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ON THE MULTI-GRID ITERATION FOR THE EIGENVALUE PROBLEM AND THE DEGREE OF INTERPOLATION WHICH IT REQUIRES (I)

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Abstract

In [8], [10] we presented an approach method to realise a third and fourth order interpolation in two dimensions. In [8] we showed that interpolations of these type can be used as prolongation operator in the multi-grid method and we proved that the accuracy of the multi-grid method can be increased in this way. In this paper we study the optimal degree of the prolongation operator which the second order elliptic eigenvalue problem requires in the point of view of the accuracy. We realise the implementation in Matlab of the multi-grid method with finite difference discretization. Numerical results are also given.

1 Introduction

To develope new numerical techniques with good efficiency and accuracy this is a practical neccesity. The aim of the paper is to study the possibilities of the increasing the accuracy of the multi-grid method with finite difference discretization. We showed ([8]) that the order of the used prolongation operator has an influence on the accuracy of the multi-grid method.

The order of the prolongation operator is used here in Hackbusch sense.

Definition 1 *[3] p is an interpolation of order m_p if interpolates polynomials of degree $m_p - 1$ exactly.*

The prolongation operators proposed by us are of order three respectively of order four. The linear prolongation operator used generally in the multi-grid method is of the second order.

We'll study and we'll compare the effect of the used prolongation operator order with the accuracy in eigenpairs computation.

The numerical results are given for the computation of eigenvectors and eigenvalues for the second order selfadjoint problems.

We are interested in the approximation of the eigenpairs of the considered problem in variational form by multi-grid method.

For general information for multi-grid method we refer to Hackbusch [3]. For the eigenvalue approximation we refer to Babuška & Osborne [2] and Chatelin.

2 Preliminary Results

In this section we present some results obtained recently which will be necessary in the following.

At first we present the main results from [8] and [10] about the interpolation of higher order in $\mathbf{R}^2$.

In [8] we formulated the following interpolation problem:

Let $\bar{\Omega} = \Omega \cup \partial\Omega$ be a unit square in plane. We consider a function $u : \Omega \to \mathbf{R}$ with $u = 0$ on $\partial\Omega$.

Let $\bar{\Omega}_l$ and $\bar{\Omega}_{l+1}$ ($l \geq 1$ integer) be grids with mesh sizes h_l^x and h_l^y and h_{l+1}^x, h_{l+1}^y in x - and y - direction and as long as $h_l^x/h_{l+1}^x = h_l^y/h_{l+1}^y = 2$ holds.

We assumed that the mesh sizes h_l^x and h_l^y are equal, $h_l^x = h_l^y = 1/n$ ($n \in \mathbf{N}^*, n \geq 2$). We denoted for simplicity by $h_l^x = h_l^y = h_l$.

Therefore we had: $\bar{\Omega}_l = \{(ih_l, jh_l) \,|\, i, j = \overline{0, n}\}$ with $|\bar{\Omega}_l| = (n+1)^2$ and $|\bar{\Omega}_{l+1}| = (2n+1)^2$.

Using the notations $\mathcal{U}_l = \{u(x,y) \,|\, (x,y) \in \bar{\Omega}_l\}$, $\mathcal{U}_{l+1} = \{u(x,y) \,|\, (x,y) \in \bar{\Omega}_{l+1}\}$, $U_l^{k+1} \subseteq \mathcal{U}_l$ and $|U_l^{k+1}| = k+1$, we get the following problem:

(P) *Determine an operator* $p : \mathcal{U}_l \to \mathcal{U}_{l+1}$ *such that* $p(x,y) = u(x,y)$, *for each* $x \in \bar{\Omega}_{l+1} \backslash \bar{\Omega}_l$..

