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ON INTERIOR-POINT METHODS AND SIMPLEX METHOD IN LINEAR PROGRAMMING

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Abstract

In this paper we treat numerical computation methods for linear programming. Started from the analysis of the efficiency and defficiency of the simplex procedure, we present new possibilities offered by the interior-point methods, which appears from practical necessity, from the need of efficient means of solving large-scale problems. We realise the implementation in Java of the Karmarkar's method.

1 Introduction

In this Section we present some basic results obtained in the theory of linear programming.

Numerical methods for the general problem of LP have been studied by a number of authors.

At first we can mention L.V. Kantorovich, who as early as 1939 formulated and solved a linear programming problem (LPs) dealing with the organization and planning of production. It's known that, historically, the initial mathematic statement, the basic theoretical results of the general problem of LP were first made, developed and applied in 1947 by George B. Dantzig, Marshall Wood, and their associates of the U.S. Department of the Air Force along with the simplex method. The Air Force organized a research group under the title of Project SCOOP (Scientific Computation of Optimum Programs) whose the most important contribution was the formal development and application of the LP model. These early applications of LP methods fell into three major categories: military applications generated by Project SCOOP, inter-industry economics based on the Leontieff input-output model and problems involving the relationship between zero-sum two-person games and LP.

The first successful solution of a LP problem on a high speed electronic, digital computer, occured in January 1952, on the National Bureau of Standards Computer (SEAC).

The used computation method was the original simplex procedure, but the need for efficient means of solving LPs especially large-scale problems has led to the design and implementation of highly complex computerized procedures, mathematical programming systems (MPS).

The simplex method is the central computational element of MPS and this method works so well in practice, but the computational process could break down for large values of m and n, and even for reasonably sized problems.

S. Smale (1983), K.H. Borgwardt (1982), V. Klee and G.J. Minty (1972) constructed a class of problems for which the simplex algorithm could be made to pass through all extreme points. This "worst case" problems require a $2^d - 1$ iterations, where d is the number of variables in an $m = 2d$ inequality problem.

The Klee-Minty problems show that the simplex method is an exponential-time algorithm.

For detailed discussion on computational complexity one may use, for example, Papadimitriou and Stieglitz (1982), Lenstra (1982), R.M. Karp (1975), J. Telgen (1979), S. Smale (1983).

Thus the simplex algorithm does not have a polynomial bound, but works well in practice. Some papers treated the simplex method from this point of view, that useful property of the simplex method, which assure that in practise we can obtain the solution, became a very interesting subject for researchers.

After 1970, a new direction has appeared. In 1979, L.G. Khachiyan published the ellipsoid algorithm. This method embeds the solution space of the joint primal-dual inequality problem in progressively smaller ellipsoids, each of which guaranteed to contain the optimal solution, if one exists.

An iterative step is the determination of the new ellipsoid. Khachiyan showed that the optimal solution can be obtained within a polynomial-bounded number of iterations.

This method was an important theoretical development but has not replaced the simplex method (see E.L. Lawler (1980), M.W. Browne (1979)).

In 1984, Narendra Karmarkar, a mathematician at AT&T Bell Laboratories, published another polynomial time algorithm for LPs.

Karmarkar's algorithm uses spheres and projective geometry to construct a sequence of points converging to a solution of a LP problem.

For more detailed description of Karmarkar's method one can read Hooker (1986), Parker and Rardin (1988) and Murty (1989).

Since the Karmarkar's publication appeared there an extensive research of the algorithms for LP has been done. The Khachiyan and Karmarkar algorithms are interior-point (IPM) algorithms and takes a fundamentally different approach from the simplex method (S. Wright (2002)).

2 Karmarkar's Method

The famous paper of Karmarkar inspired many people. There are more than 2000 papers on IPMs and there are hundred of variations.

In this Section we present the Karmarkar's method.

Karmarkar's method is applied to an LP in the following form:

$$(KF)\begin{cases} f(x) = (c, x) \to \min \\ Ax = 0 \\ x_1 + \cdots + x_n = 1 \\ x \geq 0 \end{cases}$$

where A is an $m \times n$ matrix and $c = (c_1, \ldots, c_n)$.

The LP must satisfy the conditions that:

$$x^0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \in S,$$

where S is the feasible set, and the optimal value of the objective function must be ($f_{\min} = 0$).

The (KF) form doesn't present a restriction, since any LP can be transformed in this form. This procedure will be described in the following.

