$$

$$= Nv\int f(\overrightarrow{v})\sigma\left(v^{'},\chi\right)d^2\Omega^{'} - Nv^{'}f\int \sigma\left(v,\chi\right)d^2\Omega^{'}. \tag{1}$$

In order to solve eq. (1) in agreement with (i) and (ii) , we assume for distribution function the following expansion:

$$f\left(\overrightarrow{r}, v\overrightarrow{\Omega}, t\right) = \frac{\beta^3\left(\overrightarrow{r}, t\right)}{\pi^{\frac{3}{2}}} e^{-\beta^2 v^2} \times \sum_{i=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{r=0}^{+\infty} R_r^{l,m}\left(\overrightarrow{r}, t\right) L_r^{l+\frac{1}{2}}\left(\beta^2 v^2\right) Y_{l,m}\left(0, \varphi\right) \tag{2}$$

where:

$$\beta = \sqrt{\frac{m}{2kT\left(\overrightarrow{r}, t\right)}},$$

$L_r^{l+\frac{1}{2}}\left(\beta^2 v^2\right)$ are the Laguerre-Sonine polynomials:

$$L_r^{l+\frac{1}{2}}\left(\beta^2 v^2\right) = \frac{e^{\beta^2 v^2}\left(\beta^2 v^2\right)^{-\left(l+\frac{1}{2}\right)}}{r!} \frac{d^r}{d\left(\beta^2 v^2\right)}\left[\beta^2 v^{2r+l+\frac{1}{2}} e^{-\beta^2 v^2}\right] \tag{3}$$

and $Y_{l,m}\left(\theta, \varphi\right)$ are the spherical harmonics.

A development of the distribution function only in spherical harmonics permits to ensure that the Maxwellian character of the equilibrium distribution function be not altered by the anisotropy due to the presence of both electric and magnetic fields, even in the relativistic case. The assumed double expansion (2) maintains this property. Indeed, for $m = l = r = 0$, one obtains the maxwellian distribution function.

$$\left[L_0^{\frac{1}{2}} = Y_{0,0} = 1 \qquad and \qquad R_0^{0,0}\left(\overrightarrow{r}, t\right) = n_0\left(\overrightarrow{r}, t\right)\right]$$

2. Derivation of the cross-coupling relations

Substituting (2) into (1), we obtain:

$$\begin{aligned}
A_t^{l,m} \quad = \quad & -\frac{1}{2\left(2l+3\right)}\left(l+m+1\right)\left(l+m+2\right)\cdot \\
& \cdot\left[vA_x^{l+1,m+1} - iA_y^{l+1,m+1} - \frac{e}{m_e}\left(E_z - iE_y\right)\left(\frac{l}{v}C^{l+1,m+1} - D_v^{l+1,m}\right)\right] - \\
& -\frac{1}{2l+3}\left(l+m+1\right)\left[vA_z^{l+1,m} - \frac{c}{m_e}E_z\left(\frac{l}{v}C^{l+1,m} - D_v^{l+1,m}\right)\right] + \\
& \quad + \frac{1}{2\left(2l+3\right)}\left[v\left(A_z^{l+1,m-1} + iAl+1, m-1_y\right) - \right. \\
& \quad \left. - -\frac{e}{m_e}\left(E_z + iE_y\right)\left(\frac{l}{v}C^{l+1,n-1} - D_v^{l+1,m-1}\right)\right] +
\end{aligned}$$

$$+\frac{1}{2\left(2l-1\right)}\left(l-m-1\right)\left(l-m\right)\left[v\left(A_z^{l-1,m+1}-iA_y^{l-1,m+1}\right)\right]+$$

$$+\frac{e}{m_e}\left(E_z-iE_y\right)\left(\frac{l+1}{v}C^{l-1,m+1}+D_v^{l-1,m+1}\right)-$$

$$-\frac{1}{2l-1}\left(l-m\right)\left[vA_z^{l-1,m}+\frac{e}{m_e}E_z\left(\frac{l+1}{v}C^{l-1,m}+D_v^{l-1,m}\right)\right]-$$

$$-\frac{1}{2\left(2l-1\right)}\left[v\left(A_x^{l-1,m-1}+iA_y^{l-1,m-1}\right)+\frac{e}{m_e}\left(E_z+iE_y\right)\left(\frac{l}{v}C_{l-1,m-1}+D_v^{l-1,m-1}\right)\right]+$$

