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A NONLINEAR MODEL OF A REGENERATIVE VIBRATING MACHINE TOOL

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Abstract

We study the existence of Hopf bifurcations in a model of a regenerative vibrating machine tool, by using the same methods as in the Bifurcation Theorem and in the theory of central varieties.

In a paper published in 1984, H. M. Shi and S. A. Tobias [6] have studied the theory of finite amplitude machine tool instability. They considered there some experimental results and have shown that there exist unstable finite amplitudes that are periodically unstable moments of the machine tool in the neighborhood of its asymptotically stable states.

Following [2], we apply the subcritical Hopf bifurcation in the delay equation model for machine tool vibrations.

We have the mechanical model of a machine tool with regenerative vibrations in Figure 1, which presents the orthogonal cuts of such a model taken from [2]. Here f is the thickness of the splinter.
Consider the splinter given by:

$$\Delta l = l - l_0 + x(t), \tag{1}$$

where l is the length of the initial cut, l_0 is the length of the cut made in equilibrium state, $x(t)$ characterizes the position of the knife and is depending on the component F_x of the cutting force.

Key Words: nonlinear model; regenerative vibrating machine tool.

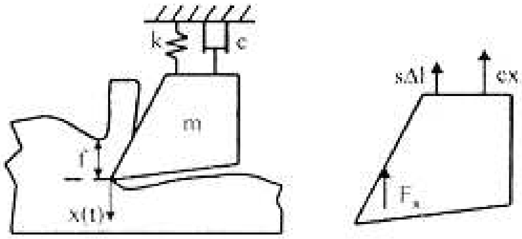

Figure 1:

The equation describing the machine's movement is:

$$m\ddot{x}(t) = -s\Delta l - F_x - C\ddot{x}(t). \tag{2}$$

If we take:

$$F_x = F_x(f_0) + \Delta F_x, \tag{3}$$

the Eq. (2) can be written as

$$\ddot{x}(t) + 2\varsigma\omega_v\dot{x}(t) + \omega_n^2 x(t) = -\frac{1}{m}\Delta F_x, \tag{4}$$

where $\omega_n = \sqrt{\dfrac{s}{m}}$ is the natural frequency of the oscillating system without amortization and $\varsigma = \dfrac{c}{2m\omega_n}$ is the relative term of the amortization. The Equation (4) is exactly Equation (3.2) in [1], with $\alpha = \omega_n$. The cutting force was obtained empirically, by using cutting test.

In [6], the cutting function is a polynomial function of third degree. We know that the cutting force depends upon the thickness of the splinter and it is given by

$$F_x(f) = KWf^{3/4}, \tag{5}$$

while its graph is given in Figure 2.

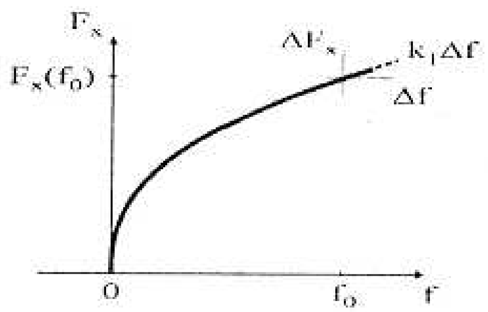

Figure 2:

In (5), K is a parameter depending on the technological parameters and it is considered constant. Developing f in Taylor series and keeping only terms until the third degree, it results:

$$F_x(f) \cong KW(f_0^{3/4}+\frac{3}{4}\,(f\text{-}f_0)\,f_0^{-1/4}\text{-}\frac{3}{32}\,(f\text{-}f_0)^2\,f_0^{-5/4}+\frac{5}{128}\,(f\text{-}f_0)^3\,f_0^{-9/4}). \tag{6}$$

Denoting $\Delta F_x = F_x(f) - F_x(f_0)$ and $\Delta f = f - f_0$, we get

$$\Delta F_x(\Delta f) \cong kW\left(\frac{3}{4}f_0^{-1/4}\Delta f - \frac{3}{32}f_0^{-5/4}\Delta f^2 + \frac{5}{128}f_0^{-9/4}\Delta f^3\right). \tag{7}$$

The coefficient of Δf in (7) is called the coefficient of the cutting force and it is

$$k_1 = \frac{3}{4}kWf_0^{-1/4}. \tag{8}$$

There the relation (7) becomes

$$\Delta F_x(\Delta f) \cong k_1\Delta f = \frac{1}{8}\frac{k_1}{f_0}\Delta f^2 + \frac{5}{96}\frac{k_1}{f_0^2}\Delta f^3. \tag{9}$$

The variation Δf of the thickness of the splinter can be expressed as the difference between the position of the maximum knife $x(t)$ and its deviation $x(t-\tau)$ under the form

$$\Delta f = x(t) - x(t-\tau), \tag{10}$$

where the delay $\tau = \dfrac{2\pi}{\Omega}$ is the time period of a rotation and Ω is the angular speed of the rotation, moment of the piece.

