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ON SOLVING A FORMAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION IN THE COMPLEX FIELD

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

Several papers of K. W. Tomantschger deals with the solving of linear partial differential equations of second order in complex field by using the Bergman integral operator. The Bergman operator contains the Bergman kernel. To determine the Bergman kernel, is not an easy work. This paper contains an approach to construct an algorithm to find the Bergman kernel for a formal hyperbolic partial differential equation with two independent complex variables.

1. Mathematical Background

In [7] a list of integral operators is given: J. Le Roux (1895), I.N. Vekua (1937), St. Bergman (1937), M. Eichler (1942), U. Stessel (1992). The Stessel operator comprises all these operators as special cases. Nevertheless the most useful integral operator is Bergman operator [1]. By this operator can be obtained a solution of the linear partial differential equation of second order in the complex field.

In this paper we deal with the following formal hyperbolic pde:

$$\frac{\partial^2 v}{\partial z \partial \zeta} - z^{p-1}\zeta^{q-1}v = 0, \quad v = v\left(z,\zeta\right), \ p,q \in \mathbb{Z}, \quad z,\zeta \text{ - complex variables.} \tag{1}$$

Notation 1. Let L denote the differential operator

$$L = \frac{\partial^2}{\partial z \partial \zeta} - z^{p-1}\zeta^{q-1}, \ \ Lv\left(z,\zeta\right) = 0. \tag{2}$$

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A solution of the equation $Lv = 0$ has the form [1], [6]

$$v(z,\zeta) = (Tf)(z,\zeta) = \int_{-1}^{1} E(z,\zeta,t)\, f\left(\frac{z}{2}\left(1-t^2\right)\right) \frac{1}{\sqrt{1-t^2}} dt\ , \qquad (3)$$

where $E(z,\zeta,t)$ is a solution of the pde

$$\left(1-t^2\right) \frac{\partial^2 E}{\partial z \partial t} - \frac{1}{t} \cdot \frac{\partial E}{\partial \zeta} + 2tLE = 0 \qquad (4)$$

and $f(z)$ is any **holomorphic function** in a neighborhood of $z = 0$.

Definition 2. $E(z,\zeta,t)$ is named **Bergman kernel** and $(Tf)(z,\zeta)$ given by (3) is called the **Bergman operator.**

Remark 3. The function f is independent of the equation $Lv = 0$ while the Bergman kernel depends upon this equation.

The most useful Bergman kernels are **polynomial kernels**. The Bergman kernel of **first type** has the form:

$$E(z,\zeta,t) = 1 + \sum_{n=1}^{\infty} z^n P_n^*(z,\zeta)\, t^n, \quad E(0,\zeta,t) = 1, \qquad (5)$$

where the functions $P_n^*(z,\zeta)$ have to determined.

Remark 4. The first problem of solving the pde (2) is to find the kernel (5) i.e. to find the functions $P_n^*(z,\zeta)$ which verify the equation (4). The second problem is to compute the integral (3) for any holomorfic function $f(z)$.

Now we are interested in finding the Bergman kernel.

Proposition 5. *The functions $P_n^*(z,\zeta)$ satisfy the system of equations*

$$\frac{2n+1}{2} \cdot \frac{\partial P_{n+1}^*}{\partial \zeta} + LP_n^* = 0, \ \ P_o^*(z,\zeta) = 1, \ n \geq 0. \qquad (6)$$

Proof. We introduce (5) in equation (4) and follow the coefficient of t^{2n+1}.

The system (5) is called a **Bergman recurrence system**. By solving this system one obtains the functions $P_n^*(z,\zeta)$ and the kernel $E(z,\zeta,t)$.

2. Bergman Recurrence System Properties

There are at least two methods to solve the Bergman recurrence system.

Method 1. One uses the system (6) for $n = 0,1,2,3,$ and the general solution $P_n^*(z,\zeta)$ is obtained by induction.

