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A NUMERICAL METHOD FOR ELLIPTIC PROBLEMS

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

Our aim is to present a numerical method for solving elliptical problems by theoretical discretization. In order to do it, a complete system of eigenfunctions of the Laplacean and the compact imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ are used in the paper.

Let Ω be a bounded domain in $\mathbf{R}^M$, with a quite smooth boundary such that we can apply the Green's formula and the Sobolev-Kondrashov imbedding theorem (see [PS]). Consider the following mixed problem:

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= u_0 \text{ on } \Gamma \subseteq \partial\Omega, meas(\Gamma) > 0 \\ \frac{\partial u}{\partial \nu} &= \sum_{i=1}^{M} \frac{\partial u}{\partial x_i}\nu_i = g \text{ on } \partial\Omega \setminus \Gamma, \end{aligned} \tag{1}$$

where L is a linear elliptic operator of divergence form:

$$Lu\,(x) := -\sum_{i,j=1}^{M} \frac{\partial}{\partial x_j}\left(a_{ij}\,(x) \cdot \frac{\partial u}{\partial x_i}\,(x)\right) + c\,(x) \cdot u\,(x)\,,\ x \in \Omega,$$

and ν is the versor of the exterior normal to $\partial\Omega$.

Suppose that L satisfies the conditions:

$$a_{ij} \in C^1(\bar{\Omega}), c \in C(\bar{\Omega}), c \geq 0,$$

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$$\sum_{i,j=1}^{M} \alpha_{ij}(x)\,\xi_i\xi_j \geq \lambda\,|\xi|^2,\ \lambda > 0,\ \xi \in R^M,\ x \in \Omega,$$

where by $|\cdot|$ we denote the euclidean norm on $\mathbf{R}^M$.

We want to find the weak solution of the problem (1), namely a function $u \in V$ such that

$$\int_\Omega \left[\sum_{i,j=1}^{M} \alpha_{ij}(x)\,\frac{\partial u}{\partial x_i}(x)\,\frac{\partial \varphi}{\partial x_j}(x) + c(x)\,u(x)\,\varphi(x)\right] dx = (f,\varphi)_{L^2(\Omega)},\ \forall \varphi \in V, \tag{2}$$

where $V := \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma \subseteq \partial\Omega\}$.

We have supposed, without losing the generality, that $u_0 = 0$, because making the translation $u - u_0$, we arrive to homogeneous conditions on Γ.

Also, we have supposed that $g = 0$ on $\partial\Omega - \Gamma$, since in the contrary case, we define $\hat{f} \in V^*$ by $(\hat{f},\varphi) := (f,\varphi)_{L^2(\Omega)} + \int_{\partial\Omega\setminus\Gamma} g\varphi ds, \forall \varphi \in V$.

It is known that this problem has a unique solution in V (see [SM]).

We shall find this solution using a discretization of the problem. For this, we need the following result (see [SM]):

Theorem 1. *Let V and H be two real Hilbert spaces, V being compactly imbedded in H. Then there exist the sequences $\{\varphi_n\}$ in V and $\{\lambda_n\}$ in $(0,\infty)$ such that:*
(i) $\{\varphi_n\}$ is an orthogonal basis in V;
(ii) $\{\sqrt{\lambda_n}e_n\}$ is an orthogonal basis in H;
(iii) $\{\lambda_n e_n\}$ is an orthogonal basis in V^*;
(iv) $\{\lambda_n\}$ is a monotone increasing sequence that diverges to $+\infty$.

From the proof of this theorem (see [SM]), we know that λ_n are the eigenvalues of the duality mapping $J : V \to V^*$, and φ_n are the corresponding eigenfunctions.

We denote by $< \cdot,\cdot >_V$ and $\|\cdot\|_V$ the inner product and respectively the norm on V.

Remember the following well-known results:

Lemma 1. *If V_n is a finite dimensional subspace of V with the basis $\varphi_1, ..., \varphi_n$, then for any $u \in V$, there exists an unique $u_n \in V_n$ satisfying:*

$$< u - u_n, \varphi >_V = 0,\ \forall \varphi \in V_n. \tag{3}$$

u_n is called the orthogonal projection of u on V_n.

Equivalently, we say that u_n is the best approximation of u in V_n in the norm of V, i.e.

$$\|u - u_n\|_V = \inf_{\varphi \in V_n} \|u - \varphi\|_V\,. \tag{4}$$

Taking $H = L^2(\Omega)$ and $V = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma \subseteq \partial\Omega\}$, we obtain the system $\{\varphi_n\}$ from the Theorem 1, formed by the eigenfunctions of the Laplacean.

Denote by

$$a(u,v) = \int_\Omega \left[\sum_{i,j=1}^{M} \alpha_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) + c(x) u(x) v(x) \right] dx, \ u, v \in V. \tag{5}$$

We easily see that $a(u,v)$ is a scalar product on V, and denote this product by $(\cdot,\cdot)_V$, and the induced norm by $|||\cdot|||_V$.

