An. Şt. Univ. Ovidius Constanţa Vol. 11(2), 2003, 25–42

$\mathcal{F}$-MULTIPLIERS AND THE LOCALIZATION OF MV-ALGEBRAS

Dumitru Buşneag and Dana Piciu

Abstract

The aim of the present paper is to define the localisation of MV-algebra of an MV-algebra A with respect to a topology F on A. In the last part of the paper it is proved that the maximal MV-algebra of quotients (defined in [6]) and the MV-algebra of fractions relative to an $\wedge-$closed system (defined in [5]) are MV - algebra of localisation.

The concept of multiplier for distributive lattices was defined by W. H. Cornish in [9]. J. Schmid used the multipliers in order to give a non–standard construction of the maximal lattice of quotients for a distributive lattice (see [14]). A direct treatment of the lattices of quotients can be found in [15]. In [11], G. Georgescu exhibited the localization lattice $L_{\mathcal{F}}$ of a distributive lattice L with respect to a topology $\mathcal{F}$ on L in a similar way as for rings (see [13]) or monoids (see [16]). For the case of Hilbert and Heyting algebras, see [1], [2] and respectively [10].

The concepts of MV-algebra of fractions relative to an $\wedge-$ closed system of MV-algebra of fractions and of maximal MV-algebra of quotients were defined by the authors ([5], [6]).

1 Definitions and preliminaries

Definition 1.1 *([7], [8]) An MV-algebra is an algebra* $(A, +, ^*, 0)$ *of type* $(2, 1, 0)$ *satisfying the following equations:*

(a_1) $x + (y + z) = (x + y) + z$,

(a_2) $x + y = y + x$,

(a_3) $x + 0 = x$,

(a_4) $x^{**} = x$,

(a_5) $x + 0^* = 0^*$,

(a_6) $(x^* + y)^* + y = (y^* + x)^* + x$.

MV - algebras were originally introduced by Chang in [7] in order to give an algebraic counterpart of the Lukasiewicz many valued logic (MV = many valued). Note that axioms a_1-a_3 state that $(A, +, 0)$ is an abelian monoid; following tradition, we denote an MV-algebra $(A, +, ^*, 0)$ by its universe A.

Remark 1.1 *If in a_6 we put $y = 0$ we obtain $x^{**} = 0^{**} + x$, so, if $0^{**} = 0$, then $x^{**} = x$ for each $x \in A$. Hence, the axiom a_4 is equivalent with (a_4') $0^{**} = 0$.*

Examples:

E_1) A singleton $\{0\}$ is a trivial example of an MV-algebra; an MV-algebra is said *nontrivial* provided its universe has more that one element.

E_2) Let $(G, \oplus, -, 0, \leq)$ be an l-group. For each $u \in G$, $u > 0$, let

$$[0, u] = \{x \in G : 0 \leq x \leq u\}$$

and for each $x, y \in [0, u]$, let $x+y = u \wedge (x \oplus y)$ and $x^* = u - x$. Then $([0, u], +, ^*, 0)$ is an MV - algebra. In particular, if we consider the real unit interval $[0, 1]$ and, for all $x, y \in [0, 1]$, we define $x + y = \min\{1, x + y\}$ and $x^* = 1 - x$, then $([0, 1], +, ^*, 0)$ is an MV-algebra.

E_3) If $(A, \vee, \wedge, ^*, 0, 1)$ is a Boolean lattice, then $(A, \vee, ^*, 0)$ is an MV-algebra.

E_4) The rational numbers in $[0, 1]$, and, for each integer $n \geq 2$, the n-element set $L_n = \left\{0, \frac{1}{(n-1)}, ..., \frac{(n-2)}{(n-1)}, 1\right\}$ yield examples of subalgebras of $[0, 1]$.

E_5) Given an MV-algebra A and a set X, the set A^X of all functions $f : X \longrightarrow A$ becomes an MV-algebra if the operations $+$ and * and the element 0 are defined pointwise. The continuous functions from $[0, 1]$ into $[0, 1]$ form a subalgebra of the MV-algebra $[0, 1]^{[0,1]}$.

In the rest of this paper, by A we denote an MV -algebra.

On A we define the constant 1 and the operations ,,$\cdot$" and ,,$-$" as follows $1 = 0^*$, $x \cdot y = (x^* + y^*)^*$ and $x - y = x \cdot y^* = (x^* + y)^*$ (we consider the ,,*" operation more binding that any other operation, and the ,,$\cdot$" more binding that $+$ and $-$).

Lemma 1.1 *([3]-[8], [12]) For $x, y \in A$, the following conditions are equivalent:*

(i) $x^* + y = 1$.

(*ii*) $x \cdot y^* = 0$.

(*iii*) $y = x + (y - x)$.

(*iv*) *There is an element* $z \in A$ *such that* $x + z = y$.

For any two elements $x, y \in A$ let us agree to write $x \leq y$ iff x and y satisfy the equivalent conditions (i)-(iv) in the above lemma. So, $\leq$ is a partial order relation on A (which is called the *natural order* on A).

Theorem 1.1 *([3]-[8], [12]) If* $x, y, z \in A$, *then the following hold:*

(c_1) $1^* = 0$,

(c_2) $x + y = (x^* \cdot y^*)^*$,

(c_3) $x + 1 = 1$,

(c_4) $(x - y) + y = (y - x) + x$,

(c_5) $x + x^* = 1, x \cdot x^* = 0$,

(c_6) $x - 0 = x$, $0 - x = 0$, $x - x = 0$, $1 - x = x^*$, $x - 1 = 0$,

(c_7) $x + x = x$ *iff* $x \cdot x = x$,

(c_8) $x \leq y$ *iff* $y^* \leq x^*$,

(c_9) *If* $x \leq y$, *then* $x + z \leq y + z$ *and* $x \cdot z \leq y \cdot z$,

(c_{10}) *If* $x \leq y$, *then* $x - z \leq y - z$ *and* $z - y \leq z - x$,

(c_{11}) $x - y \leq x, x - y \leq y^*$,

(c_{12}) $(x + y) - x \leq y$,

(c_{13}) $x \cdot z \leq y$ *iff* $z \leq x^* + y$,

(c_{14}) $x + y + x \cdot y = x + y$.

Remark 1.2 *([3]-[8], [12]) On* A, *the natural order determines a bounded distributive lattice structure. Specifically, the join* $x \vee y$ *and the meet* $x \wedge y$ *of the elements* x *and* y *are given by:*

$$x \vee y = (x - y) + y = (y - x) + x = x \cdot y^* + y = y \cdot x^* + x$$

$$x \wedge y = (x^* \vee y^*)^* = x \cdot (x^* + y) = y \cdot (y^* + x).$$

Clearly, $x \cdot y \leq x \wedge y \leq x, y \leq x \vee y \leq x + y$.

We shall denote this distributive lattice with 0 and 1 by $L(A)$ (see [7], [8]). For any MV - algebra A we shall write $B(A)$ as an abbreviation of set of all complemented elements of $L(A)$. Elements of $B(A)$ are called the *boolean* elements of A.