Method 2. We choose a particular form of the functions $P_n^*(z,\zeta)$. This form depends on the pde2 we have to solve. In the case of equation (2) we choose a polynomial form in ζ

$$\begin{aligned} P_n^*(z,\zeta) &= \zeta^n P_n(z\zeta), \\ E(z,\zeta,t) = 1 + \sum_{n=1}^{\infty} (z\zeta)^n P_n(z\zeta). \end{aligned} \tag{7}$$

If we denoted $z\zeta = u$, then the Bergman kernel becomes:

$$E(z,\zeta,t) = 1 + \sum_{n=1}^{\infty} u^n P_n(u), \; P_o(u) = 1 \tag{8}$$

where $P_n(u)$, $n \geq 1$ are **unknown functions.** We denote $N = \{1,2,3,...\}$.

Remark 6. For the equation (2) there are the following cases: $p = q = 0$;

$$\begin{aligned} p &= q, \; p,q \in \mathbb{N}; \; p \neq q, \; p,q \in \mathbb{N}; \\ p &= q, \; p,q \in \mathbb{Z}-\{0\}; \; p \neq q, \; p,q \in \mathbb{Z}-\{0\}. \end{aligned}$$

Now we deal with the case $p = q \in \mathbb{N}$ and we denote $p = q = m, \; m \geq 1$. The equation (2) becomes

$$Lv = 0, \; L = \frac{\partial^2}{\partial z \partial \zeta} - (z\zeta)^{m-1}, \; v = v(z,\zeta).$$

Proposition 7. *Using (7), the Bergman recurrence system (6) becomes:*

$$\begin{aligned} \frac{2n+1}{2}\left[uP'_{n+1}(u) + (n+1)P_{n+1}(u)\right] &= u^{m-1}P_n(u) - (n+1)P'_n(u) - \\ -uP''_n(u), \; u &= z\zeta, \; m \in \mathbb{N}. \end{aligned} \tag{9}$$

We call (9), Bergman particular system.

Proof. The system (6) has the form:

$$\begin{aligned} \frac{2n+1}{2} \cdot \frac{\partial P^*_{n+1}}{\partial \zeta} + \frac{\partial^2 P^*_n}{\partial z \partial \zeta} - (z\zeta)^{m-1} P^*_n &= 0, \\ P^*_{n+1}(z,\zeta) &= \zeta^{n+1} P_{n+1}(z\zeta). \end{aligned} \tag{10}$$

From (10), by computing all partial derivatives, one obtains (9).

Proposition 8. *The Bergman particular system (9) yields the recurrence formula:*

$$\begin{aligned} P_{n+1}(u) &= \frac{2}{n+1}\cdot\frac{1}{u^{n+1}}\int u^{n+1}\left(u^{m-2}P_n - \frac{n+1}{u}P_n' - P_n''\right)du,\ n\geq 0, \\ P_o(u) &= 1 \end{aligned} \tag{11}$$

Proof. One writes the general solution differential linear equation (9) with unknown function P_{n+1}.

3. Solution of the Bergman Particular System

We look for the solution $P_n(u)$ of the Bergman particular system, (11). One starts with $n = 0,1,2,3,4$ and then we propose the general form of unknown functions $P_n(u)$.

a) The particular values of n yield **particular functions**

$$P_0(u) = 1,$$

$$P_1(u) = \frac{2}{m}\cdot\frac{1}{u}u^m = -\frac{2^{2\cdot1}1!}{(2\cdot1)!}\cdot\frac{1}{u}\cdot\frac{-(m)_1}{1!m^2}u^m,$$

$$P_2(u) = \frac{2^{2\cdot2}2!}{(2\cdot2)!}\cdot\frac{1}{u^2}\left[-\frac{(m-1)_2}{1!m^2}u^m + \frac{(m)_1}{2!m^3}u^{2m}\right],$$

$$P_3(u) = \frac{2^{2\cdot3}3!}{(2\cdot3)!}\cdot\frac{1}{u^3}\left[-\frac{(m-2)_3}{1!m^2}u^m + \frac{3(m-1)_2}{2!m^3}u^{2m} - \frac{(m)_1}{3!m^4}u^{3m}\right],$$

