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A BASIC DECOMPOSITION RESULT RELATED TO THE NOTION OF THE RANK OF A MATRIX AND APPLICATIONS

Cristinel Mortici

To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In this paper we present a basic decomposition theorem for matrices of rank r. Then we use this result to establish interesting properties and other results regarding the notion of rank of a matrix.

1. Introduction

Here, for sake of simplicity, we often assume that the matrices we are dealing with are square matrices. Indeed, an arbitrary matrix can be transformed into a square matrix by attaching zero rows (columns), without changing its rank. Let us consider for the beginning the following operations on a square matrix, which invariate the rank:

1. permutation of two rows (columns);
2. multiplication of a row (column) with a nonzero real number;
3. addition of row (column) multiplied by a real number to another row (column).

We will call these operations elementary operations. We set the following problem: Are these elementary operations of algebraic type? For example, we ask if the permutation of the rows (columns) i and j of an arbitrary matrix A is in fact the result of multiplication to left (right) of the matrix A with a special matrix denoted U_{ij}. If such a matrix U_{ij} does exist, then it should have the same effect on the identity I_n. Hence the matrix $U_{ij}I_n$ is obtained from the identity matrix by permutation the rows i and j. But $U_{ij}I_n = U_{ij}$, so

$$U_{ij} = \left(\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 0 & \dots & \dots & \dots & 1 & \dots & \dots & \dots \\
 & & & \vdots & 1 & & & \vdots & & & \\
 & & & \vdots & & \ddots & & \vdots & & & \\
 & & & \vdots & & & 1 & \vdots & & & \\
 & & & 1 & \dots & \dots & \dots & 0 & \dots & \dots & \dots \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right)
\begin{array}{l}
\\
\\
\\
\dots \; i \\
\quad\;\vdots \\
\quad\;\vdots \\
\quad\;\vdots \\
\dots \; j \\
\\
\\
\\
\end{array} .$$

Now it can be easily seen that the matrix $U_{ij}A$, respective AU_{ij} is the matrix A with the rows, respective the columns i and j permutated. The matrix U_{ij} is invertible, because $U_{ij}^2 = I_n$. Moreover, $\det U_{ij} = -1$, since the permutation of two rows (columns) changes the sign of the determinant.

In an analogous way, we search now a matrix $V_i(\alpha)$ for which the multiplication with an arbitrary matrix A leads to the multiplication of the i-th row (column) of A by a nonzero real α. In particular, $V_i(\alpha)I_n$ will be the identity matrix having the i-th row multiplied by α. But $V_i(\alpha)I_n = V_i(\alpha)$, so we must have

$$V_i(\alpha) = \left(\begin{array}{ccccccc}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & & \alpha & \dots & \dots & \dots \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{array}\right)
\begin{array}{l}
\\
\\
\\
\dots \; i \\
\\
\\
\\
\end{array} .$$

Now it can be easily seen that the matrix $V_i(\alpha)A$, respectively $AV_i(\alpha)$ is the matrix A having the i-th row, respectively the i-th column multiplied by α. Obviously, $\det V_i(\alpha) = \alpha \neq 0$, so the matrix $V_i(\alpha)$ is invertible.

Similarly, let us remark that if we add the j-th row multiplied by λ of the identity matrix to the i-th row, we obtain the matrix

$$W_{ij}(\lambda) = \begin{array}{cc} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \ldots & \lambda & \ldots & \ldots \\ & & & 1 & \vdots & & \\ & & & & 1 & \ldots & \ldots \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \ldots \; i \\ \vdots \\ \ldots \; j \\ \\ \\ \end{array} \end{array}.$$

Now, we can easily see that the matrix $W_{ij}(\lambda)A$, respective $AW_{ij}(\lambda)$ is obtained from the matrix A by adding the j-th row multiplied by λ to the i-th row, respectively by adding the j-th column multiplied by λ to the i-th column.

All the matrices U_{ij}, $V_{ij}(\alpha)$, $W_{ij}(\lambda)$, $\alpha \in \mathbf{R}^*$, $\lambda \in \mathbf{R}$, are invertible and we will call them elementary matrices. Now we can give the following basic result:

Theorem 1. *Each matrix $A \in M_n(\mathbf{C})$ can be represented in the form*

$$A = PQR,$$

where $P, R \in M_n(\mathbf{C})$ are invertible and $Q = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_n(\mathbf{C})$, with $r = rank(A)$.