Theorem 1.2 *([7]) For every element x in an MV - algebra A, the following conditions are equivalent:*

(i) $x \in B(A)$.

(ii) $x \vee x^* = 1$.

(iii) $x \wedge x^* = 0$.

(iv) $x + x = x$.

(v) $x \cdot x = x$.

(vi) $x + y = x \vee y$, *for all* $y \in A$.

(vii) $x \cdot y = x \wedge y$, *for all* $y \in A$.

Corollary 1.1 *([7], [8], [12])*

(i) *$B(A)$ is subalgebra of the MV - algebra A. A subalgebra B of A is a boolean algebra iff $B \subseteq B(A)$.*

(ii) *An MV - algebra A is a boolean algebra iff the operation $+$ is idempotent, i.e., the equation $x + x = x$ is satisfied by A.*

Theorem 1.3 *([7], [8], [12]) If $x, y, z, (x_i)_{i \in I}$ are elements of A, then the following hold:*

(c_{15}) $x + y = (x \vee y) + (x \wedge y)$,

(c_{16}) $x \cdot y = (x \vee y) \cdot (x \wedge y)$,

(c_{17}) $x + \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I}(x + x_i)$,

(c_{18}) $x + \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I}(x + x_i)$,

(c_{19}) $x \cdot \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I}(x \cdot x_i)$,

(c_{20}) $x \cdot \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I}(x \cdot x_i)$,

(c_{21}) $x \wedge \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I}(x \wedge x_i)$,

(c_{22}) $x \vee \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I}(x \vee x_i)$ *(if all suprema and infima exist).*

Lemma 1.2 *If a, b, x are elements of A, then:*

(c_{23}) $[(a \wedge x) + (b \wedge x)] \wedge x = (a + b) \wedge x,$

(c_{24}) $a^* \wedge x \geq x \cdot (a \wedge x)^*.$

Proof. (c_{23}). By c_{18} we have $[(a \wedge x) + (b \wedge x)] \wedge x = ((a \wedge x) + b) \wedge ((a \wedge x) + x) \wedge x = ((a \wedge x) + b) \wedge x = (a + b) \wedge (x + b) \wedge x = (a + b) \wedge x.$

(c_{24}). We have $x \cdot (a \wedge x)^* = x \cdot (a^* \vee x^*) \overset{c_{19}}{=} (x \cdot a^*) \vee (x \cdot x^*) \overset{c_5}{=} (x \cdot a^*) \vee 0 = x \cdot a^* \leq a^* \wedge x.$

Corollary 1.2 *If $a \in B(A)$ and $x, y \in A$, then:*

(c_{25}) $a^* \wedge x = x \cdot (a \wedge x)^*,$

(c_{26}) $a \wedge (x + y) = (a \wedge x) + (a \wedge y),$

(c_{27}) $a \vee (x + y) = (a \vee x) + (a \vee y).$

Proof. (c_{25}). See the proof of c_{24}.

(c_{26}). We have: $(a \wedge x) + (a \wedge y) \overset{c_{18}}{=} [(a \wedge x) + a] \wedge [(a \wedge x) + y] = [(a \wedge x) \vee a] \wedge [(a + y) \wedge (x + y)] = a \wedge (a + y) \wedge (x + y) = a \wedge (x + y).$

(c_{27}). We have $(a \vee x) + (a \vee y) = (a + x) + (a + y) = (a + a) + (x + y) = a + (x + y) = a \vee (x + y).$

Definition 1.2 *([3]-[8], [12]) Let A and B be $MV-$ algebras. A function $f : A \to B$ is a morphism of $MV-$ algebras iff it satisfies the following conditions, for every $x, y \in A$:*

(a_7) $f(0) = 0,$

(a_8) $f(x + y) = f(x) + f(y),$

(a_9) $f(x^*) = (f(x))^*.$

Remark 1.3 *It follows that:*

$$f(1) = 1, f(x \cdot y) = f(x) \cdot f(y), f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

Definition 1.3 *([3]-[8], [12]) An ideal of an MV - algebra A is a subset I of A satisfying the following conditions:*

(a_{10}) *If* $x \in I$, $y \in A$ *and* $y \leq x$, *then* $y \in I$,

(a_{11}) *If* $x, y \in I$, *then* $x + y \in I$.

We denote by $Id(A)$ the set of all ideals of A and by $I(A)$ the set

$$I(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{ then } x \in I\}.$$

Remark 1.4 *Clearly,* $Id(A) \subseteq I(A)$ *and if* $I_1, I_2 \in I(A)$, *then* $I_1 \cap I_2 \in I(A)$. *Also, if* $I \in I(A)$, *then* $0 \in I$.

For $M \subseteq A$ we denote by $(M]$ the *ideal of* A *generated by* M. If $M = \{a\}$ with $a \in A$, we denote by $(a]$ the ideal generated by $\{a\}$($(a]$ is called *principal*).

Proposition 1.1 *([7], [8]) If* $M \subseteq A$, *then*

$$(M] = \{x \in A : x \leq x_1 + ... + x_n \textit{ for some } x_1, ..., x_n \in M\}.$$

In particular, for $a \in A$, $(a] = \{x \in A : x \leq na \textit{ for some integer } n \geq 0\}$; *if* $e \in B(A)$, *then* $(e] = \{x \in A : x \leq e\}$.

2 Topologies on an MV-algebra

Definition 2.1 *A non-empty set* $\mathcal{F}$ *of elements of* $I \in I(A)$ *will be called a topology on* A *if the following properties hold:*

(a_{12}) *If* $I_1 \in \mathcal{F}, I_2 \in I(A)$ *and* $I_1 \subseteq I_2$, *then* $I_2 \in \mathcal{F}$ *(hence* $A \in \mathcal{F}$ *).*

(a_{13}) *If* $I_1, I_2 \in \mathcal{F}$, *then* $I_1 \cap I_2 \in \mathcal{F}$.

Any intersection of topologies on A is a topology; hence the set $T(A)$ of all topologies of A is a complete lattice with respect to inclusion.

Examples

1. If $I \in I(A)$, then the set

$$\mathcal{F}(I) = \{I' \in I(A) : I \subseteq I'\}$$

is clearly a topology on A.

2. A non-empty set $I \subseteq A$ will be called *regular* (see [6]) if for every $x, y \in A$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$, we have $x = y$. If we denote $R(A) = \{I \subseteq A : I \text{ is a regular subset of } A\}$, then $I(A) \cap R(A)$ is a topology on A (see [6]).

3. A subset $S \subseteq A$ is called $\wedge-$ *closed* if $1 \in S$ and if $x, y \in S$ implies $x \wedge y \in S$ (see [5]). For any $\wedge-$ closed subset S of A we set $\mathcal{F}_S = \{I \in I(A) :$

$I \cap S \cap B(A) \neq \oslash\}$. Then $\mathcal{F}_S$ is a topology on A . Clearly, if $I \in \mathcal{F}_S$ and $I \subseteq J$ (with $J \in I(A)$), then $I \cap S \cap B(A) \neq \oslash$, hence $J \cap S \cap B(A) \neq \oslash$, that is $J \in \mathcal{F}_S$.