Let us consider the following LP model:

$$\begin{cases} f(x) = (x, c) \to \min \\ Ax \leq b \\ x \geq 0 \end{cases} \tag{2.1}$$

and its dual formulation

$$\begin{cases} g(y) = (y, b) \to \max \\ A^T y \geq c \\ y \geq 0, \end{cases} \tag{2.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathcal{M}_{m,n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Using (2.1) and (2.2), we can formulate a new LP model which objective function will be the duality gap. Thus, we can write

$$\begin{cases} h(x,y) = f(x) - g(y) \to \min \\ Ax \le b \\ A^T y \ge c \\ x \ge 0, y \ge 0. \end{cases} \tag{2.3}$$

Certainly, if (x^0, y^0) is the optimal solution for (2.3), then we'll have $h_{\min} = 0$ and x^0 will be optimal for the considered problem.

Thus it's sufficient to solve only the LP model (2.3).

To bring (2.3) in the standard form, we'll introduce auxilliary variables, $u \in \mathbb{R}^m_+$, and $v \in \mathbb{R}^n_+$, namely:

$$\begin{cases} h \to \min \\ Ax + u = b \\ A^T y - v = c \\ x \ge 0, y \ge 0, u \ge 0, v \ge 0 \end{cases} \tag{2.4}$$

If we denote by

$$M = \max_{i,j}\{b_i, c_j\},$$

then we can introduce a new restriction:

$$\begin{cases} h \to \min \\ Ax + u = b \\ A^T y - v = c \\ \displaystyle\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \le 2(n+m)M \\ x \ge 0, y \ge 0, u \ge 0, v \ge 0. \end{cases} \tag{2.5}$$

Next we introduce a new variable denoted by $w_1 (w_1 \ge 0$ is necessary to preserve a standard form of the problem) and the variable w_2 such that we have $w_2 = 1$ and we can write:

$$\begin{cases} h(z) \to \min \\ Ax + u - bw_2 = 0 \\ A^T y - v - cw_2 = 0 \\ \displaystyle\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j + w_1 - 2(n+m)Mw_2 = 0 \\ \displaystyle\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_i + \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j + w_1 + w_2 = 2(n+m)M + 1 \\ x \ge 0, y \ge 0, u \ge 0, v \ge 0, w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{2.6}$$

Using the substitutions $x_i = (M+1)x'_i,\ i=\overline{1,n},\ y_j = (M+1)y'_j,\ j=\overline{1,m}$, $u_i = (M+1)u'_i,\ i=\overline{1,n},\ v_j = (M+1)v'_j,\ j=\overline{1,m},\ w_l = (M+1)w'_l,\ l=\overline{1,2}$, we can write:

$$\begin{cases} h(z') \to \min \\ Ax' + u' - bw'_2 = 0 \\ A^T y' - v' - cw'_2 = 0 \\ \sum\limits_{i=1}^{n} x'_1 + \sum\limits_{j=1}^{m} y'_j + \sum\limits_{j=1}^{m} u'_j + \sum\limits_{j=1}^{n} v'_i - 2(n+m)Mw'_2 = 0 \\ \sum\limits_{i=1}^{n} x'_i + \sum\limits_{j=1}^{m} y'_j + \sum\limits_{j=1}^{m} u'_j + \sum\limits_{i=1}^{n} v'_i + w'_1 + w'_2 = 1 \\ x' \geq 0, y' \geq 0, u' \geq 0, v' \geq 0, w' \geq 0, \end{cases} \tag{2.7}$$

which is the (KF) form associated to the considered LP model.

To apply the Karmarkar's method, first we must determine the projection of a vector onto the set of x satisfying $Ax = 0$.

Next, for a given feasible point $x^k \in S$, we must use the Karmarkar's centering transformation associated with the point x^k as $T((x_1, \ldots, x_n); x_k) \overset{not}{=} (y_1, \ldots, y_n)$ where $y_j = \left(x_j / x_j^k\right) / \left(\sum_{r=1}^n \left(x_r / x_r^k\right)\right)$. To obtain a new point x^{k+1} in the original space by using the centering transformation to determine the point corresponding to y^{k+1}, the restriction

$$\sum_{i=1}^{n} x'_i + \sum_{j=1}^{m} y'_j + \sum_{j=1}^{m} u'_j + \sum_{i=1}^{n} v'_i + w'_1 + w'_2 = 1$$

is essential. We will obtain a new point: $x_j^{k+1} = \left(x_j^k y_j\right) / \left(\sum_{r=1}^n x_r^k y_r\right),\quad j = \overline{1,n}$.

In the following we present the algorithm.