The equation (4) is written as

$$\ddot{x}(t) + 2\varsigma\omega\dot{x}(t) + \left(\omega_n^2 + \frac{k_1}{m}\right)x(t) - \frac{k_1}{m}x(t-\tau) =$$

$$= \frac{k_1}{8f_0 m}\left((x(t) - x(t-\tau))^2 - \frac{5}{12f_0 m}(x(t) - x(t-\tau))^3\right). \tag{11}$$

We introduce the notations $\tilde{t}, \tilde{x}, \tilde{\tau}$ and p for the following expressions:

$$\tilde{t} = \omega_n t,\ \tilde{x} = \frac{5}{12f_0}x,\ \tilde{\tau} = \omega_n\tau,\ P = \frac{k_1}{m\omega_n^2}. \tag{12}$$

Replacing them in (11) and using t, x, τ instead of $\tilde{t}, \tilde{x}, \tilde{\tau}$ we obtain the equation

$$\ddot{x}(t) + 2\varsigma\dot{x} + (1+p)x(t) - px(t-\tau) = \frac{3p}{10}((x(t) - x(t-\tau))^2 - (x(t) - x(t-\tau)^3)). \tag{13}$$

The equation (13) is changed into a system of differential equations, by putting $x_1(t) = x(t),\ x_2(t) = \dot{x}(t)$ and $X(t) = (x_1(t),\ x_2(t))^T$. We get the PDE system:

$$\dot{X}(t) = L(p)X(t) + R(p)X(t-\tau) + F(X(t),\ X(t-\tau), p), \tag{14}$$

where

$$L(p) = \begin{pmatrix} 0 & 1 \\ -(1+p) & -2\varsigma \end{pmatrix},\ R(p) = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix},$$

$$F(X(t),\ X(t-\tau), p) = \frac{3p}{10}\begin{pmatrix} 0 \\ (x_1(t) - x_1(t-\tau))^2 - (x_1(t) - x_1(t-\tau))^3 \end{pmatrix}. \tag{15}$$

We are doing the investigation of the system (14), by considering p as a bifurcation parameter and by applying the method used in [3].

More precisely, we use some steps (stages), which will be described bellow together with the Maple program for getting the orbits of the system.

Step 1: Analyze of the linear part of the system (14).

We make the linearization of the system (14) in the equilibrium point $0 : (0,0)^T$:

$$\dot{X}(t) = L(p)X(t) + R(p)X(t-\tau), \tag{16}$$

where $L(p)$ and $R(p)$ were given in (15). The characteristic equation of (16) is obtained by considering a solution of the form $X(t) = e^{\lambda t}C$, where $C = (C_1, C_2)^T \in R^2$, $c \neq 0$.

The characteristic function

$$D(\lambda, p) = \det\left(\lambda I - L(p) - e^{-\lambda\tau} R(p)\right) = \lambda^2 + 2\varsigma\tau + (1+p) - pe^{-\lambda r},$$

with the condition $c \neq 0$, leads us to the characteristic equation:

$$D(\lambda, p) = 0 \tag{17}$$

For finding the boundary of the stability set of the solution $0 = (0,0)^T$, we study the roots of the characteristic equation. For getting the D-curve, we replace $\lambda = i\omega$, with $\omega > 0$, in the characteristic equation (17):

$$1 + p - \omega^2 - p\cos\omega\tau + i\,(2\zeta\omega + p\sin\omega\tau) = 0.$$

This equation is equivalent to the following system of equations:

$$1 - \omega^2 + p\,(1 - \cos\omega\tau) = 0,\ 2\zeta\omega + p\sin\omega\tau = 0. \tag{18}$$

By eliminating the trigonometric functions in (18), we obtain:

$$p = \frac{\left(1-\omega^2\right)^2 + 4\zeta^2\omega^2}{2\left(\omega^2 - 1\right)}. \tag{19}$$

Since $p > 0$, it results $\omega > 1$. From (18), we get also:

$$\tau = \frac{2}{\omega}\left(j\pi - \arctan\frac{\omega^2 - 1}{2\zeta\omega}\right),\ j = 1, 2, \ldots\ . \tag{20}$$

