An. Şt. Univ. Ovidius Constanţa Vol. 11(2), 2003, 69–74

SOME EXAMPLES OF REAL DIVISION ALGEBRAS

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Abstract

It is known, by Frobenius Theorem, that the only division associative algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$. In 1958 Bott and Milnor showed that the finite-dimensional real division algebra can have only dimensions 1, 2, 4, 8. The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, first, second and third are associative and the fourth is non-associative, are the only finite-dimensional alternative real division algebras. In [Ok, My; 80] is given a construction of division non-unitary non-alternative algebras over an arbitrary field K with $charK \neq 2$. In this paper we analyse a case when these algebras are isomorphic.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are **flexible** (i.e. $(xy)\,x = x\,(yx)$, for all x, y) and every element of these algebras satisfies the quadratic equation: $x^2 - t(x)x + n\,(x)\,e = 0$, where t is a linear and n is a quadratic form.

Each of these algebras is a **composition** algebras, i.e. has an associated symmetric non-degenerate bilinear form $(x, y) = \frac{1}{2}\left[n\,(x + y) - n\,(x) - n\,(y)\right]$, permitting composition:

$$(xy, xy) = (x, y)\,(y, y)\,. \tag{1}$$

Let A be an arbitrary algebra. A vector spaces morphism $f : A \to A$ is an **involution** if $f\,(xy) = f\,(y)\,f\,(x)$ and $f\,(f\,(x)) = x$, $\forall x \in A$.

Proposition 1.[Ok, My; 80] *Let A be a finite dimensional composition algebra over a field K with $charK \neq 2$ and let (x, y) be its associated symmetric non-degenerate bilinear form defined on A. If we have the relations:*

$$x(yx) = (xy)\,x = (x, x)\,y, \tag{2}$$

Key Words: composition algebra, division algebra, flexible algebra, Hurwitz algebra, power-associative algebra

Mathematical Reviews subject classification: 17D05, 17D99

then A *has the dimension* $1,2,4$ *or* $8.\square$

Proposition 2. [Ok, My; 80] *Let* A *be an algebra over the field* K *with* $charK \neq 2$, *and* (x,y) *the associated symmetric non-degenerate bilinear form. Then* A *satisfies the relation* (2) *if and only if* (x,y) *is associative, that means:*

$$(xy,z) = (x,yz)\,, x,y,z \in A, \tag{3}$$

and (x,y) *permits composition.*$\square$

Proposition 3. [Ok, My; 80] *Let* A *be a finite-dimensional composition algebra over the field* K, *with* $charK \neq 2$ *and* (x,y) *a symmetric bilinear form on* A. *Then* A *is a division algebra if and only if* $(x,x) \neq 0$ *for* $x \neq 0$, $x \in A$.

Let $sl\,(3,\mathbb{C})$ be the Lie algebra of the complex matrices of order three with the zero trace.

We define the multiplication $x * y$ in $sl\,(3,\mathbb{C})$:

$$x * y = \mu xy + (1-\mu)\,yx - \frac{1}{3}Tr\,(xy)\,I, \tag{4}$$

where xy is the multiplcation of the matrices x and y, $\mu \in \mathbb{C}$, $\mu \neq \frac{1}{2}$ and I is the identity matrix.

Since, for $x,y \in sl\,(3,\mathbb{C})\,, Tr(xy) = 0$, $sl\,(3,\mathbb{C})$ becomes an algebra over $\mathbb{C}$ with the multiplication defined by the relation (4).