To prove this, let us remark that every matrix A can be transformed into a matrix Q by applying the elementary operations 1-3. If for example $a_{11} \neq 0$, then multiply the first column by a_{11}^{-1} to obtain 1 on the position $(1,1)$. Then add the first line multiplied by $-a_{i1}$ to the i-th row, $i \geq 2$ to obtain zeros on the other places of the first column. Similarly, we can obtain zeros on the other places of the first row. Finally, a matrix Q is obtained and in algebraic formulation, we can write

$$S_1 \ldots S_p A T_1 \ldots T_q = Q,$$

where S_i, T_j, $1 \leq i \leq p$, $1 \leq j \leq q$ are elementary matrices. Hence

$$A = (S_1 \ldots S_p)^{-1} Q (T_1 \ldots T_q)^{-1}$$

and we can take

$$P = (S_1 \ldots S_p)^{-1} \quad , \quad R = (T_1 \ldots T_q)^{-1}.$$

The rank is invariant under elementary operations, so

$$rank(A) = rank(Q) = r.$$

We can see that for every matrix X, the matrix QX, respectively XQ is the matrix X having all elements of the last $n-r$ rows, respectively the last $n-r$ columns equal to zero. Theorem 1 is equivalent with the following

Proposition 1. *Let there be given $A, B \in M_n(\mathbf{C})$. Then $rank(A) = rank(B)$ if and only if there exist invertible matrices $X, Y \in M_n(\mathbf{C})$ such that $A = XBY$.*

If $rank(A) = r$, then $rank(A) = rank(Q)$ and according to the proposition, there exist X, Y invertible such that $A = XQY$. By multiplication with an invertible matrix X, the rank remains unchanged. Indeed, this follows from the proposition and from the relations

$$XB = XBI_n \quad , \quad BX = I_nBX.$$

As a direct consequence, we give

Proposition 2. *If $A, B \in M_n(\mathbf{C})$ then*

$$rank(AB) \geq rank(A) + rank(B) - n.$$

Let $r_1 = rank(A)$, $r_2 = rank(B)$ and let us consider the decompositions

$$A = P_1Q_1R_1 \quad , \quad B = P_2Q_2R_2,$$

with P_i, R_i invertible, $rank(Q_i) = r_i$, $i = 1, 2$. Then

$$AB = P_1\left(Q_1R_1P_2Q_2\right)R_2,$$

so

$$rank(AB) = rank\left(Q_1R_1P_2Q_2\right).$$

The matrix $Q_1R_1P_2Q_2$ is obtained from the (invertible) matrix R_1P_2 by replacing the last $n-r_1$ rows and last $n-r_2$ columns with zeros. In consequence,

$$rang(AB) \geq n - (n - r_1) - (n - r_2) = r_1 + r_2 - n.$$

2. Applications

Now we can show how the above the theoretical results can be applied in concrete cases.

A1. *Let $A \in M_n(\mathbf{C})$ be singular. Then the rank of the adjoint matrix A^* is equal to 0 or 1.*

If $rank(A) \leq n-2$, then $A^* = 0_n$, since all minors of order $n-1$ of the matrix A are equal to zero. If $rank(A) = n-1$, then

$$rang(AA^*) \geq rang(A) + rang(A^*) - n$$

and from $AA^* = 0_n$, we derive $rank(A^*) \leq 1$.

A2. *Let $A \in M_n(\mathbf{C})$ be with $rank(A) = r$, $1 \leq r \leq n-1$. Then there exists $B \in M_{n,r}(\mathbf{C})$, $C \in M_{r,n}(\mathbf{C})$ with $rank(B) = rank(C) = r$, such that $A = BC$. Deduce that A satisfies a polynomial equation of order $r+1$.*

Let $A = PQR$, where P, R are invertible and $Q = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. We can assume without loss of generality that the first r rows of P are linear independent and the first r columns of R are linear independent. Indeed, we can have this situation by permutation of rows, respective columns, thus by extramultiplication to left (right) of matrices P, Q, respectively with elementary matrices of the form U_{ij}. Remark that $Q^2 = Q$, so we have

$$A = (PQ)(QR).$$

The matrix PQ has the last $n-r$ columns equal to zero and the matrix QR has the last $n-r$ rows equal to zero. If denote by $B \in M_{n,r}(\mathbf{C})$, $C \in M_{r,n}(\mathbf{C})$ the matrices obtained by ignoring the last $n-r$ columns of PQ, respective the last $n-r$ rows of QR, then

$$PQ = \begin{pmatrix} B & 0 \end{pmatrix} \quad , \quad QR = \begin{pmatrix} C \\ 0 \end{pmatrix}$$