If $I_1, I_2 \in \mathcal{F}_S$ then there exist $s_i \in I_i \cap S \cap B(A), i = 1, 2$. If we set $s = s_1 \wedge s_2$, then $s \in (I_1 \cap I_2) \cap S \cap B(A)$, hence $I_1 \cap I_2 \in \mathcal{F}_S$.

3 $\mathcal{F}$-multipliers and localization MV-algebras

Let $\mathcal{F}$ be a topology on A. Let us consider the relation $\theta_{\mathcal{F}}$ of A defined in the following way:

$(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

Lemma 3.1 *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. The reflexivity and the symmetry of $\theta_{\mathcal{F}}$ are immediate; to prove the transitivity of $\theta_{\mathcal{F}}$ let $(x, y), (y, z) \in \theta_{\mathcal{F}}$. Then there exists $I_1, I_2 \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1 \cap B(A)$, and $f \wedge y = f \wedge z$ for every $f \in I_2 \cap B(A)$. If the set $I = I_1 \cap I_2 \in \mathcal{F}$, then for every $g \in I \cap B(A)$, $g \wedge x = g \wedge z$, hence $(x, z) \in \theta_{\mathcal{F}}$.

To prove the compatibility of $\theta_{\mathcal{F}}$ with the operations $+$ and *, let (x, y) and $(z, t) \in \theta_{\mathcal{F}}$, that is there exists $I, J \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$, and $f \wedge z = f \wedge t$ for every $f \in J \cap B(A)$. If we denote $K = I \cap J$, then $K \in \mathcal{F}$ and for every $g \in K \cap B(A)$, $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge t$.

By c_{26} we deduce that for every $g \in K \cap B(A)$:

$$g \wedge (x + z) = (g \wedge x) + (g \wedge z) = (g \wedge y) + (g \wedge t) = g \wedge (y + t),$$

hence $(x + z, y + t) \in \theta_{\mathcal{F}}$,that is $\theta_{\mathcal{F}}$ is compatible with the operation $+$.

Also, since $x \wedge e = y \wedge e$ for every $e \in I \cap B(A)$, we deduce that $x^* \vee e^* = y^* \vee e^*$, hence $e \cdot (x^* \vee e^*) = e \cdot (y^* \vee e^*) \Leftrightarrow e \cdot (e^* + x^*) = e \cdot (e^* + y^*)$ (since $e^* \in B(A)$)$\Leftrightarrow$ $e \wedge x^* = e \wedge y^*$ for every $e \in I \cap B(A)$, hence $(x^*, y^*) \in \theta_{\mathcal{F}}$, that is $\theta_{\mathcal{F}}$ is compatible with the operations *, so $\theta_{\mathcal{F}}$ is a congruence on A.

We shall denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x \in A$ and by

$$p_{\mathcal{F}} : A \to A/\theta_{\mathcal{F}}$$

the canonical morphism of MV - algebras.

Proposition 3.1 *For $a \in A, a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Proof. For $a \in A$, we have $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}) \Leftrightarrow a/\theta_{\mathcal{F}} + a/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}} \Leftrightarrow (a+a)/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $(a+a)\wedge e = a \wedge e$ for every $e \in I \cap B(A) \overset{c_{26}}{\Leftrightarrow} (a\wedge e)+(a\wedge e) = a\wedge e$ for every $e \in I\cap B(A) \Leftrightarrow a\wedge e \in B(A)$ for every $e \in I \cap B(A)$.

So, if $a \in B(A)$, then for every $I \in \mathcal{F}$, $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$, hence $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.

Corollary 3.1 *If $\mathcal{F} = I(A) \cap R(A)$, then for $a \in A$, $a \in B(A)$ iff $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Definition 3.1 *Let $\mathcal{F}$ be a topology on A. An $\mathcal{F}-$ multiplier is a mapping $f : I \to A/\theta_{\mathcal{F}}$ where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:*

(a_{14}) $f(e \cdot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \cdot f(x)$.

(a_{15}) $f(x) \leq x/\theta_{\mathcal{F}}$.

(a_{16}) *If $e \in I \cap B(A)$, then $f(e) \in B(A/\theta_{\mathcal{F}})$.*

(a_{17}) $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, *for every $e \in I \cap B(A)$ and $x \in I$.*

By $dom(f) \in \mathcal{F}$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify the language, we will use *multiplier* instead of *partial multiplier,* using *total* to indicate that the domain of a certain multiplier is A.

If $\mathcal{F} = \{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so an $\mathcal{F}-$ multiplier is a total multiplier in the sense of [6].

The maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are multipliers in the sense of Definition 3.1 (see [6] for the case of multipliers).

Also, for $a \in B(A)$ and $I \in \mathcal{F}$, $f_a : I \to A/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is an $\mathcal{F}-$ multiplier (see [6] for the case of multipliers). If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and

$$M(A/\theta_{\mathcal{F}}) = \cup_{I\in\mathcal{F}} M(I, A/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$, we have a canonical mapping

$$\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}}),$$

defined by

$$\varphi_{I_1,I_2}(f) = f_{|I_1} \text{ for } f \in M(I_2, A/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\left\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I\in\mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2\in\mathcal{F},I_1\subseteq I_2} \right\rangle$$

and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$A_{\mathcal{F}} = \lim_{\substack{\to \\ I\in\mathcal{F}}} M(I, A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$, we shall denote by $\widehat{(I,f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

Remark 3.1 *We recall that, if $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$ (with $I_i \in \mathcal{F}$, $i = 1, 2$) be $\mathcal{F}-$multipliers. Let us consider the mapping

$$f_1 \oplus f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}},$$

defined by

$$(f_1 \oplus f_2)(x) = (f_1(x) + f_2(x)) \wedge x/\theta_{\mathcal{F}},$$

for any $x \in I_1 \cap I_2$, and let $\widehat{(I_1, f_1)} \oplus \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \oplus f_2)$.

Also, for any multiplier $f : I \to A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$), let us consider the mapping

$$f^* : I \to A/\theta_{\mathcal{F}},$$

defined by

$$f^*(x) = x/\theta_{\mathcal{F}} \cdot (f(x))^*,$$

for any $x \in I$ and let $\widehat{(I, f)}^* = \widehat{(I, f^*)}$.

Clearly the definitions of the operations $\oplus$ and * on $A_{\mathcal{F}}$ are correctly.

