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ON THE ORDER OF APPROXIMATION OF FUNCTIONS BY THE BIDIMENSIONAL OPERATORS FAVARD-SZÁSZ-MIRAKYAN

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Abstract

We will present an approximation result concerning the order of approximation of bivariate function by means of the bidimensional operator of Favard-Szász-Mirakyan.

1 Introduction

Let X be the interval $[0,+\infty)$ and $\mathbb{R}^X$ the space of real functions defined on X. The Favard-Szász-Mirakyan operator M_m defined on $\mathbb{R}^X$ is given by

$$(M_n f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right). \tag{1.1}$$

Consider a bivariate function defined an $I = [0,+\infty) \times [0,+\infty)$.
The parametric extensions for the operators (1.1) are given by

$$\left({}_xM_m f\right)(x,y) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}, y\right), \tag{1.2}$$

$$\left({}_yM_n f\right)(x,y) = e^{-ny} \sum_{j=0}^{\infty} \frac{(ny)^j}{j!} f\left(x, \frac{j}{n}\right). \tag{1.3}$$

Since M_m is a positive linear operator, it follows that its parametric extensions ${}_xM_m$, ${}_yM_n$ are also positive and linear operators.

Key Words: Taylor series, Mirakyan operator, Voronovskaja theorem, parametric extension, modulus of smoothness
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Proposition 1.1 *The parametric extension ${}_xM_m$, ${}_yM_n$ satisfy the relation*

$${}_xM_m \cdot {}_yM_n = {}_y M_n \cdot {}_xM_m.$$

Their product is the bidimensional operator $M_{m,n}$ which, for any function $f \in R^I$, gives the approximant

$$M_{m,n}\left(f\right)\left(x,y\right) = e^{-mx}e^{-ny}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(mx\right)^k}{k!}\frac{\left(ny\right)^j}{j!}f\left(\frac{k}{m},\frac{j}{n}\right). \tag{1.4}$$

Properties of the bidimensional operator of Mirakyan were studied in [1], [2], [3], [4]. Next, we will present some of them.

Theorem 1.1 *The bidimensional operator of Favard-Szász-Mirakyan has the following properties:*

(i) *It is linear and positive.*

(ii) $M_{m,n}\left(e_{0,0}\right)\left(x,y\right) = 1$;
$M_{m,n}\left(e_{1,0}\right)\left(x,y\right) = x$;
$M_{m,n}\left(e_{0,1}\right)\left(x,y\right) = y$;
$M_{m,n}\left(e_{2,0}\right)\left(x,y\right) = x^2 + \frac{x}{m}$;
$M_{m,n}\left(e_{0,2}\right)\left(x,y\right) = y^2 + \frac{y}{m}$.

(iii) *If $a > 0$, $b > 0$ and $f \in C\left([0,a]\times[0,b]\right)$, then the sequence $\{M_{m,n}\left(f\right)\}_{(m,n)\in N^*\times N^*}$ is uniformly convergent to f on $[0,a]\times[0,b]$.*

(iv) *If $a > 0$, $b > 0$ and $f \in C\left([0,a]\times[0,b]\right)$, then we have the estimation*

$$\left|f\left(x,y\right) - M_{m,n}\left(f\right)\left(x,y\right)\right| \le 4\omega\left(\sqrt{\frac{a}{m}},\sqrt{\frac{b}{n}}\right),$$

where by ω we denote the first modulus of smoothness.

2 Main results

Lemma 2.1 *If $\delta > 0$ and $a > 0$, then*

$$\lim_{m\to\infty} me^{-mx}\sum_{\left|\frac{k}{m}-x\right|\ge\delta}\frac{\left(mx\right)^k}{k!}\left(\frac{k}{m}-x\right)^2 = 0.$$

Proof. By (1.4) we have the next inequality

$$me^{-mx}\sum_{\left|\frac{k}{m}-x\right|\geq\delta}\frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2\leq\frac{m}{\delta^2}M_{m,n}\left((x-\cdot)^4;x,y\right). \tag{2.1}$$