$$\begin{aligned} P_4(u) = \frac{2^{2\cdot4}4!}{(2\cdot4)!}\cdot\frac{1}{u^4}\Bigg[&-\frac{(m-3)_4}{1!m^2}u^m + \frac{(m-2)_2(7m-11)}{2!m^3}u^{2m} - \\ &-\frac{6(m-1)_2}{3!m^4}u^{3m} - \frac{(m)_1}{4!m^5}u^{4m}\Bigg], \end{aligned} \tag{12}$$

where $(C)_n = C(C+1)(C+2)\ldots(C+n-1) = A^n_{C+n-1}$. All functions are written in the same form in order to obtain the generalization. The quantity in brackets is a polynomial in u.

b) We propose **the general form** of unknown functions

$$P_n(u) = (-1)^n\frac{2^{2n}n!}{(2n)!}\cdot\frac{1}{u^n}\sum_{k=1}^{n}\frac{b_{n,k}}{k!m^{k+1}}u^{km},\ n\geq 1 \tag{13}$$

where $b_{n,k}$ are unknown coefficients which depend on n and k.

c) We'll find some **particular values** for these coefficients.

Proposition 9. *The first coefficients* $b_{n,k}$ $n \geq 1,\ 1 \leq k \leq n$ *have the values:*

$n = 1,\ b_{1,1} = -m;$

$n = 2,\ b_{2,1} = -(m-1)\,m,\ b_{2,2} = m;$

$n = 3,\ b_{3,1} = -(m-2)(m-1)\,m,\ b_{3,2} = 3(m-1)\,m,\ b_{3,3} = m;$

$n = 4,\ b_{4,1} = -(m-3)(m-2)(m-1)\,m,\ b_{4,2} = (m-1)\,m\,(7m-11)\,,$
$b_{4,3} = -6(m-1)\,m,\ b_{4,4} = m.$

Proof. One identifies the relation (12) and (13).

d) We find **recurrence relations** for the coefficients $b_{n,k}$.

Proposition 10. *For* $n \geq 1$ *coefficients* $b_{n,k}$ *verify the recurrence relations:*

$$b_{o,o} = 1,\ b_{n+1,1} = (m-n)\,b_{n,1} \tag{14}$$

$$b_{n+1,n+1} = -b_{n,n}, \tag{15}$$

$$b_{n+1,k} = (km-n)\,b_{n,k} - b_{n,k-1},\ k = \overline{2,n}. \tag{16}$$

Proof. We introduce the general form (13) into Bergman particular system (9) and we put u^{-n}, u^{-n-1} in the sum. In the first equality which we obtain, one changes $u^{(k+1)m-n-1}$ in u^{km-n-1} by index removals. One obtains an equality having three sums. We isolate $k = 1,\ k = n+1$ in the first sum, $k = n+1$ in the second sum, $k = 1$ in the third sum.

$$\begin{array}{rcl} u^{m-n-1} & \Rightarrow & -\dfrac{b_{n+1,1}}{m^2}m = -\dfrac{b_{n,1}}{m^2}m\,(m-n) \\ u^{(n+1)m-n-1} & \Rightarrow & -\dfrac{b_{n+1,n+1}}{(n+1)!m^{n+2}}\,(n+1)\,m = \dfrac{b_{n,n}}{n!m^{n+1}} \\ u^{km-n-1} & \Rightarrow & -\dfrac{b_{n+1,k}}{k!m^{k+1}}km = \dfrac{b_{n,k-1}}{(k-1)!m^k} - \dfrac{b_{n,k}}{k!m^{k+1}}km\,(km-n)\,,\ k = \overline{2,n}. \end{array}$$