Lemma 3.2 $f_1 \oplus f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then $(f_1 \oplus f_2)(e \cdot x) = [f_1(e \cdot x) + f_2(e \cdot x)] \wedge (e \cdot x)/\theta_{\mathcal{F}} = [(e/\theta_{\mathcal{F}} \cdot f_1(x)) + (e/\theta_{\mathcal{F}} \cdot f_2(x))] \wedge (e/\theta_{\mathcal{F}} \cdot x/\theta_{\mathcal{F}}) = [(e/\theta_{\mathcal{F}} \wedge f_1(x)) + (e/\theta_{\mathcal{F}} \wedge f_2(x))] \wedge (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}) \overset{c_{26}}{=} [e/\theta_{\mathcal{F}} \wedge (f_1(x) + f_2(x))] \wedge [e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}] = e/\theta_{\mathcal{F}} \wedge [(f_1(x) + f_2(x)) \wedge x/\theta_{\mathcal{F}}] = e/\theta_{\mathcal{F}} \cdot (f_1 \oplus f_2)(x)$.

Clearly, $(f_1 \oplus f_2)(x) \leq x/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \oplus f_2)(e) = [f_1(e) + f_2(e)] \wedge e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \oplus f_2)(e) = x/\theta_{\mathcal{F}} \wedge [(f_1(e) + f_2(e)) \wedge e/\theta_{\mathcal{F}}] = (f_1(e) + f_2(e)) \wedge x/\theta_{\mathcal{F}} \wedge e/\theta_{\mathcal{F}}$$
$$\overset{c_{26}}{=} (f_1(e) + f_2(e)) \wedge x/\theta_{\mathcal{F}},$$

and

$$e/\theta_{\mathcal{F}} \wedge (f_1 \oplus f_2)(x) = e/\theta_{\mathcal{F}} \wedge [(f_1(x) + f_2(x)) \wedge x/\theta_{\mathcal{F}}] = e/\theta_{\mathcal{F}} \cdot [(f_1(x) + f_2(x)) \wedge x/\theta_{\mathcal{F}}]$$
$$\overset{c_{20}}{=} [e/\theta_{\mathcal{F}} \cdot (f_1(x) + f_2(x))] \wedge (e \cdot x)/\theta_{\mathcal{F}} \overset{c_{26}}{=} [(e/\theta_{\mathcal{F}} \cdot f_1(x)) + (e/\theta_{\mathcal{F}} \cdot f_2(x))] \wedge (e \cdot x)/\theta_{\mathcal{F}}$$
$$= [x/\theta_{\mathcal{F}} \cdot f_1(e) + x/\theta_{\mathcal{F}} \cdot f_2(e)] \wedge (e \cdot x)/\theta_{\mathcal{F}}$$
$$= [(f_1(e) \wedge x/\theta_{\mathcal{F}}) + (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge (e \wedge x)/\theta_{\mathcal{F}}$$
$$= [[(f_1(e) \wedge x/\theta_{\mathcal{F}}) + (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$

$$\overset{c_{23}}{=} ((f_1(e) + f_2(e)) \wedge x/\theta_{\mathcal{F}}) \wedge e/\theta_{\mathcal{F}} \overset{c_{26}}{=} (f_1(e) + f_2(e)) \wedge x/\theta_{\mathcal{F}},$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \oplus f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \oplus f_2)(x),$$

that is $f_1 \oplus f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Lemma 3.3 $f^* \in M(I, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I$ and $e \in B(A)$, then $f^*(e \cdot x) = (e \cdot x)/\theta_{\mathcal{F}} \cdot (f(e \cdot x))^* = e/\theta_{\mathcal{F}} \cdot x/\theta_{\mathcal{F}} \cdot (e/\theta_{\mathcal{F}} \cdot f(x))^* = e/\theta_{\mathcal{F}} \cdot x/\theta_{\mathcal{F}} \cdot [(e/\theta_{\mathcal{F}})^* + (f(x))^*] = x/\theta_{\mathcal{F}} \cdot (e/\theta_{\mathcal{F}} \cdot ((e/\theta_{\mathcal{F}})^* + (f(x))^*)) = x/\theta_{\mathcal{F}} \cdot (e/\theta_{\mathcal{F}} \wedge (f(x))^*) = x/\theta_{\mathcal{F}} \cdot (e/\theta_{\mathcal{F}} \cdot (f(x))^*) = e/\theta_{\mathcal{F}} \cdot (x/\theta_{\mathcal{F}} \cdot (f(x))^*) = e/\theta_{\mathcal{F}} \cdot f^*(x)$.

Clearly, $f^*(x) \leq x/\theta_{\mathcal{F}}$ for every $x \in I$.

Clearly, if $e \in I \cap B(A)$, then

$$f^*(e) = e/\theta_{\mathcal{F}} \cdot [f(e)]^* \in B(A/\theta_{\mathcal{F}}).$$

Since $f \in M(I, A/\theta_{\mathcal{F}})$, for $e \in I \cap B(A)$ and $x \in I$ we have:

$$x/\theta_{\mathcal{F}} \wedge f(e) = e/\theta_{\mathcal{F}} \wedge f(x) \Rightarrow (x/\theta_{\mathcal{F}})^* \vee (f(e))^* = (e/\theta_{\mathcal{F}})^* \vee (f(x))^*$$
$$\Rightarrow (x/\theta_{\mathcal{F}})^* + (f(e))^* = (e/\theta_{\mathcal{F}})^* + (f(x))^*$$
$$\Rightarrow e/\theta_{\mathcal{F}} \cdot x/\theta_{\mathcal{F}} \cdot [(x/\theta_{\mathcal{F}})^* + (f(e))^*] = x/\theta_{\mathcal{F}} \cdot e/\theta_{\mathcal{F}} \cdot [(e/\theta_{\mathcal{F}})^* + (f(x))^*] \Rightarrow$$
$$\Rightarrow e/\theta_{\mathcal{F}} \cdot [x/\theta_{\mathcal{F}} \wedge (f(e))^*] = x/\theta_{\mathcal{F}} \cdot [e/\theta_{\mathcal{F}} \wedge (f(x))^*]$$
$$\Rightarrow e/\theta_{\mathcal{F}} \cdot x/\theta_{\mathcal{F}} \cdot (f(e))^* = x/\theta_{\mathcal{F}} \cdot e/\theta_{\mathcal{F}} \cdot (f(x))^*$$
$$\Rightarrow x/\theta_{\mathcal{F}} \cdot [e/\theta_{\mathcal{F}} \cdot (f(e))^*] = e/\theta_{\mathcal{F}} \cdot [x/\theta_{\mathcal{F}} \cdot (f(x))^*]$$
$$\Rightarrow x/\theta_{\mathcal{F}} \wedge [e/\theta_{\mathcal{F}} \cdot (f(e))^*] = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \cdot (f(x))^*] \Rightarrow x/\theta_{\mathcal{F}} \wedge f^*(e) = e/\theta_{\mathcal{F}} \wedge f^*(x),$$

hence f^* verify and a_{17}, that is $f^* \in M(I, A/\theta_{\mathcal{F}})$.

Proposition 3.2 $(A_{\mathcal{F}}, \oplus, ^*, \widehat{(A, \mathbf{0})})$ *is an* MV *- algebra.*

Proof. We verify the axioms of MV - algebras.

a_1). Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1,2,3$ and denote $I = I_1 \cap I_2 \cap I_3 \in \mathcal{F}$.

