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ON THE APPROXIMATION OF INCONSISTENT INEQUALITY SYSTEMS

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Abstract

In this paper is analyzed the minimal correction problem for an inconsistent linear inequality system. By the correction we mean avoiding its contradictory nature by means of relaxing the constraints. When the system of inequalities $Ax \leq b$ has no solutions, we are interested in a vector that satisfies the system in a Least Squares (LS) sense, i.e. a vector $x \in \mathbf{R}^n$ that minimizes the quantity $\left\|(Ax-b)_+\right\|_2$, where $(Ax-b)_+$ is the vector whose i^{th} component is $\max\left\{(Ax-b)_i, 0\right\}$. In fact, the right-hand side (RHS) vector is corrected. Often, in the real world it is more expedient to correct some submatrix of the augmented matrix (A, b), i.e. the RHS vector as well as some rows and some columns of the matrix A.

1. Correction of RHS vector. Least Squares problem for linear inequalities

Consider the system of linear inequalities

$$Ax \leq b, \tag{1}$$

where $A \in \mathcal{M}_{m\times n}(\mathbf{R})$, $b \in \mathbf{R}^m$, $x \in \mathbf{R}^n$. Because in the real life the numerical data for the system (1) are not exactly determined, they are known by approximation. Approaches to the problem of solving linear systems and different solution concepts are presented in $[PS1]$. A method to obtain the minimum-norm solution of a large-scale system of linear inequalities, when the vector b is perturbed is included in $[Po1]$. When the system is inconsistent, we are interested in vectors satisfying the system (1) in LS sense, that is, vectors $x \in \mathbf{R}^n$ solving

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$$\min \frac{1}{2} \left\| (Ax-b)_+ \right\|_2^2, \tag{2}$$

where $(Ax-b)_+$ is the $m-$ vector whose i^{th} component is $\max\{(Ax-b)_i, 0\}$ and $\|.\|_2$ is the Euclidean norm.The problem to find an LS solution to the system (1) is a natural extension of the equality linear LS problem.
In $[Ha]$, S. P. Han characterized the LS solutions for linear inequalities and proposed a method for finding one of these solutions in a finite number of iterations.
When the variables are restricted to lie in certain prescribed intervals, which may reflect some a priori information about the desired solution, we have

$$\left\{ \begin{array}{c} \min \frac{1}{2} \left\| (Ax-b)_+ \right\|_2^2 \\ \text{subject to} \\ l_i \leq x_i \leq u_i \, (i = 1, ..., n), \end{array} \right. \tag{3}$$

where $l_i,\ u_i \in \mathbf{R}\ (i = 1, ..., n)$.

In the papers $[Po2]$ and $[PS2]$ we present the constrained LS problem for linear inequalities. Two types of constraints are considered: linear equalities and lower and upper bounds. A direct method based on QR - decomposition for the least-squares problems of linear inequalities with linear equality constraints is presented. In the case where the variables are within meaningful intervals, the QR - factorization is updated when columns are added to, or removed from the matrix.

2. Least Squares problem for linear inequalities with linear equality constraints.

Consider the following problem

$$\left\{ \begin{array}{c} \min \frac{1}{2} \left\| (Ax-b)_+ \right\|_2^2 \\ \text{subject to} \\ Bx = d, \end{array} \right. \tag{4}$$

where $A \in \mathcal{M}_{m\times n}(\mathbf{R})$, $b \in \mathbf{R}^m$, $x \in \mathbf{R}^n, B \in \mathcal{M}_{r\times n}(\mathbf{R})$, $d \in \mathbf{R}^r$ and $rank(B) = r$.
We start by computing the QR - factorization of the matrix $B^T \in \mathcal{M}_{n\times r}(\mathbf{R})$:

$$Q^T B^T = \left(\begin{array}{c} R \\ 0 \end{array} \right) \begin{array}{c} r \\ n-r \end{array}, \tag{5}$$

where $Q \in \mathcal{M}_{n\times n}(\mathbf{R})$ is an orthogonal matrix ($Q^T Q = I_n$) and
$R \in \mathcal{M}_{r\times r}(\mathbf{R})$ is upper triangular. We partition AQ and $Q^T x$ as follows:

$$AQ = \begin{array}{cc} (A_1 & A_2) \\ r & n-r \end{array}$$

$$\text{and } \; Q^T x = \left(\begin{array}{c} u \\ w \end{array} \right) \begin{array}{c} r \\ n-r \end{array} .$$