We can write the problem (P) as:

(P') *For* $\{(x_i, y_i), u_i\}_{i=1}^N$ *and* $k < n$ *considered, with* $P_0(x_0, y_0) = u_0$ *we have to determine a recurrence:*

$$P_j(x,y) = P_{j-1}(x,y) + b_j g(x,y) \quad \text{with} \quad j = \overline{0,k},$$

$g(x,y) = \prod_{i=0}^{k-1}[(x,y) - (x_i, y_i)]$, *for* $(x,y) \in \bar{\Omega}_l$ *such that* $u(x,y) \in \mathcal{U}_l^{k+1}$, *where we denote by* $u(x_i, y_i) = u_i$, *and* $N = (n+1)^2$.

The accuracy of the polynomial interpolation formula P_k, depends on the choice of k.

In [8] we got an interpolation operator, realised with nine points for $k = 8$.

If $k = 8$, then $|\mathcal{U}_l^9| = 9$, and we can consider a nine points basic square.

Theorem 1 *[8] The polynomial interpolation formula, for* $k = 8$, *has the following form:*

$$\begin{gathered}P_1(x,y) = u_{i-1,j-1} + h[u_{\overline{x},ij-1}x + u_{\overline{y},i-1,j}y] + \\ + \frac{h^2}{2}[u_{\overline{x}x,ij-1} \cdot x(x-1) + u_{\overline{y}y,i-1j}y(y-1) + 2u_{\overline{xy},ij}xy] + \\ + \frac{h^3}{2}[u_{\overline{x}x\overline{y},ij} \cdot xy(x-1) + u_{\overline{y}y\overline{x},ij}xy(y-1)] + \\ + \frac{h^4}{4}u_{\overline{y}y\overline{x}x,ij}xy(x-1)(y-1).\end{gathered}$$

Theorem 2 *[8] If* $u \in C^k(\bar{\Omega})$, *where* $k \geq 2$ *are integers then the interpolation polynom has a global error order of* $\mathcal{O}(h^2)$. Thus
$P_1(x,y) = u((i-1)h,(j-1)h) + \frac{h}{1!}\left[\frac{\partial u(ih,(j-1)h)}{\partial x} \cdot x + \frac{\partial u((i-1)h,jh)}{\partial y}y\right] + \mathcal{O}(h^2)$.
In [10] we realised a fourth order interpolation operator.

Theorem 3 *[10] The polynomial interpolation formula, for* $k = 16$, *has the following form:*

$$\begin{aligned}&P_2(x,y) = u_0 + (u_1 - u_0)x + \frac{u_2 - 2u_1 + u_0}{2}x(x-1) + \\ &+(u_3 - 3u_2 + 3u_5 - u_0)x(x-1)(x-2) + (u_4 - u_0)y + \\ &+(u_5 - u_4 - u_1 + u_0)xy + \frac{u_6 - 2u_5 + u_4 - (u_2 - 2u_1 + u_0)}{2}xy(x-1) + \\ &+\frac{u_7 - 3u_6 + 3u_5 - u_4 - u_3 + 3u_2 - 3u_1 + u_0}{6}xy(x-1)(x-2) + \\ &+\frac{u_8 - 2u_4 + u_0}{2}y(y-1) + \frac{u_9 - 2u_5 + u_1 - (u_8 - 2u_4 + u_0)}{2}xy(y-1) + \\ &+\frac{(u_{10} - 2u_9 + u_8) - 2(u_6 + u_4 - 2u_5) + (u_0 + u_2 - 2u_1)}{4} \cdot \\ &\cdot xy(x-1)(y-1) + \left[\frac{u_{11} - 3u_{10} + 3u_9 - u_8 - 2(u_7 - 3u_6) + 2(u_4 - 3u_5)}{12} + \right. \\ &\left. + \frac{u_3 - 3u_2 + 3u_1 - u_0}{12}\right] x(x-1)(x-2)y(y-1) + \frac{u_{12} - 3u_8 + 3u_4 - u_0}{6} \cdot \\ &\cdot y(y-1)(y-2) + \frac{u_{13} - 3u_9 - u_{12} + 3u_8 + 3u_5 - u_1 - 3u_4 + u_0}{6} \cdot \\ &\cdot xy(y-1)(y-2) + \left[\frac{u_{14} - 3u_{10} - 2(u_{13} - 3u_9) + u_{12} - 3u_8 + 3u_6 - u_2}{12} + \right. \\ &\left. + \frac{2(u_1 - 3u_5) - u_0 + 3u_4}{12}\right] x(x-1)y(y-1)(y-2) + \left[\frac{u_{15} - 3u_{14}}{36} + \right.\end{aligned}$$