$$+\left(l+m+1\right)\left(l-m\right)\frac{e}{m_e c}\frac{i}{2}\left(B_x-iB_y\right)C^{l,m}+m\frac{e}{m_e c}iB_zC^{l,m}-$$

$$-\frac{e}{m_e c}i\left(B_x-iB_y\right)C^{l,m-1}+4\pi Nv\left[\frac{(l+m)!}{(l-m)!}\sigma_l-\sigma_0\right]C^{l,m}, \tag{4}$$

where we have written:

$$A_x^{l,m}=\left(3-2\beta^2v^2\right)\beta^2\frac{\partial\beta}{\partial x_i}e^{-\beta^2v^2}\sum_{r=0}^{\infty}R_r^{l,m}L_r^{l+\frac{1}{2}}-$$

$$-\beta^3e^{-\beta^2v^2}\sum_{r=0}^{\infty}\frac{\partial R_r^{l,m}}{\partial x_i}L_r^{l+\frac{1}{2}}+2\beta^4v^2e^{-\beta^2v^2}\sum_{r=0}^{\infty}R_r^{l,m}\frac{\partial L_r^{l+\frac{1}{2}}}{\partial\left(\beta^2v^2\right)}\frac{\partial\beta}{\partial x_1} \tag{5}$$

(here $i=1,2,3,4,$ so : $x_1=x,x_2=y,x_3=z,x_4=l$)

$$C^{l,m}=\beta^3e^{-\beta^2v^2}\sum_{r=0}^{\infty}R_r^{l,m}L_r^{l+\frac{1}{2}} \tag{6}$$

and

$$D_v^{l,m}=2\beta^5ve^{-\beta^2v^2}\sum_{r=0}^{\infty}R_r^{l,m}\left(\frac{\partial L_r^{l+\frac{1}{2}}}{\partial\left(\beta^2v^2\right)}-L_r^{l+\frac{1}{2}}\right). \tag{7}$$

Taking into account the following recurrence relations for Laguerre polynomials:

$$\beta^3v^2\frac{\partial L_r^l\left(\beta^2v^2\right)}{\partial\left(\beta^2v^2\right)}=rL_r^l\left(\beta^2v^2\right)-\left(r+l\right)L_{r-1}^l\left(\beta^2v^2\right) \tag{8}$$

and

$$\beta^2v^2L_r^l\left(\beta^2v^2\right)=\left(2r+l+1\right)L_r^l\left(\beta^2v^2\right)-$$

$$-\left(r+l\right)L_{r-1}^l\left(\beta^2v^2\right)-\left(r+1\right)L_{r+1}^l\left(\beta^2v^2\right), \tag{9}$$

the expressions (5) and (7) become:

$$A_{x_i}^{l,m} = 2\beta^2 \frac{\partial \beta}{\partial x_i} e^{-\beta^2 v^2} \sum_{r=0}^{\infty} R_r^{l,m} \left[(r+1) L_{r+1}^{l+\frac{1}{2}} - (r+l) L_r^{l+\frac{1}{2}} \right] +$$

$$+\beta^3 e^{-\beta^2 v^2} \sum_{r=0}^{\infty} \frac{\partial R_0^{l,m}}{\partial x} L_r^{l+\frac{1}{2}} \tag{$5'$}$$

and

$$D_v^{l,m} = 2\beta^3 e^{-\beta^2 v^2} \sum_{r=0}^{\infty} R_r^{l,m} \left[-\frac{1}{v} r L_r^{l+\frac{1}{2}} + \beta^3 v L_r^{l+\frac{1}{2}} + \frac{1}{v} (r+l) L_{r-1}^{l+\frac{1}{2}} \right]. \tag{$7'$}$$