Corollary 11. By index removal $n+1 \to n$ for $n \geq 1,\ m \geq 1$, one obtains:

$$(14) \quad \Rightarrow \quad b_{n,1} = -(m-n+1)_n = -A_m^n = -n!C_m^n = -n!\begin{pmatrix} m \\ n \end{pmatrix} \tag{17}$$

$$(15) \quad \Rightarrow \quad b_{n,n} = (-1)^n\,m \tag{18}$$

$$(16) \quad \Rightarrow \quad b_{n,k} = (km-n+1)\,b_{n-1,k} - b_{n-1,k-1},\ n \geq 2,\ k \geq 2,\ n. \tag{19}$$

e) Using (19) we have to obtain the **non-recurence relation** for coefficiens $b_{n,k}$. We try to obtain the simplest form for each coefficient. At the beginning we evaluate the particular cases $k = 2, 3, 4$ (because $b_{n,1}$ is known).

Proposition 12. *I.1) $k = 2$; for $n \geq 2$, $b_{n,2}$ has the equivalent expressions*

$$b_{n,2} = -\sum_{j=1}^{n-1} (2m-n+1)_{n-j-1}\, b_{j,1} \tag{20}$$

$$b_{n,2} = -\sum_{j=1}^{n-1} A_{2m-j-1}^{n-j-1} A_m^j,\ n \geq 2. \tag{21}$$

I.2) The simplest form is

$$b_{n,2} = \frac{n!}{m}\left(\frac{1}{2}C_{2m}^n - C_m^n\right),\ n \geq 2. \tag{22}$$

Proof. From (19), we express $b_{n,2}$ with $b_{2,2} = -m$ and $b_{j,1}$. We obtain:

$$b_{n,2} = (2m-n+1)_{n-2}\, b_{2,2} - (2m-n+1)_{n-3}\, b_{2,1} - \ldots - (2m-n+1)_1\, b_{n-2,1} - b_{n-1,1}.$$

The result is (20). For (21), one takes into account that:

$$(2m-n+1)_{n-j-1} = A_{2m-j-1}^{n-j-1},\ \ b_{j,1} = -A_m^j.$$

For (22), one proves the equality:

$$\sum_{j=1}^{n-1} A_{2m-j-1}^{n-j-1} = \frac{n!}{m}\left(\frac{1}{2}C_{2m}^n - C_m^n\right),\ n \geq 2.$$

Proposition 13. *II.1). $k = 3$; for $n \geq 3$, $b_{n,3}$ has the equivalent expressions*

$$b_{n,3} = -\sum_{j=2}^{n-1} (3m-n+1)_{n-j-1}\, b_{j,2}, \tag{23}$$

i.e.

$$b_{n,3} = \frac{(m-1)!\,(2m-1)!}{(3m-n)!}\left[\sum_{j=2}^{n-1} C_{3m-j-1}^{2m-1} - \sum_{j=2}^{n-1} C_{3m-j-1}^{m-1}\right]. \tag{24}$$

II.2) The simplest form is:

$$b_{n,3} = \frac{(m-1)!\,(2m-1)!}{(3m-n)!}\left[C_{3m-2}^{2m} - C_{3m-n}^{2m} - C_{3m-2}^{m} + C_{3m-n}^{m}\right], \tag{25}$$

i.e.

$$b_{n,3} = \frac{n!}{m^2}\left[-\frac{1}{6}C_{3m}^{n} + \frac{1}{2}C_{2m}^{n} - \frac{1}{2}C_{m}^{n}\right]. \tag{26}$$

Proof. From (19), we express $b_{n,3}$ with $b_{3,3} = -b_{2,2}$ and $b_{j,2}$ in the form:

$$b_{n,3} = (3m-n+1)_{n-3}\, b_{3,3} - (3m-n+1)_{n-4}\, b_{3,2} - \ldots - (3m-n+1)_{1}\, b_{n-2,2} - b_{n-1,2}.$$

The result is (23). For (24), in (23) one uses $(c)_n$ and $b_{j,2}$ given by (22). Then one writes all the terms with $k!$ and we try to obtain C_i^j.