***Karmarkar Algorithm*:**

□

Step 1

Let $x^0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$;
$k = 0$.

Step 2

If $(c, x^k) < \varepsilon$ then x^k optimal
else go to Step 3

Step 3

Determine the next feasible solution x^{k+1} such that:

Calculate

$$y^{k+1} = \left(\frac{1}{n} \dots \frac{1}{n}\right)^T - \frac{\theta(I - P^T(PP^T)^{-1}P)[\mathrm{Diag}(x^k)]c^T}{\|c\|\sqrt{n(n-1)}}$$

and use the centering transformation:

$$x^{k+1} = T^{-1}(y^{k+1}, x^k)$$

$$k = k + 1;$$

return to Step 2.

□

In algorithm P denote $(m+1)\times n$ matrix whose first m rows are $A[\mathrm{Diag}(x^k)]$ and $\theta \in (0, 1)$ is chosen to ensure convergence of the algorithm.

Karmarkar showed that if we project $[\mathrm{Diag}(x^k)]c^T$ (not c^T) onto the feasible region in the transformed space, then, for some $\delta > 0$, it will be true that, for $k = 0, 1, 2, \dots,$

$$f(x^k) - f(x^{k+1}) \geq \delta.$$

Since $f(x^k)$ is decreased by at least δ per iteration, it follows that by choosing k sufficiently large, we can ensure that the value of the objective function for x^k is less than ε.

In the following we present the first iteration numerically.

We considered the following LP model in (KF) form:

$$\begin{cases} f(x) = -x_1 + 2x_2 \to \min \\ x_1 - 2x_2 + x - 3 = 0 \\ x_1 + x_2 + x - 3 = 1 \\ x \in \mathbb{R}^3_+. \end{cases}$$

We have: $A = (1 \ -2 \ \ 1)$, $n = 3$, $x^0 = \left(\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}\right)$. Let $\varepsilon = \frac{1}{10}$.

For $k = 0$ we get

$$f(x^0) = \frac{1}{3} > \frac{1}{10}, \quad \mathrm{Diag}(x^0) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$A \cdot \mathrm{Diag}(x^0) = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 1 & 1 & 1 \end{pmatrix}$$

$$P \cdot P^T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 3 \end{pmatrix}$$

$$P_0 = (I_3 - P^T(PP^T)^{-1}P) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$(\mathrm{Diag}\, x^0)\ C^T = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}$$

$$y^1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} - \frac{\theta \begin{pmatrix} -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}^T}{\sqrt{3 \cdot 2}\sqrt{2/36}}.$$

We chose $\theta = \frac{1}{4} \in (0, 1)$ and we get:

$$y^1 = (0,4055 \quad 0,3333 \quad 0,2611).$$

To obtain a new point x^1 in the original space we used the centering transformation. We can write

$$x_1^1 = \frac{x_1^0 y_1^1}{x_1^0 y_1^1 + x_2^0 y_2^1 + x_3^0 y_3^1}$$

$$x_1^1 = 0,4055$$

$$x_2^1 = \frac{x_2^0 y_2^1}{x_1^0 y_1^1 + x_2^0 y_2^1 + x_3^0 y_3^1}$$

$$x_2^1 = 0,3333$$

$$x_3^1 = \frac{x_3^0 y_3^1}{x_1^0 y_1^1 + x_2^0 y_2^1 + x_3^0 y_3^1}$$

$$x_3^1 = 0,2611$$

We got:

$$x^1 = (0,4055 \;\; 0,3333 \;\; 0,2611)$$

$$f(x^1) = -0,4055 + 2 \cdot 0,3333 = 0,2611 > \frac{1}{10}$$

With the obtained new point x^1, we can start the next iteration.

3 Conclusion

If we would like to compare in runtime the simplex method with Karmarkar's method, we can observe than neither is faster then the other in all problems.

The practical efficiency of both methods depends strongly on the details of their implementation.

The number of iterations required by Karmarkar's method is typically between 10 and 100. The simplex method needs $2n - 3n$ iterations, where n is the number of primal variables.

Thus generally the IPMs are better for large-scale problems.

References

[1] S. Wright: *Interior - Point Methods*, Computer Sciences Department, University of Wisconsin - Madison, 2002.

[2] Wayne L. Winson: *Operations Research*, PWS - Kent Publishing Company, 1991.

[3] Saul L. Gass: *Linear Programming*, An International Thomson Publishing Company, 1985.

[4] G. Meinsma: *Interior - Point Methods*, University of Twente, 1997.

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