$$+\frac{3u_{13}-u_{12}+3(3u_{10}-u_{11})+3(u_8-3u_9)+3(u_7-3u_6)}{36}+$$
$$\left.+\frac{3(3u_5-u_4)-u_3+3u_2-3u_1+u_0}{36}\right]x(x-1)(x-2)y(y-1)(y-2).$$

Theorem 4 *[10] Let $\Omega=(0,1)\times(0,1)$ with the boundary $\partial\Omega=\{0,1\}\times\{0,1\}$ and $u:\bar{\Omega}\to\mathbf{R}$ be a sufficiently smooth function, $u\in C^k(\bar{\Omega})$, $k\geq 2$, then the interpolation operator P_2 is of the third order.*

Thus we can write:

$$\begin{aligned}
&P_2(x,y)=u((i-1)h,\,(j-1)h)+\\
&+\frac{h}{1!}\left[\frac{\partial u(ih,\,(j-1)h)}{\partial x}\cdot x+\frac{\partial u((i-1)h,\,jh}{\partial y}\cdot y\right]+\\
&+\frac{h^2}{2!}\left[C_2^0\,\frac{\partial^2 u(ih,(j-1)h)}{\partial x^2}\cdot x(x-1)+C_2^1\,\frac{\partial^2 u(ih,jh)}{\partial x\partial y}\cdot x\cdot y+\right.\\
&\left.+\ C_2^2\,\frac{\partial^2 u((i-1)h,jh)}{\partial y^2}\cdot y(y-1)\right]+\frac{h^3}{3!}\left[C_3^0\frac{\partial^3 u(ih,(j-1)h)}{\partial x^3}\cdot\right.\\
&\cdot x(x-1)(x-2)+C_3^1\,\frac{\partial^3 u(ih,jh)}{\partial x^2\partial y}\cdot xy(x-1)+\ C_3^2\,\frac{\partial^3 u(ih,jh)}{\partial y^2\partial x}\cdot\\
&\left.\cdot xy(y-1)\ +C_3^3\frac{\partial^3 u((i-1)h,jh)}{\partial y^3}y(y-1)(y-2)\right]+\mathcal{O}(h^3).
\end{aligned}$$

To verify the obtained theoretical results, we have written a program in MATLAB.

For tests, we consider functions defined on $\Omega=(0,1)\times(0,1)$, which on the boundary of the domain denoted by $\Gamma=\{0,1\}\times\{0,1\}$, verify the condition $u=0$.

We studied the order of the error in connection with the number of grid-points on $\Omega_h=\{(ih,jh)\,|\,i,j=\overline{0,N}\}$, where $h=\frac{1}{N}$ is the meshsize and $N>0$ is a given integer number.

We can observe that if we increase the number of meshpoints, then the error decreases semnificatively.

The obtained error orders in both cases we present in the following tables, where N means the number of the grid points. The choice of N is unlimitated, depends only on the configuration of the computer.

N	error order
4	10^{-3}
16	10^{-4}
32	10^{-5}
64	10^{-6}
128	10^{-7}
256	10^{-8}

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4	10^{-3}
16	10^{-5}
32	10^{-6}
64	10^{-8}
128	10^{-9}

3 Numerical results

In this Section, to test the numerical results obtained with different numerical methods, we use the Laplace-operator on the unit square, with first order boundary conditions on the boundary of the domain. It's a good example for our purpose, because the eigenvalues of the Laplace operator can be determinated exactly.