Let $N \in \mathbf{N}^*$ and $S_N(\Omega)$ be the space generated by the functions $\varphi_1, \varphi_2, ..., \varphi_N$.

Consider now instead of V_N from the above theorem, the space $S_N(\Omega)$. In this case, the matrix $A = (A_{ij})$, $A_{ij} = < \varphi_i, \varphi_j >_V$ is the unity matrix, because $\{\varphi_i\}_{i=1,2,...,N}$ form an orthonormal system.

Denote by $T_N : V \to S_N(\Omega)$ the operator which satisfies:

$$(u - T_N u, \varphi)_V = 0, \ \forall \varphi \in S_N(\Omega) \tag{6}$$

or equivalently,

$$|||u - T_N u|||_V = \inf_{\varphi \in S_N(\Omega)} |||u - \varphi|||_V \ . \tag{7}$$

Now we state the approximation problem corresponding to the problem (1):

Find $u_N \in S_N(\Omega)$ such that:

$$(u_N, \varphi)_V = (f, \varphi)_{L^2} \ \text{ for any } \varphi \in S_N(\Omega). \tag{8}$$

Because $u_N \in S_N(\Omega)$, we have that

$$u_N = \sum_{i=1}^{N} a_i \varphi_i \tag{9}$$

and the relations (8) and (9) lead us to the algebraic system:

$$\sum_{i=1}^{N} a_i (\varphi_i, \varphi_j)_V = (f, \varphi_j)_{L^2}, \ j = 1, 2, ..., N,$$

where a_i are not known and must be determined.

Further, we shall prove the existence, the uniqueness and the estimation of the errors for the approximation problem (8).

Theorem 2. *In the above conditions, we have that*
(i) For any $f \in L^2(\Omega)$, there exists an unique $u_N \in S_N(\Omega)$ satisfying (8);

(ii) If u satisfies (1) and $u_N \in S_N(\Omega)$ satisfies (8), then $u - u_N$ satisfies the relation (6), i. e. $u_N = T_N u$ and we have:

$$|||u - u_N||| = \inf_{\varphi \in S_N(\Omega)} |||u - \varphi||| . \tag{10}$$

Proof. (i) As $S_N(\Omega) \subset V$, $(\cdot,\cdot)$ is also a scalar product on $S_N(\Omega)$. For a fixed f in $L^2(\Omega)$, $\hat{f}(\varphi) := (f,\varphi)_{L^2(\Omega)}$ is a linear and continous functional on S_N, and by the Riesz-Frechet theorem, it results that the equation (8) has a unique solution in $S_N(\Omega)$, for any $f \in L^2(\Omega)$.

(ii) By (1) and (8), u_N satisfies:

$$(u - u_N, \varphi) = 0\,,\ \forall \varphi \in S_N(\Omega) \tag{11}$$

We have that

$$|||u - u_N|||^2 = (u - u_N, u - u_N)\ .$$

From (11), for any $\varphi \in S_N(\Omega)$, we have:

$$|||u - u_N|||^2 = (u - u_N, u - \varphi + \varphi - u_N) = (u - u_N, u - \varphi) + (u - u_N, \varphi - u_N)\,.$$

But $(u - u_N, \varphi - u_N) = 0$, because $\varphi - u_N \in S_N(\Omega)$ (see relation (10)). So,

$$|||u - u_N|||^2 = (u - u_N, u - \varphi) \le |||u - u_N||| \cdot |||u - \varphi|||\,,$$

from the Cauchy-Buniakowski-Schwartz inequality. From this it results that

$$|||u - u_N||| \le |||u - \varphi|||$$

i.e. (10), so $u_N = T_N u$.

Now we can estimate the error as follows:

Theorem 3. *For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbf{N}^*$ such that for any $N \ge N_\varepsilon$, then*

$$\|u - T_N u\|^2_{L^2(\Omega)} \le \varepsilon \cdot \|u\|^2_{H^1(\Omega)}\ .$$

Proof. From the Theorem 1, we have

$$J\varphi_n = \lambda_n \varphi_n,\ J : V \to V^*. \tag{12}$$

We have that $\left\{\sqrt{\lambda_n}\varphi_n\right\}$ is an orthonormal basis in $H := L^2(\Omega)$, so in $L^2(\Omega)$ we can write:

$$u = \sum_{n=1}^{\infty} c_n \sqrt{\lambda_n}\varphi_n,$$

where $c_n = < u, \sqrt{\lambda_n}\varphi_n >_{L^2(\Omega)} = \sqrt{\lambda_n} < u, \varphi_n >_{L^2(\Omega)}$.