Using a trick, we get:

$$\begin{aligned} b_{n,3} \quad &= \quad \frac{(m-1)!}{(3m-n)!}\sum_{j=2}^{n-1}\frac{(2m-1)!\,(3m-j-1)!}{(m-j)!\,(2m-1)!} - \\ &\quad -\frac{(2m-1)!}{(3m-n)!}\sum_{j=2}^{n-1}\frac{(m-1)!\,(3m-j-1)!}{(2m-j)!\,(m-1)!}. \end{aligned}$$

Using C_i^j and $C_i^j = C_i^{i-j}$ $(*)$, one obtain (24). For (25) we use $C_{i-1}^{j-1} = C_i^j - C_{i-1}^j$ $(**)$. For (25), we simplify (25), and the coefficients become:

$$b_{n,3} = -\frac{(3m)!}{6m^2\,(3m-n)!} + \frac{(2m)!}{2m^2\,(2m-n)!} - \frac{m!}{2m^2\,(m-n)!}.$$

By multiplication with $n!/n!$ one obtains the desired formula (26).

Proposition 14. *III.1) $k = 4$; for $n \geq 4$, $b_{n,4}$ has the equivalent expressions*

$$b_{n,4} = -\sum_{j=3}^{n-1}(4m-n+1)_{n-j-1}\, b_{j,3} \tag{27}$$

$$\begin{aligned} b_{n,4} \quad &= \quad \frac{1}{2m}\cdot\frac{(3m-1)!\,(m-1)!}{(4m-n)!}\sum_{j=3}^{n-1}C_{4m-j-1}^{m-1} - \\ &\quad -\frac{1}{m}\cdot\frac{(2m-1)!\,(2m-1)!}{(4m-n)!}\sum_{j=3}^{n-1}C_{4m-j-1}^{2m-1} + \\ &\quad +\frac{1}{2m}\cdot\frac{(m-1)!\,(3m-1)!}{(4m-n)!}\sum_{j=3}^{n-1}C_{4m-j-1}^{3m-1}. \end{aligned} \tag{28}$$

III.2) The simplest form is:

$$\begin{aligned} b_{n,4} &= \frac{1}{2m}\cdot\frac{(3m-1)!\,(m-1)!}{(4m-n)!}\left(C_{4m-3}^{m}-C_{4m-n}^{m}\right)- \\ &-\frac{1}{m}\cdot\frac{\left[(2m-1)!\right]^2}{(4m-n)!}\left(C_{4m-3}^{2m}-C_{4m-n}^{2m}\right)+ \\ &+\frac{1}{2m}\cdot\frac{(m-1)!\,(3m-1)!}{(4m-n)!}\left(C_{4m-3}^{3m}-C_{4m-n}^{3m}\right) \end{aligned} \tag{29}$$

$$b_{n,4}=\frac{n!}{m^3}\left[\frac{1}{24}C_{4m}^{n}-\frac{1}{6}C_{3m}^{n}+\frac{1}{4}C_{2m}^{n}-\frac{1}{6}C_{m}^{n}\right]. \tag{30}$$

Proof is done in the same way as for the previous Proposition.

Remark 15. The simplest form determined for each particular coefficient was obtained in many steps, generally four steps.

Our main purpose is to find the non-recurrence form of $b_{n.k}$ from (19), using the particular coefficients $b_{n.k}$, $k=1,2,3,4$ given by their simplest form (17), (22), (26), (30) respectively. We'll do it in the next proposition.

Proposition 16. *The general form of coefficients $b_{n,k}$ from (13) is:*

$$b_{n,k}=\frac{n!}{k!m^{k-1}}\sum_{j=1}^{k}(-1)^j\,C_k^jC_{jm}^n\ ,\ \ n\geq k,\ m\geq 1,\ n\geq 0,\ m\geq n. \tag{31}$$

Proof. We introduce (31) in (19) and obtain

$$\sum_{j=1}^{k}(-1)^j\,C_k^jC_{jm}^n=\frac{km-n+1}{n}\sum_{j=1}^{k}(-1)^j\,C_k^jC_{jm}^{n-1}-\frac{km}{n}\sum_{j=1}^{k-1}(-1)^j\,C_{k-1}^jC_{jm}^{n-1}.$$