Substituting ($5'$) and ($7'$) into (4), we obtain a set of eqs. which contains Laguerre-Sonine polynomials corresponding to the indices $l+\frac{1}{2}, l+\frac{3}{2}, l-\frac{1}{2}$. In order to utilize the orthogonality relation for these polynomials

$$\int e^{-\beta^2 v^2} L_r^{l+\frac{1}{2}} \left(\beta^2 v^2\right) L_k^{l+\frac{1}{2}} d\left(\beta^2 v^2\right) =$$

$$= \begin{cases} 0, \quad for \quad k \neq 0 \\ \frac{\Gamma\left(r+l+\frac{3}{2}\right)}{r!} \quad for \quad k = r \ , \end{cases} \tag{10}$$

they must have the upper index. Multiplying now (5'),(6) and ($7'$) by v, then applying relations:

$$\beta^2 v^2 L_r^l \left(\beta^2 v^2\right) = (r+l) L_r^{l-1} \left(\beta^2 v^2\right) - L_r^{l-1} \left(\beta^2 v^2\right) \tag{11}$$

and

$$\beta^2 v^2 L_r^l \left(\beta^2 v^2\right) = 3 (r+l+2) L_r^{l+1} \left(\beta^2 v^2\right) - (3r+2l+1) L_{r-1}^{l+1} \left(\beta^2 v^2\right) -$$

$$- (r+1) L_{r+1}^{l+1} \left(\beta^2 v^2\right) + (r+l) L_{r+2}^{l+2} \left(\beta^2 v^2\right) \tag{12}$$

we obtain for (4) an expansion which contains Laguerre-Sonine polynomials of the same upper index. Multiplying this result by $\left(\beta^2 v^2\right)^{l+\frac{1}{2}} d\left(\beta^2 v^2\right) L_k^{l+\frac{1}{2}}$ and subsequently performing the integration over the interval $(0, \infty)$ and summing relatively to the index mk, we obtain the following infinite set of equations:

$$2\beta^2 \frac{\partial \beta}{\partial t} R_t^{l,m} \left[(r+1) L_{r+1,k+1}^l - (r+l) L_{r,k}^l \right] + \beta^3 \frac{\partial R_r^{l,m}}{\partial t} L_r^l =$$

$$\frac{1}{2(2l+r)} \sum_{\pm} \left[\begin{array}{l} +(l+m+1)(l+m+2) \\ -1 \end{array} \right]$$

$$\left\{ \sum_{j=0,1,2} (-1)^j \Re_j R_{r+j}^{l+1,m\pm1} \left(\frac{\partial \beta}{\partial x} \mp i \frac{\partial \beta}{\partial y} \right) - \right.$$

$$\left. - \sum_{j=0,1} (-1)^j \mathcal{B}_j \left[\beta \left(\frac{\partial R_{r+j}^{l+1,m\pm1}}{\partial x} \mp \frac{\partial R_{r+j}^{l+1,m\pm1}}{} \right) + \frac{2e}{m_e} \beta^3 \left(E_z \mp i E_y \right) R_{r+j}^{l+1,m\pm1} \right] \right\} +$$

$$+ \frac{1}{2(2l-1)} \sum_{\pm} \left[\begin{array}{l} +(l-m-1)(l-m) \\ -1 \end{array} \right] \cdot$$

$$\left\{ \sum_{j=-2,-1,0,1,2} (-1)^j \mathcal{U}_{2+j} R_{r+j}^{l-1,m\pm1} \left(\frac{\partial \beta}{\partial x} \mp \frac{\partial \beta}{\partial y} \right) + \right.$$

$$\left. + \sum_{j=-2,-1,0,1} (-1)^j \mathcal{D}_{2+j} \left[\beta \left(\frac{\partial R_{r+j}^{l-1,m\pm1}}{\partial x} \mp i \frac{\partial R_{r+j}^{l-1,m\pm1}}{\partial y} \right) - 2 \frac{e}{m_e} \beta^3 \left(E_x \mp i E_y \right) R_{r+j}^{l-1,m\pm1} \right] \right\} +$$

$$+ \frac{1}{2l-1} (l-m) \left[\sum_{j=-2,-1,0,1,2} (-1)^j \mathcal{U}_{2+j} R_{r+j}^{l-1,m} \frac{\partial \beta}{\partial z} + \sum_{j=-2,-1,0,1} (-1)^j \mathcal{D}_{2+j} \frac{\partial R_{r+j}^{l-1,m}}{\partial z} \right] +$$

$$+ \frac{ie\beta^3}{m_e c} I_{r,k}^l \left\{ \sum_{\pm} \left[\begin{array}{l} +(l+m+1)(l-m) \\ -1 \end{array} \right] \cdot \left[\left(B_z \mp i B_y \right) R_r^{l,m\pm1} \right] + m B_z R_r^{l,m} \right\} +$$