Then

$$\left\|(Ax-b)_+\right\|_2^2 = \left\|\left(AQQ^T x - b\right)_+\right\|_2^2 = \left\|(A_1 u + A_2 w - b)_+\right\|_2^2 .$$

On the other hand, from (5), it follows that:

$$B = \left(R^T \; 0\right) Q^T$$

and

$$Bx = \left(R^T \; 0\right) \left(\begin{array}{c} u \\ w \end{array} \right) = R^T u.$$

Thus, u is determined by *forward elimination*, from the lower triangular system $R^T u = d$. The vector w is obtained by solving the unconstrained LS problem

$$\min_w \left\|(A_2 w - e)_+\right\|_2^2 ,$$

where $e = b - A_1 u$, i. e. w is an LS solution for the system of linear inequalities $A_2 w \leq e$. The solution of the problem (4) is $x = Q \left(\begin{array}{c} u \\ w \end{array} \right)$.

3. Least Squares problem for linear inequalities with linear equality constraints and bounds on the variables

Consider the following problem

$$\left\{ \begin{array}{l} \qquad \min \frac{1}{2} \left\|(Ax-b)_+\right\|_2^2 \\ \text{subject to} \\ \qquad Bx = d \\ \qquad l_i \leq x_i \leq u_i \; (i = 1, ..., n), \end{array} \right. \tag{6}$$

where $A \in \mathcal{M}_{m\times n}(\mathbf{R})$, $b \in \mathbf{R}^m$, $x \in \mathbf{R}^n, B \in \mathcal{M}_{r\times n}(\mathbf{R})$, $d \in \mathbf{R}^r$ and $rank(B) = r$. Some developments in general nonlinear optimization with bounds on the variables have generated methods for the problem

$$\left\{ \begin{array}{l} \qquad \min f(x) \\ \text{subject to} \\ \qquad c_i(x) = 0 \; (i = 1, ..., r) \\ \qquad l_i \leq x_i \leq u_i \; (i = 1, ..., n), \end{array} \right.$$

where $f(x)$ and $c_i(x)$ are twice continuously differentiable functions. The function

$$f(x) = \frac{1}{2} \left\| (Ax-b)_+ \right\|_2^2 \tag{7}$$

is a convex continuously differentiable function. Unfortunately, it is not twice differentiable and these methods are not applicable to the problem (6). In the paper $[MP]$ is presented a method for solving convex programs subject to linear constraints and bounds on the variables:

$$\begin{cases} \qquad \min f(x) \\ \text{subject to} \\ \qquad Bx = d \\ \qquad l_i \le x_i \le u_i \, (i = 1, ..., n) \, , \end{cases} \tag{8}$$

where f is a convex continuously differentiable function, B is an $r \times n$ real matrix and $d \in \mathbf{R}^r$. It is an adaptive method for determining the constraints that are active at optimal solution $\widehat{x}$, i. e. the components of $\widehat{x}$ which are exactly at one of their bounds.

For any feasible point x, we denote by $N_l(x)$ and $N_u(x)$ the sets of indices for which the corresponding components of the point x are fixed at one of its bounds, that is

$$\begin{aligned} N_l(x) &= \{i / x_i = l_i\} \\ N_u(x) &= \{i / x_i = u_i\} \, . \end{aligned}$$

Let $N(x) = N_l(x) \cup N_u(x)$. If $\widehat{x}$ is an optimal point for the problem (8), then $\widehat{x}$ is also optimal for the "restraint" problem:

$$\begin{cases} \qquad \min f\,(x) \, , \\ \text{subject to} \\ \qquad Bx = d, \\ \qquad x_i = l_i, \ i \in N_l(\widehat{x}) \\ \qquad x_i = u_i, \ i \in N_u(\widehat{x}), \end{cases}$$

i. e. the i^{th} constraints for which $l_i < \widehat{x}_i < u_i$ can be excluded without no changing the optimal solution. In order to determine the sets $N_l(\widehat{x})$ and $N_u(\widehat{x})$, the method proceeds by solving a finite number of smaller subproblems consisting of only equality constraints, such a subproblem having the form of convex minimization over a linear subspace.

The considered method develops on two levels. At each iteration of a higher level it is decided which variables are fixed at one of its bounds and which of them are free, that is, strictly between its bounds.
At lower level, a subproblem is solved at a time, only in the subspace of the free variables while keeping the fixed variables unchanged.