We have, using (12):

$$< u, \varphi_n >_{L^2(\Omega)} = \int_\Omega u(x)\,\varphi_n(x)\,dx = \frac{1}{\lambda_n}\int_\Omega u(x)\,\lambda_n\varphi_n(x)\,dx =$$

$$= \frac{1}{\lambda_n}\int_\Omega u(x)\,J\varphi_n(x)\,dx = \frac{1}{\lambda_n}\int_\Omega Ju(x)\cdot\varphi_n(x)\,dx$$

$$< u, \varphi_n >_{L^2(\Omega)} = \frac{1}{\lambda_n}\int_\Omega Ju(x)\cdot\varphi_n(x)\,dx,$$

so

$$u = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left(\int_\Omega Ju(x)\,\sqrt{\lambda_n}\varphi_n(x)\,dx\right)\sqrt{\lambda_n}\varphi_n = \sum_{n=1}^{\infty}\frac{1}{\lambda_n} < Ju, \sqrt{\lambda_n}\varphi_n >_{L^2(\Omega)} \cdot\varphi_n .$$

Because $T_N u = \sum_{n=1}^{N} c_n\sqrt{\lambda_n}\varphi_n$, we obtain:

$$u - T_N u = \sum_{n=N+1}^{\infty}\frac{1}{\lambda_n}\cdot < Ju, \sqrt{\lambda_n}\varphi_n >_{L^2(\Omega)} \cdot\sqrt{\lambda_n}\varphi_n .$$

It results from this that:

$$\|u - T_N u\|^2_{L^2(\Omega)} = \sum_{n=N+1}^{\infty}\frac{1}{\lambda_n^2}\cdot < Ju, \sqrt{\lambda_n}\varphi_n >^2_{L^2(\Omega)} .$$

Let now be $\varepsilon > 0$ arbitrary fixed. Because $\lambda_n \nearrow \infty$, we have that there exists $N_\varepsilon \in \mathbf{N}$ such that $\frac{1}{\lambda_n} < \sqrt{\varepsilon}$, $\forall n \geq N_\varepsilon$.

If $N \geq N_\varepsilon$, then:

$$\|u - T_N u\|^2_{L^2(\Omega)} \leq \varepsilon \sum_{n=N+1}^{\infty} < Ju, \sqrt{\lambda_n}\varphi_n >^2_{L^2(\Omega)} \leq$$

$$\leq \varepsilon \sum_{n=1}^{\infty} < Ju, \sqrt{\lambda_n}\varphi_n >^2_{L^2(\Omega)} = \varepsilon\,\|Ju\|_{L^2(\Omega)} ,$$

so

$$\|u - T_N u\|^2_{L^2(\Omega)} \leq \varepsilon\,\|Ju\|^2_{L^2(\Omega)} = \varepsilon\,\|u\|^2_{H^1(\Omega)} .$$

Remark. This method can easily be generalized to the case where u is a vectorial function, $u = (u_1, u_2, ..., u_p)$ and belongs to the space $V := \{u \in [H^1(\Omega)]^p | u_i = 0 \text{ on } \Gamma \subseteq \Omega, meas(\Gamma) > 0\}$.

Application - The linear system of elasticity in the static case. The deformation of a body that occupies a bounded region Ω in the space $\mathbf{R}^p$, ($p = 2$ or 3), is characterized by the displacement vector $u : \Omega \to \mathbf{R}^p$ and the strain tensor $\varepsilon = \varepsilon(u)$. In the case of small (infinitesimal) deformation, $\varepsilon(u)$ has the form: $\varepsilon(u) := \{\varepsilon_{ij}(u)\}_{1\le i,j\le p}$, where $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

The constitutive relation that characterizes the elasticity in the static case is a dependence of the stress tensor

$$\sigma := \{\sigma_{ij}\}_{1\le i,j\le p}, \sigma_{ij} = \sigma_{ji},$$

namely:

$$\sigma_{ij} = \sum_{k,l=1}^{p} a_{ijkl}\varepsilon_{kl}(u) \text{ in } \Omega,$$

$$\sum_{j=1}^{p} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, 1 \le i \le p. \tag{13}$$

The coefficients a_{ijkl} satisfy the symmetry conditions

$$a_{ijkl} = a_{jikl} = a_{klij},$$

and the ellipticity conditions

$$\sum_{k,l=1}^{p} a_{ijkl}\xi_k\xi_l \ge \lambda|\xi|^2, \lambda > 0, \xi \in \mathbf{R}^p, 1 \le i,j \le p.$$

The boundary conditions are:

$$u_i = U_i \text{ on } \Gamma \subseteq \partial\Omega,$$

$$\sum_{j=1}^{p} \sigma_{ij}\nu_j = F_i \text{ on } \partial\Omega \setminus \Gamma, meas(\Gamma) > 0.$$

Here $f = \{f_i\}_{1\le i\le p}$ is the vector of the density of the volume forces given on Ω, U is the field of the displacement given on Γ, and F is the vector of the surface forces, given on $\partial\Omega \setminus \Gamma$.

Denote by

$$Lu := -\sum_{j=1}^{p} \frac{\partial \sigma_{ij}}{\partial x_j}(u) = -\frac{\partial}{\partial x_j}[\sum_{k,l=1}^{p} a_{ijkl}\varepsilon_{kl}(u)].$$

Then, the system (13) becomes: $Lu = f$ in Ω,where L is an elliptic operator of divergential form, and we can apply the above theory, and find the weak solution.

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