A simple computation yields

$$\begin{aligned} M_{m,n}\left(\phi^3\right)(x,y) &= \frac{x}{m^3}+7\frac{x^2}{m^2}+6\frac{x^3}{m}+x^4, \\ M_{m,n}\left(\phi^4\right)(x,y) &= \frac{x}{m^2}+3\frac{x^2}{m}+x^3. \end{aligned}$$

These equalities and (i), (ii) from Theorem 1.1. imply:

$$M_{m,n}\left((x-\cdot)^4;x,y\right)=\frac{1}{m^3}\left(3mx^2+x\right). \tag{2.2}$$

By (2.1) and (2.2), we have

$$me^{-mx}\sum_{\left|\frac{k}{m}-x\right|\geq\delta}\frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2\leq\frac{3mx^2+x}{m^2\delta^2}.$$

The proof of the lemma is complete.

Theorem 2.1 *Consider* $a>0$, $b>0$, $(x,y)\in[0,a]\times[0,b]$ *and* $f\in C([0,a]\times[0,b])$. *Assume that*

(i) the function f *has the second partial derivatives;*

(ii) the function f *has the mixt partial derivatives in* (x,y).

Then

$$\lim_{m,n\to\infty}\min\{m,n\}\left[M_{m,n}(f)(x,y)-f(x,y)\right]\leq\frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y).$$

The equality holds when $m=n$.

Proof. Using the corresponding Taylor series, we obtain

$$\begin{aligned} f(s,t) &= f(x,y)+\frac{1}{1!}\left[(s-x)\frac{\partial f}{\partial x}(x,y)+(t-y)\frac{\partial f}{\partial y}(x,y)\right] \\ &+ \frac{1}{2!}\left[(s-x)^2\frac{\partial^2 f}{\partial x^2}(x,y)+(s-x)(t-y)\frac{\partial^2 f}{\partial x\,\partial y}(x,y)\right. \\ &+ \left.(s-x)(t-y)\frac{\partial^2 f}{\partial y\partial x}(x,y)+(t-y)^2\frac{\partial^2 f}{\partial y^2}(x,y)\right] \\ &+ (s-x)^2\mu_1(s-x)+(s-x)(t-y)\mu_2(s-x,t-y) \\ &+ (t-y)^2\mu_3(t-y), \end{aligned} \tag{2.3}$$

where the mappings μ_1, μ_2, μ_3 are bounded and

$$\lim_{h\to 0} \mu_1(h) = 0, \quad \lim_{h_1,h_2\to 0} \mu_2(h_1,h_2) = 0, \quad \lim_{h\to 0} \mu_3(h) = 0\,.$$

In (2.3), we choose $s = \frac{k}{m}$, $t = \frac{j}{n}$ and multiply by $\frac{(mx)^k}{k!}\frac{(ny)^j}{j!}$, we get

$$\begin{aligned}(M_{m,n}f)(x,y) - f(x,y) &= \frac{\partial f}{\partial x}(x,y)\, M_{m,n}((\cdot - x);x,y) \\ + \frac{\partial f}{\partial y}(x,y)\, M_{m,n}((* - y);x,y) &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y)\, M_{m,n}\left((\cdot - x)^2;x,y\right) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(x,y)\, M_{m,n}((\cdot - x)(* - y);x,y) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial y \partial x}(x,y)\, M_{m,n}((\cdot - x)(* - y);x,y) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,y)\, M_{m,n}\left((* - y)^2;x,y\right) + \big(R_{m,n}f\big)(x,y)\,, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\big(R_{m,n}f\big)(x,y) &= e^{-mx}\sum_{k=0}^{\infty}\frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2\mu_1\left(\frac{k}{m}-x\right) \\ + e^{-mx}e^{-ny}&\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^k}{k!}\frac{(ny)^j}{j!}\left(\frac{k}{m}-x\right)\left(\frac{j}{n}-y\right)\mu_2\left(\frac{k}{m}-x,\frac{j}{n}-y\right) \\ &+ e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}\left(\frac{j}{n}-x\right)^2\mu_3\left(\frac{j}{n}-x\right). \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} M_{m,n}((\cdot - x);x,y) &= 0 \\ M_{m,n}((* - y);x,y) &= 0 \\ M_{m,n}((\cdot - x)(* - y);x,y) &= 0 \\ M_{m,n}\left((\cdot - x)^2;x,y\right) &= \frac{x}{m} \\ M_{m,n}\left((* - y)^2;x,y\right) &= \frac{y}{n}, \end{aligned}$$