The numerical results presents a study of the accuracy of multi-grid method and it can be used to get information about the degree of the used prolongation operator and the accuracy of the used method.

We showed that the accuracy depends on the order of the used prolongation operator (see [8]).

Automatically we can formulate the following problem:

Can we improve semnificatively the accuracy, without limit, with the increasing of the order of the prolongation operator, or we can determine a step m_o for that ?

Prolongation operator of order m_o means that for each prolongation of order $m_p \geq m_0$ the accuracy doesn't increase semnificatively.

In Section 2 we gave the third and the fourth order interpolation operator, as a consequence of the definition formulated by Hackbusch.

In the following we present its use as prolongation in the multi-grid method with finite difference discretization (MGFDD) in eigen-computation.

We'll use the Rayleigh quotient, which is most important from numerical point of view because, if we determine the global minimum (maximum) of the quotient, then we can calculate the suitable least (greatest) eigenvector.

For the computation we use the preconditioned gradient method. The convergency of the method was proved in 1984 by V.G. Prikazcsikov, A.N. Himics.

The multi-grid method permits the computation of a selected eigenvalue and eigenvector. If approximations to the first n eigenvalues and eigenvectors are desired, the eigenvector iteration can be improved by means of the Ritz projection as proposed by Ruge (1981).

We use the multi-grid method in the sense of transfering the eigenvectors obtained on each grid to the finest grid and with that we start the next iteration.

The interpolation operator denoted by P_1 is of the third order and the interpolation operator denoted by P_2 is of the fourth order. The linear prolongation operator, generally used in the multi-grid method, is of the second order and we'll denote it by P.

In the following we present the obtained maximal errors in connection with the used prolongation operator.

The MGFDD results in eigenvalues approximating those of the differential operator from below.

We will denote by $\lambda_i, i = \overline{1,m}$ the exact eigenvalues and with $\lambda_i', i = \overline{1,m}$ the approximate eigenvalues.

It's known that the exact values are: $\lambda_1 = 2\pi^2$, $\lambda_2 = \lambda_3 = 5\pi^2$ $\lambda_4 = 8\pi^2$ etc.

In Table 1, Table 2, Table 3 and Table 4 we present the first two approximate eigenvalue, on $2, 3, 4, 5$ grids and the obtained maximal errors in each case, denoted by $e_{\lambda_i'}$.

We used 2 interior, 2 exterior iterations and we started the iterations with 5 gridpoints on the first grid (on the 5th grid we got 16129 gridpoints).

Table 1	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,6990	0,6004	48,8540	1,0287		
P_1	19,6902	0,4497	48,8484	0,9621		
P_2	19,6907	0,4179	48,8358	0,6522		

Table 2	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7322	0,2686	49,2288	0,4908		
P_1	19,7280	0,1807	49,2252	0,3662		
P_2	19,7279	0,1589	49,2192	0,2198		

Table 3	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7418	0,2296	49,3256	0,3949		
P_1	19,7384	0,1232	49,3225	0,2680		
P_2	19,7384	0,1226	49,3178	0,1352		

Table 4	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7442	0,2139	49,3498	0,3710		
P_1	19,7411	0,0915	49,3469	0,2571		
P_2	19,7411	0,1143	49,3426	0,1171		

We note that we could improve the accuracy, if we increase the number of grids and the order of the prolongation operator, but in the same case with P_2 the improvement it's not semnificatively.

On the other hand, we can improve the accuracy semnificatively if we increase the number of the interior and exterior iterations.

The next Tables presents the obtained maximal errors on $2, 3, 4, 5$ grids with 8 exterior and 4 interior iterations. We used 5 gridpoints also, on the first grid.