and consequently

$$A = BC.$$

As we have assumed, $rank(B) = r$ and $rank(C) = r$. For the second part we use Cayley-Hamilton theorem. For the matrix $\overline{A} = CB \in M_r(\mathbf{C})$, we can find complex numbers $a_1, ..., a_r$ such that

$$\overline{A}^r + a_1\overline{A}^{r-1} + ... + a_r I = 0.$$

By multiplying with B to the left and with C to the right, we obtain

$$B\overline{A}^r C + a_1 B\overline{A}^{r-1} C + ... + a_r BC = 0.$$

Now, $B\overline{A}^k C = (BC)^{k+1} = A^{k+1}$, $1 \leq k \leq r$, so

$$A^{r+1} + a_1 A^r + ... + a_r A = 0.$$

A3. *If* $A=(a_{ij})_{1\le i,j\le n}\in M_n(\mathbf{C})$ *is a matrix with* $rank(A)=1$, *then*

$$a_{ij}=x_iy_j \quad , \quad \forall\, 1\le i,j\le n,$$

for some complex numbers $x_1,...,x_n,y_1,...,y_n$.

Indeed, there exist matrices

$$B=\begin{pmatrix} x_1\\ ...\\ x_n\end{pmatrix}\in M_{n,1}(\mathbf{C}) \quad , \quad C=\begin{pmatrix} y_1 & ... & y_n\end{pmatrix}\in M_{1,n}(\mathbf{C})$$

such that

$$A=BC=\begin{pmatrix} x_1\\ ...\\ x_n\end{pmatrix}\begin{pmatrix} y_1 & ... & y_n\end{pmatrix}=\begin{pmatrix} x_1y_1 & x_1y_2 & ... & x_1y_n\\ x_2y_1 & x_2y_2 & ... & x_2y_n\\ ... & ... & ... & ...\\ x_ny_1 & x_ny_2 & ... & x_ny_n\end{pmatrix}.$$

A4. *Let* A *be of* $rank(A)=1$. *Then* $\det(I_n+A)=1+Tr(A)$. *Moreover,*

$$\det(\lambda I_n+A)=\lambda^n+\lambda^{n-1}\cdot Tr(A),$$

for all complex numbers λ.

With the previous notations, we also have $A=B'C'$, where

$$B'=\begin{pmatrix} x_1 & 0 & ... & 0\\ ... & ... & ... & ...\\ x_n & 0 & ... & 0\end{pmatrix}\in M_n(\mathbf{C}) \quad , \quad C'=\begin{pmatrix} y_1 & ... & y_n\\ 0 & ... & 0\\ ... & ... & ...\\ 0 & ... & 0\end{pmatrix}\in M_n(\mathbf{C}).$$

Then

$$\det(I_n+A)=\det(I_n+B'C')=\det(I_n+C'B')=$$

$$=\begin{vmatrix} 1+\sum\limits_{k=1}^{n}x_ky_k & 0 & ... & 0\\ 0 & 1 & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & 1\end{vmatrix}=1+\sum_{k=1}^{n}x_ky_k=1+Tr(A).$$

The other equality can be obtained by changing A with $\lambda^{-1}A$.

A5. *Let there be given* $A\in M_n(\mathbf{C})$. *Denote by* B *a matrix obtained by permutation of the rows of the matrix* A. *Then* $\det(A+B)=0$ *or* $\det(A+B)=2^r\cdot\det A$, *for some nonnegative integer* r.

As we have already seen, we have $B = UA$, where U is obtained by permutating the rows of the identity matrix. Thus

$$\det(A+B) = \det(A+UA) = \det(I+U) \cdot \det A$$

and we will prove that $\det(I+U) \in \{0, 2^r\}$. To do this, remind that the determinant of a matrix is equal to the product of all its eigenvalues so the problem is solved if we prove that the eigenvalues of the matrix $I+U$ are 0 or 2. Let us suppose that $\lambda \in \mathbf{C}$ satisfies $(I+U)x = \lambda x$, for some nonzero vector $x = (x_1, ..., x_n)^t \in \mathbf{R}^n$. This system can be written as

$$Ux = (\lambda - 1)x.$$

In the left hand of the system the unknowns $x_1, ..., x_n$ appear in some order. By squaring the equations and then adding, we obtain

$$x_1^2 + ... + x_n^2 = (\lambda - 1)^2 \left(x_1^2 + ... + x_n^2\right),$$

so $(\lambda - 1)^2 = 1 \Rightarrow \lambda \in \{0, 2\}$.

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"Valahia" University of Targoviste,
Department of Mathematics,
Bd. Unirii 18,
0200 Targovişte,
Romania
e-mail: cmortici@valahia.ro