Also, denote $f = f_1 \oplus (f_2 \oplus f_3)$, $g = (f_1 \oplus f_2) \oplus f_3$ and for $x \in I$, $a = f_1(x), b = f_2(x), c = f_3(x)$.

Clearly $a, b, c \leq x/\theta_{\mathcal{F}}$. Thus, for $x \in I$:

$f(x) = (f_1(x) + (f_2 \oplus f_3)(x)) \wedge x/\theta_{\mathcal{F}} = (f_1(x) + ((f_2(x) + f_3(x)) \wedge x/\theta_{\mathcal{F}})) \wedge x/\theta_{\mathcal{F}} =$
$= (a + (b+c) \wedge x/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}} = ((a \wedge x/\theta_{\mathcal{F}}) + ((b+c) \wedge x/\theta_{\mathcal{F}})) \wedge x/\theta_{\mathcal{F}} \overset{c_{23}}{=} (a+b+c) \wedge x/\theta_{\mathcal{F}}$.

Analogously, $g(x) = (a+b+c) \wedge x/\theta_{\mathcal{F}}$, hence $f = g$, so

$$\widehat{(I_1, f_1)} \oplus [\widehat{(I_2, f_2)} \oplus \widehat{(I_3, f_3)}] = [\widehat{(I_1, f_1)} \oplus \widehat{(I_2, f_2)}] \oplus \widehat{(I_3, f_3)},$$

that is the operation $\oplus$ is associative on $A_{\mathcal{F}}$.

a_2). Obviously.

a_3). Let $f \in M(I, A/\theta_{\mathcal{F}})$ with $I \in \mathcal{F}$. If $x \in I$, then $(f \oplus \mathbf{0})(x) = (f(x) + \mathbf{0}(x)) \wedge x/\theta_{\mathcal{F}} = f(x) \wedge x/\theta_{\mathcal{F}} = f(x)$, hence $f \oplus \mathbf{0} = f$, that is

$$\widehat{(I, f)} \oplus \widehat{(A, \mathbf{0})} = \widehat{(I, f)}.$$

a_4). For $x \in A$, we have $\mathbf{0}^*(x) = x/\theta_{\mathcal{F}} \cdot (\mathbf{0}(x))^* = x/\theta_{\mathcal{F}} \cdot (0/\theta_{\mathcal{F}})^* = x/\theta_{\mathcal{F}} \cdot 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x)$, hence $\mathbf{0}^* = \mathbf{1}$ and $\mathbf{1}^*(x) = x/\theta_{\mathcal{F}} \cdot (\mathbf{1}(x))^* = x/\theta_{\mathcal{F}} \cdot (x/\theta_{\mathcal{F}})^* = 0/\theta_{\mathcal{F}} = \mathbf{0}(x)$. So, $\mathbf{0}^{**} = \mathbf{1}^* = \mathbf{0}$ that is

$$\widehat{(A, \mathbf{0})}^{**} = \widehat{(A, \mathbf{0})}$$

and by Remark 11, a_4) is verified.

a_5). Since $\mathbf{0}^* = \mathbf{1}$, for $f \in M(I, A/\theta_{\mathcal{F}})$ (with $I \in \mathcal{F}$) and $x \in I$, we have: $(f \oplus \mathbf{0}^*)(x) = (f \oplus \mathbf{1})(x) = (f(x) + x/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x) = \mathbf{0}^*(x)$, hence $f \oplus \mathbf{0}^* = \mathbf{0}^*$, that is

$$\widehat{(I, f)} \oplus \widehat{(A, \mathbf{0})}^* = \widehat{(A, \mathbf{0})}^*.$$

a_6). Let $f \in M(I, A/\theta_{\mathcal{F}}), g \in M(J, A/\theta_{\mathcal{F}})$ (with $I, J \in \mathcal{F}$) and $x \in I \cap J$.

If denote $h = (f^* \oplus g)^* \oplus g$, $t = (g^* \oplus f)^* \oplus f$, and $a = f(x), b = g(x)$, then $a, b \leq x/\theta_{\mathcal{F}}$ and we have:

$h(x) = ((f^* \oplus g)^* \oplus g)(x) = ((f^* \oplus g)^*(x) + g(x)) \wedge x/\theta_{\mathcal{F}} = ((x/\theta_{\mathcal{F}} \cdot ((f^* \oplus g)(x))^*) + g(x)) \wedge x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \cdot ((f^*(x) + g(x)) \wedge x/\theta_{\mathcal{F}})^* + g(x)) \wedge x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \cdot (((x/\theta_{\mathcal{F}} \cdot (f(x))^*) + g(x)) \wedge x/\theta_{\mathcal{F}})^* + g(x)) \wedge x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \cdot (((x/\theta_{\mathcal{F}} \cdot a^*) + b) \wedge x/\theta_{\mathcal{F}})^* + b) \wedge x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \cdot (((x/\theta_{\mathcal{F}} \cdot a^*) + b)^* \vee (x/\theta_{\mathcal{F}})^*) + b) \wedge x/\theta_{\mathcal{F}}$

$= (x/\theta_{\mathcal{F}} \cdot ((x/\theta_{\mathcal{F}} \cdot a^*) + b)^* + b) \wedge x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \cdot ((x/\theta_{\mathcal{F}})^* + a) \cdot b^* + b) \wedge x/\theta_{\mathcal{F}} = (((x/\theta_{\mathcal{F}} \wedge a) \cdot b^*) + b) \wedge x/\theta_{\mathcal{F}} = ((a \cdot b^*) + b) \wedge x/\theta_{\mathcal{F}} = (a \vee b) \wedge x/\theta_{\mathcal{F}} = a \vee b$.

Analogously, $t(x) = a \vee b = h(x)$, hence $h = t$, so

$$((\widehat{I,f})^* \oplus (\widehat{J,g}))^* \oplus (\widehat{J,g}) = ((\widehat{J,g})^* \oplus (\widehat{I,f}))^* \oplus (\widehat{I,f}).$$

Remark 3.2 *$(M(A/\theta_{\mathcal{F}}), \oplus, ^*, \mathbf{0})$ is an MV - algebra.*

Lemma 3.4 *Let $f_1, f_2 \in M(A/\theta_{\mathcal{F}})$ with $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ $(I_i \in \mathcal{F})$, $i = 1, 2$. Then for every $x \in I_1 \cap I_2$:*

(i) $(f_1 \odot f_2)(x) = f_1(x) \cdot [(x/\theta_{\mathcal{F}})^* + f_2(x)] = f_2(x) \cdot [(x/\theta_{\mathcal{F}})^* + f_1(x)]$.

(ii) $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$.

(iii) $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$.