We isolate terms for $j=k$ and it results:

$$\sum_{j=1}^{k-1}(-1)^j\,C_k^jC_{jm}^n=\frac{km-n+1}{n}\sum_{j=1}^{k-1}(-1)^j\,C_k^jC_{jm}^{n-1}-\frac{km}{n}\sum_{j=1}^{k-1}(-1)^j\,C_{k-1}^jC_{jm}^{n-1}.$$

Here we use some known formulas as: $C_{k-1}^j=\dfrac{k-j}{k}C_k^j$, and $C_{jm}^{n-1}=$ $=\dfrac{n}{jm-n+1}C_m^n$, and we get

$$\begin{aligned}\sum_{j=1}^{k-1}(-1)^j C_k^j C_{jm}^n &= \frac{km-n+1}{n}\cdot\sum_{j=1}^{k-1}(-1)^j C_k^j \frac{n}{jm-n+1}C_{jm}^n - \\ &-\frac{km}{n}\cdot\sum_{j=1}^{k-1}(-1)^j\,\frac{k-j}{k}C_k^j\frac{n}{jm-n+1}C_{jm}^n,\end{aligned}$$

therefore, we get:

$$\begin{aligned}\sum_{j=1}^{k-1}(-1)^j C_k^j C_{jm}^n &= \sum_{j=1}^{k-1}(-1)^j C_k^j \frac{km-n+1}{jm-n+1}C_{jm}^n - \\ &-\sum_{j=1}^{k-1}(-1)^j C_k^j\frac{m\,(k-j)}{jm-n+1}C_{jm}^n.\end{aligned}$$

The last equality is an identity. Hence (31) satisfies (19).

Remark 17. The whole work was done to find the relation (31). The function $P_n\,(u)$ becomes

$$\begin{aligned}P_n\,(u) &= (-1)^n\,\frac{2^{2n}n!}{(2n)!}\cdot\frac{1}{u^n}\sum_{k=1}^{n}\frac{1}{k!m^{k+1}}u^{km}\cdot\frac{n!}{k!m^{k-1}}\sum_{j=1}^{n}(-1)^j C_k^j C_{jm}^n \\ P_n\,(u) &= (-1)^n\,\frac{2^{2n}n!n!}{(2n)!}\cdot\frac{1}{u^n}\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^j C_k^j C_{jm}^n\right]\frac{1}{k!k!m^{2k}}u^{km}. \quad (32)\end{aligned}$$

4. Algorithm to Find Bergman Kernel of First Type

1) Choose the Bergman kernel of first type $E\,(z,\zeta,t)$ having the form (5). The functions $P_n^*\,(z,\zeta)$ are unknown functions.

2) Choose the functions $P_n^*\,(z,\zeta)$ having the form (7). The $P_n\,(z,\zeta)$ are unknown functions.

3) Use the Bergman particular recurrence system (11) and find several functions P_n for the particular values $n=0,1,2,3,4$. All functions P_n should be written in the same aspect, to be possible the generalization (13).

4) Using the particular functions P_n one finds the particular values of the coefficients $b_{n,k}$, $1\le k\le n$.

5) Find $b_{n,1}$ and the recurrence relation for $b_{n,k}$, $n\ge 2$ (19).

6) Find the non-recurrence relation for $b_{n,k}$, $n \geq 2$ and obtain the simplest form for each coefficient.

6.1). Find the equivalent expressions for $b_{n,2}$ and obtain the simplest forms.

6.2). Find the equivalent expressions for $b_{n,3}$ and obtain the simplest forms.

6.3). Find the equivalent expressions for $b_{n,4}$ and obtain the simplest forms.

7) Write the coefficients $b_{n,k}$, $k = 1, 2, 3, 4$ in the same simplest form (31).

8) Write the coefficients form of the coefficients $b_{n,k}$ (31).

9) Write the general form of the function P_n (32).

10) Write the general form of the function P_n^* and the Bergman kernel $E(z, \zeta, t)$ (7).

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