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A SEMILINEAR PERTURBATION OF THE IDENTITY IN HILBERT SPACES

Dinu Teodorescu

To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In this note we establish an existence and uniqueness result for the equation $u - Au + F(u) = f$ where A is linear, quasi-positive and the nonlinear function F is a Lipschitz monotone operator.

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and $f \in H$.

We consider the equation

$$u - Au + F(u) = f, \tag{1}$$

where $A : H \longrightarrow H$ is a linear operator and $F : H \longrightarrow H$ is a nonlinear Lipschitz monotone operator, i.e.

$$\langle F(x) - F(y), x - y \rangle \geq 0 \tag{2}$$

for all $x, y \in H$ and there exists $M > 0$ such that

$$\|F(x) - F(y)\| \leq M \|x - y\| \tag{3}$$

for all $x, y \in H$. We suppose moreover that A is quasi-positive i.e. there exists $c > 0$ such that

$$\langle Ax, x \rangle \geq c \|Ax\|^2, \tag{4}$$

for all $x \in H$. It$'$s clear that

Proposition 1. *The linear operator A is bounded and*

$$\|A\|_{L(H)} \leq \frac{1}{c}.$$

Proof. We have $c\,\|Ax\|^2 \leq \langle Ax, x\rangle = |\langle Ax, x\rangle| \leq \|Ax\| \cdot \|x\|$ for all $x \in H$. It results that

$$\|Ax\| \leq \frac{1}{c}\,\|x\|$$

for all $x \in H$.

An operator $B : H \longrightarrow H$ is called strongly monotone if there exists $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\,\|x - y\|^2$$

for all $x, y \in H$.

The equation (1) can be written equivalently as

$$Vu = f \tag{5}$$

where $V = I - A + F$ and I is the identity of H.

Proposition 2 *If $c > 1$, then V is a strongly monotone operator.*

Proof. We have

$$\langle (I - A)v, v\rangle = \|v\|^2 - \langle Av, v\rangle \geq \|v\|^2 - \|Av\| \cdot \|v\| \geq$$

$$\|v\|^2 - \frac{1}{c}\,\|v\|^2 = \frac{c-1}{c}\,\|v\|^2$$

for all $v \in H$. Consequently we obtain

$$\langle Vx - Vy, x - y\rangle = \langle (I - A)(x - y), x - y\rangle + \langle F(x) - F(y), x - y\rangle \geq$$

$$\langle (I - A)(x - y), x - y\rangle \geq \frac{c-1}{c}\,\|x - y\|^2$$

for all $x, y \in H$.

It is clear that the operator V is continuous on H. Also, if $c > 1$, then V is coercive(i.e. $\frac{\langle Vx, x\rangle}{\|x\|} \longrightarrow \infty$, when $\|x\| \longrightarrow \infty$) and $\langle Vx - Vy, x - y\rangle > 0$ for all $x, y \in H$ with $x \neq y$, because V is strongly monotone. By the Minty-Browder theorem, we obtain that the equation (5) has a unique solution in H. (see [1] , p. 88).

So we obtained the following

Theorem. *Let $F : H \longrightarrow H$ be a nonlinear Lipschitz monotone operator and $A : H \longrightarrow H$ a linear operator such that*

$$\langle Ax, x\rangle \geq c\,\|Ax\|^2,$$

for all $x \in H$, with $c \in R$, $c > 1$. Then the equation $u - Au + F(u) = f$ has a unique solution in H for all $f \in H$.

References

[1] H. Brezis, *Analyse fonctionelle-Theorie et applications,* Masson Editeur, Paris 1992.

[2] D. Pascali-S. Sburlan, *Nonlinear mappings of monotone type,* Sijthoff & Noordhoff, Int. Publishers,Alphen aan den Rijn, 1978.

"Valahia" University,
Department of Mathematics,
Bd. Unirii 18,
0200 Targoviste,
Romania