Proof. We recall that in the MV - algebra $M(A/\theta_{\mathcal{F}})$ we have:

$$f_1 \odot f_2 = (f_1^* \oplus f_2^*)^*,$$

$$f_1 \wedge f_2 = f_1 \odot [f_1^* \oplus f_2],$$

and

$$f_1 \vee f_2 = (f_1^* \wedge f_2^*)^*.$$

For $x \in I_1 \cap I_2$ we denote $a = f_1(x), b = f_2(x)$; clearly $a, b \leq x/\theta_{\mathcal{F}}$.

So: (i).$(f_1 \odot f_2)(x) = x/\theta_{\mathcal{F}} \cdot [(f_1^*(x) + f_2^*(x)) \wedge x/\theta_{\mathcal{F}}]^* = x/\theta_{\mathcal{F}} \cdot [(x/\theta_{\mathcal{F}} \cdot a^* + x/\theta_{\mathcal{F}} \cdot b^*) \wedge x/\theta_{\mathcal{F}}]^* = x/\theta_{\mathcal{F}} \cdot [(x/\theta_{\mathcal{F}} \cdot a^* + x/\theta_{\mathcal{F}} \cdot b^*)^* \vee (x/\theta_{\mathcal{F}})^*]$
$= x/\theta_{\mathcal{F}} \cdot [((x/\theta_{\mathcal{F}})^* + a) \cdot ((x/\theta_{\mathcal{F}})^* + b) \vee (x/\theta_{\mathcal{F}})^*] = x/\theta_{\mathcal{F}} \cdot ((x/\theta_{\mathcal{F}})^* + a) \cdot ((x/\theta_{\mathcal{F}})^* + b) = (x/\theta_{\mathcal{F}} \wedge a) \cdot ((x/\theta_{\mathcal{F}})^* + b) = a \cdot ((x/\theta_{\mathcal{F}})^* + b) = f_1(x) \cdot ((x/\theta_{\mathcal{F}})^* + f_2(x)) = f_2(x) \cdot ((x/\theta_{\mathcal{F}})^* + f_1(x))$.

(ii).$(f_1^* \oplus f_2)(x) = (x/\theta_{\mathcal{F}} \cdot a^* + b) \wedge x/\theta_{\mathcal{F}}$, hence $(f_1 \wedge f_2)(x) = f_1(x) \cdot [(x/\theta_{\mathcal{F}})^* + (f_1^* \oplus f_2)(x)] = a \cdot [(x/\theta_{\mathcal{F}})^* + (x/\theta_{\mathcal{F}} \cdot a^* + b) \wedge x/\theta_{\mathcal{F}}] \overset{c_{18}}{=} a \cdot [((x/\theta_{\mathcal{F}})^* + x/\theta_{\mathcal{F}} \cdot a^* + b) \wedge ((x/\theta_{\mathcal{F}})^* + x/\theta_{\mathcal{F}})] \overset{c_5}{=} a \cdot [((x/\theta_{\mathcal{F}})^* + x/\theta_{\mathcal{F}} \cdot a^* + b) \wedge 1] = a \cdot ((x/\theta_{\mathcal{F}})^* + x/\theta_{\mathcal{F}} \cdot a^* + b) = a \cdot [((x/\theta_{\mathcal{F}})^* \vee a^*) + b] = a \cdot (a^* + b) = a \wedge b = f_1(x) \wedge f_2(x)$.

(iii). $(f_1 \vee f_2)(x) = (f_1^* \wedge f_2^*)^*(x) = x/\theta_{\mathcal{F}} \cdot [(x/\theta_{\mathcal{F}} \cdot a^*) \wedge (x/\theta_{\mathcal{F}} \cdot b^*)]^* = x/\theta_{\mathcal{F}} \cdot [((x/\theta_{\mathcal{F}})^* + a) \vee ((x/\theta_{\mathcal{F}})^* + b)] = x/\theta_{\mathcal{F}} \cdot [(x/\theta_{\mathcal{F}})^* + (a \vee b)] = x/\theta_{\mathcal{F}} \wedge (a \vee b) = a \vee b = f_1(x) \vee f_2(x)$.

Corollary 3.2 *$(A_{\mathcal{F}}, \oplus, ^*, \mathbf{0})$ is an MV - algebra, where $\mathbf{0} = \widehat{(A, \mathbf{0})}$ and $\mathbf{1} = \mathbf{0}^* = \widehat{(A, \mathbf{1})}$. Also, for two elements $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)}$ in $A_{\mathcal{F}}$ we have*

$$\widehat{(I_1, f_1)} \odot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \odot f_2),$$

$$\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2),$$
$$\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2)$$

where $f_1 \odot f_2, f_1 \wedge f_2, f_1 \vee f_2$ are characterized as in Lemma 3.4.

Definition 3.2 *The MV - algebra $A_{\mathcal{F}}$ will be called the localization MV - algebra of A with respect to the topology $\mathcal{F}$.*

Lemma 3.5 *Let the map $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(A, \overline{f_a})}$ for every $a \in B(A)$. Then:*

(i) $v_{\mathcal{F}}$ is a morphism of MV - algebras.

(ii) For $a \in B(A)$, $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$.

(iii) $v_{\mathcal{F}}(B(A)) \in R(A_{\mathcal{F}})$.

Proof. (i). We have $v_{\mathcal{F}}(0) = \widehat{(A, \overline{f_0})} = \widehat{(A, \mathbf{0})} = \mathbf{0}$.

For $a, b \in B(A)$, we have $v_{\mathcal{F}}(a) \oplus v_{\mathcal{F}}(b) = \widehat{(A, \overline{f_a})} \oplus \widehat{(A, \overline{f_b})} = (A, \widehat{\overline{f_a} \oplus \overline{f_b}}) \overset{c_{23}}{=} \widehat{(A, \overline{f_{a+b}})} = v_{\mathcal{F}}(a+b)$ and for $x \in A$, since

$$(\overline{f}_a)^*(x) = x/\theta_{\mathcal{F}} \cdot [(a \wedge x)/\theta_{\mathcal{F}}]^* = x/\theta_{\mathcal{F}} \cdot ((x/\theta_{\mathcal{F}})^* \vee (a/\theta_{\mathcal{F}})^*)$$
$$= x/\theta_{\mathcal{F}} \cdot ((x/\theta_{\mathcal{F}})^* + (a/\theta_{\mathcal{F}})^*) = x/\theta_{\mathcal{F}} \wedge (a/\theta_{\mathcal{F}})^* = \overline{f_{a^*}}(x),$$

that is $(\overline{f}_a)^* = \overline{f_{a^*}}$ we deduce that

$$v_{\mathcal{F}}(a^*) = \widehat{(A, \overline{f_{a^*}})} = \widehat{(A, \overline{f_a})}^* = (v_{\mathcal{F}}(a))^*,$$

hence $v_{\mathcal{F}}$ is a morphism of MV - algebras.

(ii). For $a \in B(A)$ we have $a + a = a$, hence by c_{23}, $((a \wedge x) + (a \wedge x)) \wedge x = a \wedge x$ for every $x \in A$.