Suppose that $\mu \in \mathbb{C}$ satisfies the equation:

$$3\mu\,(1-\mu) = 1. \tag{5}$$

We define the non-degenerate symmetric bilinear form:

$$(x,y) = \frac{1}{6}Tr\,(xy)\,, x,y \in sl\,(3,\mathbb{C})\,, \tag{6}$$

and the associated quadratic form:

$$N\,(x) = (x,x) = \frac{1}{6}Trx^2. \tag{7}$$

Obviously, this bilinear form is associative and permits composition:

$$N\,(x * y) = N\,(x)\,N\,(y)\,. \tag{8}$$

Using the Cayley-Hamilton Theorem, the relation (8) gives us the equation:

$$x^3 - \frac{1}{2}\left(Trx^2\right)x - \frac{1}{3}\left(Trx^2\right)I = 0, \text{ for } x \in sl\,(3,\mathbb{C}) \tag{9}$$

and

$$Trx^4 = \frac{1}{2}\left(Trx^2\right)^2. \tag{10}$$

The algebra $sl\,(3,\mathbb{C})$,with the multiplication given by the relations (4) and (5) , is called the **pseudo-octonions algebra.** This algebra is a simple flexible non-associative algebra without unity element.

Let $\bar{A} = \{x \in (sl\,(3,\mathbb{C})\,,*)\ /\ \bar{x}^t = x\}$. Since $\bar{\mu} = 1-\mu$ is the conjugate of μ, it follows from (4) that $\left(\overline{x*y}\right)^t = x*y$, for all $x, y \in \bar{A}$. Therefore, $\left(\bar{A},*\right)$ becomes an algebra over $\mathbb{R}$, called the **real pseudo-octononion algebra**. So that, this algebra gives us a new example of real division algebra without unity element, with dimension 8.

Let A be a composition algebra over the field K with e the unit element. We have the relation:

$$x^2 - 2\,(e,x)\,x + (x,x)\,e = 0, \forall x \in A, \tag{11}$$

with (x,y) the associated nondegenerated bilinear form. Then the algebra A has the dimensions $1,2,4$ or 8 and it is a quaternion or octonion algebra when $\dim A = 4$ or $\dim A = 8$. Let $x \in A$. We denoted by $\bar{x} = 2\,(e,x)\,e - x$, and it is called the **conjugate** of x.

We define a new multiplication on A:

$$x \circ y = \bar{x}\bar{y} = -yx + 2\,(e,yx)\,e. \tag{12}$$

The algebra A defined in (12) is denoted A_e. It satisfies the relation (2) and:

$$x \circ e = e \circ x = \bar{x}. \tag{13}$$

It is obvious that $(e,e) = 1$ and $e \circ e = e$.

An element $e \in A$ with the properties

$$x \circ e = e \circ x = \bar{x}, (e,e) = 1 \text{ and } e \circ e = e \tag{14}$$

is called the **pseudo-unit** or **para-unit** of the algebra A.

If O is a real octonion algebra with the unit element e, then the real algebra O_e defined by (12) is called the **para-octonion algebra** and has the para-unit e. The real pseudo-octonion algebra and para-octonion algebra are division algebras.

Proposition 4.[Ok, My; 80] *Let A be an algebra over the field K, with $charK \neq 2$, wich satisfies the condition of Proposition 2. Let $\gamma \in K$ and $g \in A$ be arbitrary elements such that*:

$$\gamma \neq \frac{1}{(g,g)}. \tag{15}$$

Let $A(\gamma, g)$ be an algebra defined on the vector space A with multiplication $x * y$ given by:

$$x * y = -yx + \gamma (g, yx) g. \tag{16}$$

If $(x, x) \neq 0$ for $x \neq 0$, $x \in A$, then $A(\gamma, g)$ is a division algebra.