Consider the subproblem at iteration p:

$$\begin{cases} \quad\quad \min f(x) \\ \text{subject to} \\ \quad\quad Bx = d \\ \quad\quad x_i = l_i,\ i \in N_l^p \\ \quad\quad x_i = u_i,\ i \in N_u^p. \end{cases} \tag{9}$$

If the number of fixed variables at iteration p is k, then the number of the free variables is $q = n - k$. Assume that F^p is the complement of $N^p = N_l^p \cup N_u^p$, that is, the set of indices of free variables at iteration p. We also may assume without loss of generality, that

$$F^p = \{1, ..., q\}.$$

Then we have the following partitions:

$$x = \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, B = (C\ D),$$

where the subvector z contains the first q components of x (the free part), while $\overline{z}$ contains the last $n - q$ components (the fixed part). The matrix $C \in \mathcal{M}_{r\times q}$ $(\mathbf{R})$ contains the first q columns of B. The columns of B which are not in C form a matrix denoted by D.

Let $g(z) = f(\begin{smallmatrix} z \\ \overline{z} \end{smallmatrix})$.With these notations, the subproblem (9) reduces to finding a vector $z^{(p)} \in \mathbf{R}^q$ solving the problem

$$\begin{cases} \quad\quad \min g(z) \\ \text{subject to} \\ \quad\quad Cz = e, \end{cases} \tag{10}$$

where $e = d - D\overline{z}$. The optimal point for (9) is

$$y^p = \begin{pmatrix} z^{(p)} \\ \overline{z} \end{pmatrix}.$$

Returning to the problem (6), with f of the form (7), consider the partition $A = (G\ H)$, where $G \in \mathcal{M}_{m\times q}$ $(\mathbf{R})$ contains the first q columns of A. Then the subproblem (10) becomes the following LS problem with linear equality constraints:

$$\begin{cases} \quad\quad \min \|(Gz - h)_+\|_2^2 \\ \text{subject to} \\ \quad\quad Cz = e, \end{cases} \tag{11}$$

where $h = b - H\overline{z}$.

The key step in implementing the method for problem (8) is updating of the QR - factorization of matrix C^T from (11) when rows are added to, or removed from the matrix. În the first case of method, a fixed variable leaves a bound and becomes free. This means that a column has been appended to C. In the second case at least one additional variable hits one of their bounds. The columns corresponding to these variables are deleted from C. Since we have the QR - factorization of C^T, we may need to calculate the QR - factorization of a matrix $\overline{C}^T$ that is obtained by appending a row to C^T or by deleting a row from C^T. Methods for modifying matrix factorizations are presented in $[GL]$.

4. Correction of the augmented matrix (A, b)

Consider the linear inequality system:

$$\begin{cases} \langle a_i, x\rangle \leq b_i,\ i \in M_0 \cup M_1 \\ x_j \geq 0,\ j = 1, ..., n, \end{cases} \tag{12}$$

where a_i^T, $i = 1, ..., m$, forms the i^{th} row of the matrix A, b_i is the i^{th} component of b, M_0, M_1 are finite index sets and $\langle ., .\rangle$ stands for the standard inner product in $\mathbf{R}^n$.

With system (12) we associate the *corrected system*:

$$\begin{cases} \langle a_i, x\rangle \leq b_i,\ i \in M_0 \\ \langle a_i + h_i', x\rangle \leq b_i - h_{i,n+1},\ i \in M_1 \\ x_j \geq 0,\ j = 1, ..., n, \end{cases} \tag{13}$$

where $h_i' \in \mathbf{R}^n$ and $h_{i,n+1} \in \mathbf{R}$. Let $h_i \in \mathbf{R}^{n+1}, h_i = (h_i', h_{i,n+1}) = (h_{i1,...,}h_{i,n+1})$ be the vector correcting the i^{th} row of system (12), $i \in M_1$.

The rows with indices $i \in M_0$ are not corrected (are assumed to be fixed). We can fix also arbitrary columns of the augmented matrix (A, b), with indices $j \in J_0 \subset \{1, ..., n+1\}$. Thus we set $h_{ij} = 0,\ i \in M_1,\ j \in J_0$.

Let $M_1 = \{i_1, ..., i_p\}$ and $J_1 = \{1, ..., n+1\} \diagdown J_0$ be the complement of $J_{0,}$ i.e. the set of indices of columns to be corrected.