Since $A \in \mathcal{F}$ we deduce that $((a \wedge x)/\theta_{\mathcal{F}} + (a \wedge x)/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}} = (a \wedge x)/\theta_{\mathcal{F}}$ hence $\overline{f_a} \oplus \overline{f_a} = \overline{f_a}$, that is

$$\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}}).$$

(iii). To prove that $v_{\mathcal{F}}(B(A))$ is a regular subset of $A_{\mathcal{F}}$, let $\widehat{(I_i, f_i)} \in A_{\mathcal{F}}$, $I_i \in \mathcal{F}$, $i = 1, 2$, such that $\widehat{(A, \overline{f_a})} \wedge \widehat{(I_1, f_1)} = \widehat{(A, \overline{f_a})} \wedge \widehat{(I_2, f_2)}$ for every $a \in B(A)$. By (ii), $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$.

Then $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$ for every $x \in I_1 \cap I_2$ and $a \in B(A)$ $\Leftrightarrow$ $f_1(x) \wedge x/\theta_{\mathcal{F}} \wedge a/\theta_{\mathcal{F}} = f_2(x) \wedge x/\theta_{\mathcal{F}} \wedge a/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$ and $a \in B(A)$ $\Leftrightarrow$ $f_1(x) \wedge a/\theta_{\mathcal{F}} = f_2(x) \wedge a/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$ and $a \in B(A)$.

In particular for $a = 1$, $a/\theta_{\mathcal{F}} = \mathbf{1} \in B(A/\theta_{\mathcal{F}})$ we obtain that $f_1(x) = f_2(x)$ for every $x \in I_1 \cap I_2$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is $v_{\mathcal{F}}(B(A)) \in R(A_{\mathcal{F}})$.

4 Applications

In the following we describe the localization MV - algebra $A_{\mathcal{F}}$ in some special instances.

1. If $I \in I(A)$, and $\mathcal{F}$ is the topology

$$\mathcal{F}(I) = \{I' \in I(A) : I \subseteq I'\}$$

(see example 1 in section 2), then $A_{\mathcal{F}}$ is isomorphic with $M(I, A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

2. If $\mathcal{F} = I(A) \cap R(A)$ is the topology of regular ideals (see example 2 in section 2), then $\theta_{\mathcal{F}}$ is the identity congruence of A and

$$A_{\mathcal{F}} = \lim_{\substack{\to \\ I \in \mathcal{F}}} M(I, A),$$

where $M(I, A)$ is the set of multipliers of A having the domain I (see [6]).

In this situation we obtain:

Proposition 4.1 *In the case $\mathcal{F} = I(A) \cap R(A)$, $A_{\mathcal{F}}$ is exactly the maximal MV-algebra $Q(A)$ of quotients of A (introduced by the authors in [6] where this is denoted by A'').*

3. If $S \subseteq A$ an $\wedge-$closed system of A. Consider the following congruence on $A : (x, y) \in \theta_S \Leftrightarrow$ there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ (see [5]). $A[S] = A/\theta_S$ is called in [5] the *MV - algebra of fractions of A relative to the $\wedge-$closed system S.*

Proposition 4.2 *If $\mathcal{F}_S$ is the topology associated with a $\wedge-$closed system $S \subseteq A$ (see example 3 in section 2), then the MV - algebra $A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Proof. For $x, y \in A$ we have $(x, y) \in \theta_{\mathcal{F}_S} \Leftrightarrow$ there exists $I \in \mathcal{F}_S$ (hence $I \cap S \cap B(A) \neq \oslash$) such that $x \wedge e = y \wedge e$ for any $e \in I \cap B(A)$. Since $I \cap S \cap B(A) \neq \oslash$ there exists $e_0 \in I \cap S \cap B(A)$ such that $x \wedge e_0 = y \wedge e_0$, hence $(x, y) \in \theta_S$. So, $\theta_{\mathcal{F}_S} \subseteq \theta_S$.

If $(x, y) \in \theta_S$, there exists $e_0 \in S \cap B(A)$ such that $x \wedge e_0 = y \wedge e_0$. If we set $I = (e_0] = \{a \in A : a \leq e_0\}$, then $I \in I(A)$; since $e_0 \in I \cap S \cap B(A)$, then $I \cap S \cap B(A) \neq \oslash$, that is $I \in \mathcal{F}_S$. For every $e \in I \cap B(A)$, $e \leq e_0$, hence $e = e \wedge e_0$ and $x \wedge e = x \wedge (e_0 \wedge e) = (x \wedge e_0) \wedge e = (y \wedge e_0) \wedge e = y \wedge (e_0 \wedge e) = y \wedge e$, hence $(x, y) \in \theta_{\mathcal{F}_S}$, that is $\theta_{\mathcal{F}_S} = \theta_S$.

Then $A[S] = A/\theta_S$; therefore an $\mathcal{F}_S-$multiplier can be considered in this case (see $a_{14} - a_{17}$) as a mapping $f : I \to A[S]$ $(I \in \mathcal{F}_S)$ having the properties

$f(e \cdot x) = e/S \cdot f(x)$ and $f(x) \le x/S$, for every $x \in I$ and $e \in B(A)$, if $e \in I \cap B(A)$, then $f(e) \in B(A[S])$ and for every $e \in I \cap B(A)$ and $x \in I$,

$$(e/S) \wedge f(x) = (x/S) \wedge f(e)$$

(x/S denotes the congruence class of x relative to θ_S).

We recall ([5]) that for $x \in A$, $x/S \in B(A[S])$ iff there is $e_0 \in S \cap B(A)$ such that $e_0 \wedge x \in B(A)$. In particular if $e \in B(A)$, then $e/S \in B(A[S])$.

If $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in A_{\mathcal{F}_S} = \lim_{\rightarrow I \in \mathcal{F}_S} M(I, A[S])$, and $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ then there exists $I \in \mathcal{F}_S$ such that $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$. Since $I, I_1, I_2 \in \mathcal{F}_S$, there exist $e \in I \cap S \cap B(A), e_1 \in I_1 \cap S \cap B(A)$ and $e_2 \in I_2 \cap S \cap B(A)$. We shall prove that $f_1(e_1) = f_2(e_2)$. If denote $f = e \wedge e_1 \wedge e_2$, then $f \in I \cap S \cap B(A)$, and $f \le e_1, e_2$. Since $e_1 \wedge f = e_2 \wedge f$ then $f_1(e_1 \wedge f) = f_1(e_2 \wedge f) = f_2(e_2 \wedge f) \Leftrightarrow f_1(e_1) \wedge f/S = f_2(e_2) \wedge f/S \Leftrightarrow f_1(e_1) \wedge 1 = f_2(e_2) \wedge 1$ (since $f \in S \Rightarrow f/S = 1$) $\Leftrightarrow f_1(e_1) = f_2(e_2)$. In a similar way we can show that $f_1(s_1) = f_2(s_2)$ for any $s_1, s_2 \in I \cap S \cap B(A)$.