Table 5	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,6759	$0,3642 \cdot 10^{-6}$	48,8116	$0,0321 \cdot 10^{-6}$		
P_1	19,6759	$0,1955 \cdot 10^{-5}$	48,8116	$0,1054 \cdot 10^{-5}$		
P_2	19,6759	$0,0215 \cdot 10^{-6}$	48,8116	$0,8189 \cdot 10^{-6}$		

Table 6	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7234	$0,3931 \cdot 10^{-4}$	49,2134	$0,3775 \cdot 10^{-4}$		
P_1	19,7234	$0,0321 \cdot 10^{-4}$	49,2134	$0,2562 \cdot 10^{-4}$		
P_2	19,7234	$0,0004 \cdot 10^{-5}$	49,2134	$0,1221 \cdot 10^{-5}$		

Table 7	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7352	$0,1143 \cdot 10^{-3}$	49,3143	$0,2508 \cdot 10^{-3}$		
P_1	19,7352	$0,0241 \cdot 10^{-4}$	49,3143	$0,2795 \cdot 10^{-4}$		
P_2	19,7352	$0,0019 \cdot 10^{-6}$	49,3143	$0,4509 \cdot 10^{-6}$		

Table 8	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$		
P	19,7382	$0,1557 \cdot 10^{-3}$	49,3396	$0,4004 \cdot 10^{-3}$		
P_1	19,7382	$0,0199 \cdot 10^{-4}$	49,3396	$0,2361 \cdot 10^{-4}$		
P_2	19,7382	$0,0015 \cdot 10^{-6}$	49,3396	$0,3214 \cdot 10^{-6}$		

We can observe also, that the accuracy doesn't increase semnificatively if we use the fourth order prolongation P_2.

The program permits the calculation of more than two eigenvalues. Next Tables presents the errors for the first four approximate eigenvalue on $2, 3$ grids with 8 exterior and 4 interior iteration.

Table 9	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$	λ_3'	$e_{\lambda_3'}$	λ_4'	$e_{\lambda_4'}$
P	19,6762	0,0079	48,8122	0,0154	92,0576	19,8359	48,8579	1,3660
P_1	19,6760	0,0021	48,8119	0,0067	91,5988	22,5726	48,8783	1,6246
P_2	19,6759	0,0011	48,8119	0,0059	91,3645	23,0349	49,0075	5,7136

Table 10	λ_1'	$e_{\lambda_1'}$	λ_2'	$e_{\lambda_2'}$	λ_3'	$e_{\lambda_3'}$	λ_4'	$e_{\lambda_4'}$
P	19,7236	0,0076	49,2139	0,0169	91,7930	31,7023	49,2201	0,247
P_1	19,7234	0,0005	49,2136	0,0038	91,1140	37,3704	49,2780	1,4331
P_2	19,7234	0,0002	49,2135	0,0034	90,7750	40,1834	49,3290	4,3989

4 Conclusions

In Section 3 we studied and we compared the effect of the used prolongation operators, of different order, to the accuracy in eigenpairs computation.

The obtained maximal errors shows, as we can see in the Tables, the efficiency of the used higher order prolongation operators. So we see that the considered problem requires third order prolongation.

If we use third order prolongation than the accuracy will increase semnificatively in comparison with the second order prolongation, but with the fourth order prolongation the increasing it's innesential.

Our program permits the study of the convergence of the eigenvectors.

The suitable eigenfunctions from the space $V_h = \{w_1, \ldots, w_n\}$ are:

$$u_h^{(i)} = \sum_{k=1}^{n} v_k^{(i)} \cdot w_k, \quad i = \overline{1,n}.$$

We used the multi-grid method to increase the accuracy in the sense of transfering the eigenvectors obtained on each grid to the finest grid and with this we start the next iteration.

The main result is, that we show that if we use prolongation with high accuracy to transfer the eigenvectors obtained on each grid to the finest grid, then the decrease of the accuracy is minimal and the iteration on the finest grid will start better and for 2nd order problems from this point of view prolongation of third order suffices.

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