Proof. [Ok, My; 80]For $\gamma = 2$ and $g = e$, we obtain the para-octonion algebra. For $a \neq 0, b \in A(\gamma, g)$ the equations $a * x = b$ and $y * a = b$ become:

$$-xa + \gamma (g, xa) g = b. \tag{17}$$

We multiply the relation (17) to the left side with a and we get:

$$-(a, a) x = \gamma (g, xa) ag + ab. \tag{18}$$

We apply the $(\cdot, g)$in the relation (17) and we obtain:

$$(g, xa) [-1 + \gamma (g, g)] = (g, b). \tag{19}$$

Since $(a, a) \neq 0$, it results that the equation $a * x = b$ has a unique solution:

$$x = -\frac{1}{(a, a)} \left[ab + \frac{\gamma (b, g)}{-1 + \gamma (g, g)} ag \right].$$

Similarly, we get that the equation $y * a = b$ has the unique solution:

$$y = -\frac{1}{(a, a)} \left[ba + \frac{\gamma (b, g)}{-1 + \gamma (g, g)} ga \right].$$

Since (x, y) permits composition on A, it follows from (16) that (x, y) permits composition on $A(\gamma, g)$ if and only if we have:

$$\gamma (g, yx)^2 [\gamma (g, g) - 2] = 0. \tag{20}$$

Since (x, y) is nondegenerate if $g \neq 0$, we get:

$$\gamma = 0 \text{ or } \gamma = \frac{2}{(g, g)} \square \tag{21}$$

Proposition 5. *Let $(A, \cdot)$ be a unitary finite-dimensional algebra over the field K, with char$K \neq 2$, which satisfies the conditions in Proposition 4 and $A(\gamma, g)$ be the algebra defined by (16).*

a) In algebra $A(\gamma, e)$, the map $f(x) = \bar{x}$ is an involution.

b) If A' is an unitary finite-dimensional algebra which satisfies the conditions in Proposition 4, $f : A \to A'$ is an algebra isomorphism and $\gamma = \gamma'$, $f(g) = g'$, $(x, y) = (f(x), f(y))$, then $(A(\gamma, g)) \simeq (A'(\gamma, g'))$.

Proof. a) $\bar{x} * \bar{y} = -\bar{y}\bar{x} + \gamma (e, \bar{y}\bar{x}) e$ and $\overline{y * x} = -\bar{y}\bar{x} + \gamma (e, xy) e$. Since $(x, y) = (\bar{x}, \bar{y})$, and the bilinear form $(\cdot, \cdot)$ is associative, we get that f is an involution.

b) By calculation, we obtain $f (x * y) = -f (yx) + \gamma (g, yx) f (g)$, and $f (x) * f (y) = -f (y) f (x) + \gamma (g', f (y) f (x)) g' = -f (yx) + \gamma (f (g), f (yx)) f (g)$. By hypothesis, we get that $f (x * y) = f (x) * f (y)$ so that $(A (\gamma, g)) \simeq (A' (\gamma, g'))$, because f is a bijective map.□

The algebra $A (\gamma, g)$ is not in general flexible and associative. The associativity law $(x * y) * x = x * (y * x)$ is equivalent with

$$\gamma (g, xy) gx + \gamma (g, (x * y) x) g = \gamma (g, yx) xg + \gamma (g, x (y * x)) g.$$

Let O be a real division octonion algebra with the unit e. The associated para-octonion algebra O_e, is a division algebra with the para-unit e and satisfies the conditions on the Proposition 4. Then the multiplication $x * y$ in $O_e (\gamma, e)$ is

$$x * y = -y \circ x + \gamma (e, y \circ x) e = xy - (2 - \gamma) (e, xy) e, \tag{22}$$

with $(e, e) = 1$.

Using (22) we get then:

$$(x * y) * x - x * (y * x) = 0$$

since O is flexible and $(e, xy) = (e, yx)$. It results that $O_e (\gamma, e)$ is flexibile, but it doesn't have the identity element only if $\gamma = 2$. Indeed, we suppose that f is a unit element for $O_e (\gamma, e)$. Then, by (22), $f = \alpha e$ with $\alpha = 1 + (2 - \gamma) (e, f)$. Since $(e, e) = 1$ it results $\alpha = 1 + (2 - \gamma) \alpha$. Hence we get that f is not a unit element in $O_e (\gamma, e)$ only if $\gamma = 2$.

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