the relation (2.4) is equivalent to the following

$$M_{m,n}(f)(x,y) - f(x,y) = \frac{x}{2m}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{y}{2n}\frac{\partial^2 f}{\partial y^2}(x,y) + \big(R_{m,n}f\big)(x,y). \tag{2.6}$$

Now, by multiplying (2.6) by $\min\{m,n\}$, and crossing to limit with respect to m,n we obtain

$$\lim_{m,n\to\infty} \min\{m,n\} M_{m,n}(f)(x,y) - f(x,y) \le$$
$$\le \frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y) + \lim_{m,n\to\infty} \min\{m,n\} \big(R_{m,n}f\big)(x,y). \quad (2.7)$$

We claim that

$$\lim_{m,n\to\infty} \min\{m,n\} \big(R_{m,n}f\big)(x,y) = 0.$$

Indeed, let $\varepsilon > 0$. Since $\lim\limits_{h\to 0}\mu_1(h) = 0$, there exists $\delta' > 0$ such that, for any h, with $|h| < \delta'$, we have $|\mu_1(h)| < \varepsilon$. From $\lim\limits_{k\to 0}\mu_3(k) = 0$, we obtain that there exist an $\delta'' > 0$ such that, for any k with $|k| < \delta''$, we have $\mu_3(k) < \varepsilon$.

Considering $\delta = \max\{\delta', \delta''\}$, for every h, k with $|h| < \delta$ and $|k| < \delta$, we have

$$|\mu_1(h)| < \varepsilon \quad \text{and} \quad |\mu_3(k)| < \varepsilon\,.$$

Let us use the notations

$$\begin{aligned} I_1 &= \left\{k \in N; \left|\tfrac{k}{m} - x\right| < \delta\right\}, \\ I_2 &= \left\{k \in N; \left|\tfrac{k}{m} - x\right| \ge \delta\right\}, \\ J_1 &= \left\{j \in N; \left|\tfrac{j}{n} - y\right| < \delta\right\}, \\ J_2 &= \left\{j \in N; \left|\tfrac{j}{n} - y\right| \ge \delta\right\}. \end{aligned}$$

Since the maps μ_1, μ_2 and μ_3 are bounded, we can write

$$\begin{aligned} \left|\big(R_{m,n}f\big)(x,y)\right| &\le e^{-mx}\sum_{k\in I_1}\frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2\left|\mu_1\left(\frac{k}{m}-x\right)\right| \\ &+ (\sup|\mu_1|)\, e^{-mx}\sum_{k\in I_2}\frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 + (\sup|\mu_2|)\, M_{m,n}\left(|\cdot - x|\,|* - y|\,; x, y\right) \\ &+ e^{-nx}\sum_{j\in J_1}\frac{(ny)^j}{j!}\left(\frac{j}{n}-x\right)^2\left|\mu_3\left(\frac{j}{n}-y\right)\right| + (\sup|\mu_3|)\, e^{-nx}\sum_{j\in J_2}\frac{(ny)^j}{j!}\left(\frac{j}{n}-x\right)^2 \end{aligned} \quad (2.8)$$

For $k \in I_1$ and $j \in J_1$, we have $\left|\mu_1\left(\dfrac{k}{m} - x\right)\right| < \varepsilon$ and $\left|\mu_3\left(\dfrac{j}{n} - y\right)\right| < \varepsilon$.