$$+ \sum_{\pm} \left[+ \begin{array}{l} + \frac{(l+m)!}{(l-m)!} \frac{4\pi}{2l+1} N \beta \sigma_e \left(v' \right) \\ -4\pi N \sigma_0 \end{array} \right] \sum_{j=-1,0,1} (-1)^j - \mathcal{U}_{1+j} R_{r+j}^{l,m} , \qquad (13)$$

where we have set:

$$\left\{ \begin{array}{l} \Re_0 = 2(r+l+1)\left(r+l+\frac{3}{2}\right) \frac{\Gamma\left(r+l+\frac{3}{2}\right)}{r!} \\ \Re_1 = 2\left[(r+l+1)+(r+1)\left(r+l+\frac{5}{2}\right)\right] \frac{\Gamma\left(r+l+\frac{5}{2}\right)}{(r+1)!} \\ \Re_2 = 2(r+1) \frac{\Gamma\left(r+l+\frac{7}{2}\right)}{(r+2)!} \end{array} \right. \qquad (14)$$

$$\left\{ \begin{array}{l} \mathcal{B}_0 = \left(r+l+\frac{3}{2}\right) \frac{\Gamma\left(r+l+\frac{3}{2}\right)}{r!} \\ \mathcal{B}_1 = \frac{\Gamma\left(r+l+\frac{5}{2}\right)}{(r+1)!} \end{array} \right. \qquad (15)$$

$$\begin{cases} \mathcal{U}_0 = 2\left(r+l-1\right)\left(r+l-\frac{1}{2}\right)\frac{\Gamma\left(r+l-\frac{1}{2}\right)}{(r-2)!} \\ \mathcal{U}_1 = 2\left[\left(r+1\right)\left(r+l+\frac{1}{2}\right)+\left(r+l-1\right)\left(3r+2l\right)\right]\frac{\Gamma\left(r+l+\frac{1}{2}\right)}{(r-2)!} \\ \mathcal{U}_2 = 2\left[\left(r+1\right)\left(r+l+\frac{1}{2}\right)+\left(r+l-1\right)\left(3r+l+\frac{3}{2}\right)\right]\frac{\Gamma\left(r+l-\frac{1}{2}\right)}{(r-1)!} \\ \mathcal{U}_3 = 2\left(r+1\right)\left(r+l+\frac{7}{4}\right)\frac{\Gamma\left(r+l+\frac{5}{2}\right)}{(r+1)!} \\ \mathcal{U}_4 = 2\left(r+1\right)\left(r+2\right)\frac{\Gamma\left(r+l+\frac{7}{2}\right)}{(r+2)!} \end{cases} \tag{16}$$

$$\begin{cases} \mathcal{D}_0 = \left(r+l-\frac{1}{2}\right)\frac{\Gamma\left(r+l-\frac{1}{2}\right)}{(r-2)!} \\ \mathcal{D}_1 = \left(3r+2l\right)\frac{\Gamma\left(r+l+\frac{1}{2}\right)}{(r-1)!} \\ \mathcal{D}_2 = \left(3r+l+\frac{3}{2}\right)\frac{\Gamma\left(r+l+\frac{3}{2}\right)}{r!} \\ \mathcal{D}_3 = \mathcal{U}_2 = \left(r+1\right)\frac{\Gamma\left(r+l+\frac{5}{2}\right)}{(r+1)!} \end{cases} \tag{17}$$

$$\begin{cases} \mathcal{U}_0 = \left(r+l+\frac{1}{2}\right)\frac{\Gamma\left(r+l+\frac{1}{2}\right)}{(r+1)!} \\ \mathcal{U}_1 = \left(2r+l+\frac{3}{2}\right)\frac{\Gamma\left(r+l+\frac{3}{2}\right)}{r!} \end{cases} \tag{18}$$

and

$$I^l_{r,k} = \int_0^\infty ve^{-\beta^2v^2}\left(\beta^2v^2\right)^{l+\frac{1}{2}} L_r^{l+\frac{1}{2}} L_k^{l+\frac{1}{2}} d\left(\beta^2v^2\right) . \tag{19}$$