In accordance with these considerations we can define the mapping:

$$\alpha : A_{\mathcal{F}_S} = \lim_{\rightarrow I \in \mathcal{F}_S} M(I, A[S]) \rightarrow B(A[S]),$$

by putting

$$\alpha(\widehat{(I, f)}) = f(s) \in B(A[S]),$$

where $s \in I \cap S \cap B(A)$.

This mapping is a morphism of MV - algebras.

Indeed, $\alpha(\mathbf{0}) = \alpha(\widehat{(A, \mathbf{0})}) = \mathbf{0}(e) = 0/S = \mathbf{0}$ for every $e \in S \cap B(A)$. If $\widehat{(I, f)} \in A_{\mathcal{F}_S}$, we have $\alpha(\widehat{(I, f)^*}) = \alpha(\widehat{(I, f^*)}) = f^*(e) = (e/S) \cdot [f(e)]^* = 1 \cdot (f(e))^* = (f(e))^* = (\alpha(\widehat{(I, f)}))^*$ (with $e \in I \cap S \cap B(A)$). Also, for every $\widehat{(I_i, f_i)} \in A_{\mathcal{F}_S}, i = 1, 2$ we have: $\alpha[\widehat{(I_1, f_1)} \oplus \widehat{(I_2, f_2)}] = \alpha[\widehat{(I_1 \cap I_2, f_1 \oplus f_2)}.] = (f_1 \oplus f_2)(e) = (f_1(e) + f_2(e)) \wedge (e/S) = f_1(e) + f_2(e) = \alpha[\widehat{(I_1, f_1)}] + \alpha[\widehat{(I_2, f_2)}]$ (with $e \in I_1 \cap I_2 \cap S \cap B(A)$).

We shall prove that α is injective and surjective. To prove the injectivity of α let $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in A_{\mathcal{F}_S}$ such that $\alpha(\widehat{(I_1, f_1)}) = \alpha(\widehat{(I_2, f_2)})$. Then for any $e_1 \in I_1 \cap S \cap B(A)$, $e_2 \in I_2 \cap S \cap B(A)$ we have $f_1(e_1) = f_2(e_2)$. If $f_1(e_1) = x/S, f_2(e_2) = y/S$ with $x, y \in A$, since $x/S = y/S$, there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

If we consider $e' = e \wedge e_1 \wedge e_2 \in I_1 \cap I_2 \cap S \cap B(A)$, we have $x \wedge e' = y \wedge e'$ and $e' \le e_1, e_2$. It follows that $f_1(e') = f_1(e' \wedge e_1) = f_1(e_1) \wedge (e'/S) = x/S \wedge 1 = x/S = y/S = f_2(e_2) = f_2(e_2) \wedge (e'/S) = f_2(e_2 \wedge e') = f_2(e')$. If denote $I = (e']$ then we obtained that $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is α is injective.

To prove the surjectivity of α, let $a/S \in B(A[S])$ (hence there exists $e_0 \in S \cap B(A)$ such that $a \wedge e_0 \in B(A)$). We consider $I_0 = (e_0] = \{x \in A : x \le e_0\}$ (since $e_0 \in I_0 \cap S \cap B(A)$, then $I_0 \in \mathcal{F}_S$) and define $f_a : I_0 \to A[S]$ by putting $f_a(x) = x/S \wedge a/S = (x \wedge a)/S$ for every $x \in I_0$.

We shall prove that f_a is a $\mathcal{F}_S-$multiplier. Indeed, if $e \in B(A)$ and $x \in I_0$, since $e/S \in B(A[S])$, then

$$f_a(e \cdot x) = f_a(e \wedge x) = (e/S) \wedge (x/S) \wedge (a/S)$$

$$= (e/S) \wedge ((x/S) \wedge (a/S)) = (e/S) \wedge f_a(x) = (e/S) \cdot f_a(x);$$

Clearly, $f_a(x) \le x/S$. Also, if $e \in I_0 \cap B(A)$, then $f_a(e) = e/S \wedge a/S \in B(A[S])$.

Clearly if for every $e \in I_0 \cap B(A)$ and $x \in I_0$,

$$(e/S) \wedge f_a(x) = (x/S) \wedge f_a(e),$$

hence f_a is a $\mathcal{F}_S-$multiplier and we shall prove that $\alpha(\widehat{(I_0, f_a)}) = a/S$.

Indeed, since $e_0 \in S$ we have $\alpha(\widehat{(I_0, f_a)}) = f_a(e_0) = (e_0 \wedge a)/S = (e_0/S) \wedge (a/S) = 1 \wedge (a/S) = a/S$.

So, we have proved that α is an isomorphism of MV - algebras.

REFERENCES

[1] D. Buşneag, *Hilbert algebra of fractions and maximal Hilbert algebras of quotients,* Kobe Journal of Mathematics, 5 (1988), 161-172.

[2] D. Buşneag, *F-multipliers and the localization of Hilbert algebras,* Zeitschr. f. math. Logik und Grundlagen d. Math. Bd.36 (1990), 331-338.

[3] D. Buşneag, D. Piciu, *Meet-irreducible ideals in an MV - algebra*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXVIII, (2001), 110-119.

[4] D. Buşneag, D. Piciu, *On the lattice of ideals of an MV - algebra*, Scientiae Mathematicae Japonicae, 56, No.2 (2002), 367-372,:e6, 221-226.

[5] D. Buşneag, D. Piciu, *MV - algebra of fractions relative to a $\wedge-$closed system*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXX, (2003), 1-6.

[6] D. Buşneag, D. Piciu, *MV-algebra of fractions and maximal MV-algebra of quotients,* to appear in Journal of Multiple-Valued Logic and Soft Computing.

[7] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc., 88(1958), 467-490.

[8] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic foundation of many -valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.

[9] W. H. Cornish, *The multiplier extension of a distributive lattice*, Journal of Algebra, 32 (1974), 339-355.

[10] C. Dan, *F-multipliers and the localization of Heyting algebras*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXIV, (1997), 98-109.

[11] G. Georgescu, *F-multipliers and the localization of distributive lattices,* Algebra Universalis, 21 (1985), 181-197.

[12] G. Georgescu, A. Iorgulescu, *Pseudo MV - algebras*, Multi. Val. Logic, vol 6 (2001), 95-135.

[13] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, New York,1973.

[14] J. Schmid, *Multipliers on distributive lattices and rings of quotients*, Houston Journal of Mathematics, vol.6, no. 3 (1980).

[15] J. Schmid, *Distributive lattices and rings of quotients*, Coll. Math. Societatis Janos Bolyai, 33, Szeged, Hungary, (1980).

[16] B. Strenström, *Flatness and localization over monoids*, Math. Nachrichten 48 (1971), 315-334.

Department of Mathematics, University of Craiova,
Al.I.Cuza street, 13, Craiova, 200585, Romania
E-mail: busneag@central.ucv.ro

Department of Mathematics, University of Craiova,
Al.I.Cuza street, 13, Craiova, 200585, Romania
E-mail: danap@central.ucv.ro