Moreover, $M_{m,n}\left(|\cdot - x|\,|* - y|\,; x, y\right) = 0$. Therefore (2.8) becomes

$$\begin{aligned}
\left|\big(R_{m,n}f\big)(x,y)\right| &\leq \varepsilon e^{-mx} \sum_{k\in I_1} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \\
&\quad + \left(\sup|\mu_1|\right) e^{-mx} \sum_{k\in I_2} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \\
&\quad + \varepsilon e^{-nx} \sum_{j\in J_1} \frac{(ny)^j}{j!}\left(\frac{j}{n}-x\right)^2 + \left(\sup|\mu_3|\right) e^{-nx} \sum_{j\in J_2} \frac{(ny)^j}{j!}\left(\frac{j}{n}-x\right)^2 \\
&\leq \varepsilon\delta^2 e^{-mx} \sum_{k\in I_1} \frac{(mx)^k}{k!} + \left(\sup|\mu_1|\right) e^{-mx} \sum_{k\in I_2} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \\
&\quad + \varepsilon\delta^2 e^{-ny} \sum_{j\in I_1} \frac{(ny)^j}{j!} + \left(\sup|\mu_3|\right) e^{-ny} \sum_{j\in I_2} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 \\
&\leq 2\varepsilon\delta^2 + \left(\sup|\mu_1|\right) e^{-mx} \sum_{k\in I_2} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \\
&\quad + \left(\sup|\mu_3|\right) e^{-ny} \sum_{j\in I_2} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 .
\end{aligned}$$

Multiplying the previous inequality with $\min\{m,n\}$, we obtain

$$\begin{aligned}
\min\{m,n\}\left|\big(R_{m,n}f\big)(x,y)\right| &\leq 2\varepsilon\delta^2 \min\{m,n\} \\
+ \left(\sup|\mu_1|\right) \min\{m,n\}\, e^{-mx} &\sum_{k\in I_2} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \\
+ \left(\sup|\mu_3|\right) \min\{m,n\}\, e^{-ny} &\sum_{j\in I_2} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 . \qquad (2.9)
\end{aligned}$$

But

$$\min\{m,n\}\, e^{-mx} \sum_{k\in I_2} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 \leq m e^{-mx} \sum_{\left|\frac{k}{m}-x\right|\geq\delta} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2$$

and

$$\min\{m,n\}\, e^{-ny} \sum_{j\in I_2} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 \leq n e^{-ny} \sum_{\left|\frac{j}{n}-y\right|\geq\delta} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 .$$

Now, by using the Lemma 2.1, we obtain

$$\lim_{m,n\to\infty} \min\{m,n\}\, e^{-mx} \sum_{\left|\frac{k}{m}-x\right|\geq\delta} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right)^2 = 0$$

and

$$\lim_{m,n\to\infty} \min\{m,n\}\, e^{-ny} \sum_{\left|\frac{j}{n}-y\right|\geq\delta} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right)^2 = 0.$$

So, there exist $m_0, n_0 \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$, with $m \geq m_0$ and $n \geq n_0$, we have

$$\min\{m,n\}\, e^{-mx} \sum_{\left|\frac{k}{m}-x\right|\geq\delta} \frac{(mx)^k}{k!}\left(\frac{k}{m}-x\right) < \varepsilon \frac{1}{(\sup|\mu_1|)}$$

and

$$\min\{m,n\}\, e^{-ny} \sum_{\left|\frac{j}{n}-y\right|\geq\delta} \frac{(ny)^j}{j!}\left(\frac{j}{n}-y\right) < \varepsilon \frac{1}{(\sup|\mu_3|)}.$$

Then, according to (2.13), we can conclude that there exist $m_0, n_0 \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$, with $m \geq m_0$ and $n \geq n_0$, the next inequality holds

$$\min\{m,n\} \left|\left(R_{m,n}f\right)(x,y)\right| < 2\varepsilon\left(\delta^2 \min\{m,n\} + 1\right). \tag{2.10}$$

This is equivalent with

$$\lim_{m,n\to\infty} \min\{m,n\} \left(R_{m,n}f\right)(x,y) = 0.$$

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