The correction problem of system (12) may be expressed as

$$min\, \{\Phi(H)/H \in S\} \tag{14}$$

where $H(h_{ij})_{p\times(n+1)}$ is the matrix whose entries are h_{ij},

$$S = \{H/h_{ij} = 0,\ i \in M_1,\ j \in J_0 \textit{ and system (13) is consistent}\}.$$

$\Phi\,(H)$ is the *correction criterion* estimating the quality of correction.

In $[Va]$ A.A. Vatolin proposed an algorithm based on linear programming, which finds minimal corrections of the constraint matrix and RHS vector.

5. The LP-based algorithm for solving correction problem (14). The main difficulties in solving the problem (14) is that left-hand sides of system (13) are bilinear in h_i and x. The idea of Vatolin algorithm is to take h_i of the form:

$$h_i = t_i c, \quad i \in M_1, \quad t_i \in \mathbf{R},$$

where $c \in \mathbf{R}^{n+1}$, $c = (c_1,...,c_{n+1})$ is defined bellow.

Thus, the problem (13) is also bilinear, but it can be converted into a linear one by:

a) changing variable $x \in \mathbf{R}^n$ for variable $h_0 \in \mathbf{R}^{n+1}$ so that

$$x = h_{0,n+1}^{-1}(h_{01}, ..., h_{0,n})^T,$$

where it is assumed that $0 \notin M_1$, $h_0 = (h_{01}, ..., h_{0,n}, h_{0,n+1}), h_{0,n+1} > 0$ and by

b) introducing an additional constraint

$$\langle c, h_0 \rangle = -1.$$

Consequently, the algorithm reduces solving correction problem (14) to solving a linear programming problem.

In $[Po4]$ the correction problem is analyzed by using two criteria $\|.\|_\infty$ and $\|.\|_1$. If $\Phi(H)$ takes form:

$$\Phi(H) = \max_{i,j} |h_{ij}| \tag{15}$$

then the vector $c \in \mathbf{R}^{n+1}$ is of the form

$$c_j = \begin{cases} 0, \ j \in J_0 \\ -1, \ j \in J_1. \end{cases}$$

We have to solve one linear program:

$$\begin{cases} \quad min \ \theta \\ \text{subject to} \\ \quad \langle d_i, h_0 \rangle \le 0, \ i \in M_0 \\ \quad \langle d_i, h_0 \rangle \le t_i, \ i \in M_1 \\ \quad 0 \le t_i \le \theta, \ i \in M_1 \\ \quad \sum_{j \in J_1} h_{0,j} = 1 \\ \quad h_{0,j} \ge 0, \ j = 1, ..., n+1, \end{cases} \tag{16}$$

where $d_i = (a_i, -b_i) \in \mathbf{R}^{n+1}$, $i \in M_0 \cup M_1$. Using the criterion (15), the rows $i \in M_1$ and all columns $j \in J_1$ are effectively corrected.

If $\Phi(H)$ takes form:

$$\Phi(H) = \sum_{i,j} |h_{ij}| \,,$$

then the number of linear programming problems which will be solved is $|J_1|$. At each linear programming problem, only a column of augmented matrix (A, b) is corrected. (See $[Po4]$).

Let K be the set of feasible solutions (θ, t, h_0) of problem (16), where vector t is composed of components t_i, $i \in M_1$. If $K = \phi$ then S=ϕ. Else, for each optimal solution (θ, t, h_0) of the problem (16) it is obtained the optimal value $\sigma = \theta$, the optimal correction matrix $H = H(t)$ with (i, j) component

$$h_{ij} = \begin{cases} 0, \ j \in J_0 \\ -t_i, \ j \in J_1 \end{cases}, \ i \in M_1$$

and the solution x of the corrected system

$$x = h_{0,n+1}^{-1}(h_{01}, ..., h_{0,n})^T.$$

In the paper $[Po3]$ we use interior-point techniques for solving the associated linear program.

Interior-point methods are iterative methods that compute a sequence of strict nonnegative iterates (it is assumed that $h_{0,n+1} > 0$) and converging to an optimal solution. This is completely different from the simplex method which explores the vertices of the polyhedron and an exact optimal solution is obtained after a finite number of steps.

Interior-point iterates tend to an optimal solution but never attain it. Yet an approximate solution is sufficient for our purpose. In addition, these methods are practically efficient and can be used to solve large-scale problems. For such of problems, the chances that the system is self-contradictory (inconsistent) are high.

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