3. Application

In order to compute the integral (19)

$$I^l_{r,k} = \int_0^\infty ve^{-\beta^2v^2}\left(\beta^2v^2\right)^{l+\frac{1}{2}} L_r^{l+\frac{1}{2}} L_k^{l+\frac{1}{2}} d\left(\beta^2v^2\right) ,$$

we use the following property of Laguerre polynomials:

$$\int_0^\infty e^{-z\left(S+\frac{\alpha_1+\alpha_2}{2}\right)} x^{\mu+\beta'} L_k^\mu\left(a_1x\right) L_r^\mu\left(a_2x\right) dx =$$

$$= \begin{cases} 0, \qquad for \qquad k\neq r \\ \frac{\Gamma\left(1+\mu+\beta'\right)\Gamma(1+\mu k)}{k!k!\Gamma(1+\mu)} \times \left\{ \frac{d^k}{dh^h}\left[\frac{F\left(\frac{1+\mu+\beta'}{2},1+\frac{\mu+\beta'}{2},1+\mu,\frac{A^2}{B^2}\right)}{(1-h)^{1+\mu}B^{1+\mu+\beta}}\right]\right\}_{h=0} \end{cases}, \; for\ k=r, \tag{A.1}$$

where

$$A_2 = \frac{4a_1a_2h}{\left(1-h\right)^2}; B = S + \frac{a_1+a_2}{2}\frac{1+h}{1-h} \tag{A.2}$$

and

$$F\left(\alpha,\beta^{'},\gamma,z\right) = 1+\frac{\alpha\beta^{'}}{\gamma_1}z+\frac{\alpha\left(\alpha+1\right)\beta^{'}\left(\beta^{'}+1\right)}{\gamma\left(\gamma+1\right)1.2}z^2+ \\ +\frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\beta^{'}\left(\beta^{'}+1\right)\left(\beta^{'}+2\right)}{\gamma\left(\gamma+1\right)\left(\gamma+2\right)1.2.3}z^3+... \quad \text{(A.3)}$$

The equality (A.1) is satisfied if and only if:

$$Re\left(S+\frac{a_1+a_2}{2}\right)>0, \qquad a_1>0, \qquad a_2>0, \qquad Re\left(\mu+\beta^{'}\right)>-1. \quad \text{(A.4)}$$

With (A.1), we write:

$$I^l_{r,k}=\frac{1}{\beta}\int_0^\infty e^{-\beta^2v^2}\left(\beta^2v^2\right)^{l+\frac{1}{2}+\frac{1}{2}}L_r^{l+\frac{1}{2}}L_k^{l+\frac{1}{2}}d\left(\beta^2v^2\right). \quad \text{(19')}$$

Let us note that:

$$s=0;\ a_1=a_2=1;\mu=l+\frac{1}{2}\,,\ \beta^{'}=\frac{1}{2};\ A^2=\frac{4h}{\left(1-h\right)^2};B=\frac{1+h}{1-h}\,. \quad \text{(A.5)}$$

If we substitute (A.5) in (A.4), then all the inequalities are verified. Taking into consideration (A.5), we obtain for (A.1) the following expression:

$$\Gamma_{r,k}=0 \quad for \quad k\neq r$$

and

$$I^l_r=I^l_{r,r}=\frac{1}{\beta}\frac{\Gamma\left(l+2\right)\Gamma\left(l+r+\frac{3}{2}\right)}{r!r!\Gamma\left(l+\frac{3}{2}\right)}\cdot\left\{\frac{d^r}{dh^r}\left[\frac{F\left(\frac{l+2}{2};\frac{l+3}{2};l+\frac{3}{2};\frac{4h}{(1+h)^2}\right)}{\left(1-h\right)^{\frac{-1}{2}}\left(1+h\right)^{l+2}}\right]\right\}_{h=0}. \quad \text{(A.6)}$$

The hypergeometric function F in this case has the form :

$$F=F_1=1+\sum_{p=1}^{\infty}\frac{l+p+1}{\left[2\left(l+p+1\right)-1\right]p!}\frac{2^ph^p}{\left(1+h\right)^{2p}}. \quad \text{(A.7)}$$

For the particular case of Section 3, we have:

$$I^0_0=\frac{1}{\beta_0} \quad and \quad I^1_0=\frac{5}{\beta_0}. \quad \text{(A.8)}$$

We will note also that, because of the form of z (and hence of A and B), the integrals I^l_r, for all finite values of r and l, have a finite number of terms, although the hypergeometric function has been given by a series with an infinite number of terms.

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