Using $\Omega = \dfrac{\pi}{\tau}$ in (20), we have:

$$\Omega = \frac{\omega\pi}{j\pi - \arctan\dfrac{\omega^2 - 1}{2\zeta\omega}},\ j = 1, 2, \ldots\ . \tag{21}$$

The D-curve that characterizes the boundary separating the stability region from the instability one is given by:

$$p(\omega) = \frac{\left(1-\omega^2\right)^2 + 4\zeta^2\omega^2}{2\left(\omega^2 - 1\right)},\ \ \Omega(\omega) = \frac{\omega\pi}{j\pi - \arctan\frac{\omega^2-1}{2\zeta\omega}},\ j = 1, 2, \ldots, \tag{22}$$

with $\omega > 1$. Taking $j = 1$, the curve (22) has the graph as in Figure 3.

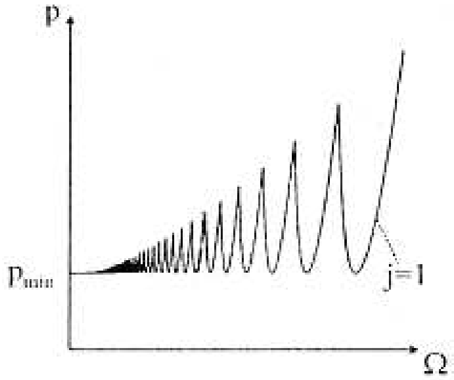

Figure 3:

Between the initial thickness f_0 of the splinter and the coefficient p there is the relation: $p = \dfrac{3}{4}\dfrac{kW}{4m\omega_n^2} f_0^{-1/4}$. From (22), it results that $f_0(\omega) = \left(\dfrac{3kW}{4m\omega_n^2}\right)^4 \dfrac{1}{p^4(\omega)}$. The value of ω for which $f_0(\omega)$ has a minimal value is the solution of the equation $\dfrac{dp(\omega)}{d\omega} = 0$. By using (22), we have

$$\omega_0 = \sqrt{1+2\zeta} \text{ and } p_0 = p(\omega_0) = 2\zeta\,(\zeta+1)\,.$$

For $\omega = \omega_0$, from (20) and (21), we obtain

$$\tau_0 = \frac{2\left(j\pi - \arctan\dfrac{1}{\sqrt{1+2\zeta}}\right)}{\sqrt{1+2\zeta}},\ \Omega_0 = \frac{\sqrt{1+2\zeta\pi}}{j\pi - \arctan\dfrac{1}{\sqrt{1+2\zeta}}},\ j = 1,2,\ldots\,. \tag{23}$$

All these results show that there is a value ω_0 such that $(p_0 = p(\omega_0),\ \Omega_0 = \Omega(\omega_0))$ lies on the D-curve separating the stable region by the unstable region and for which the initial thickness of the splinter has a maximal value given by:

$$f_0 = f(\omega_0) = \left(\frac{4kW}{4m\omega_n^2}\right)^4 \frac{1}{16\zeta^4\,(\zeta+1)^4}. \tag{24}$$

Step 2. Analyze of the characteristic equation $D(\lambda, p) = 0$ and the Hopf bifurcation.

The roots of the characteristic equation (17) are functions of the parameter p. Let $\lambda = \lambda(p)$ be such a root. We call the Hopf bifurcation point the value $p = p_{cr}$ for which $Re\lambda(p_{cr}) = 0$ and $Re\dfrac{d\lambda(p_{cr})}{dp} \neq 0$. From **Step 1**, for $p_{cr} = p_0$, we find $Re\lambda(p_0) = 0$. By derivation in $D(\lambda(p), p) = 0$ with respect to p, it results

$$\frac{d\lambda(p)}{dp} = \frac{e^{-\lambda(p)\tau} - 1}{2\lambda(p) + 2\zeta + pe^{-\lambda(p)\tau}}.$$

For $p = p_0$ and $\lambda = \pm i\omega_0$, we obtain:

$$M = \frac{d\lambda(p_0)}{dp} = \frac{e^{-\omega_0\tau_0} - 1}{2i\omega_0 + 2\zeta + p_0 e^{-i\omega_0\tau_0}}. \tag{25}$$

From (25) and (23), it results

$$\gamma = Re\frac{d\lambda(p_0)}{dp} > 0. \tag{26}$$

The relation (26) shows us that p_0 is a Hopf bifurcation point. For $p \in (0, p_0)$, the roots of the characteristic equation have their real parts negative. By the stability theorem of PDE systems. It results that the equilibrium point $0 = (0,0)^T$ of the system (14) hence of the equation (13) is asymptotically stable.

We are going on to analyze the solution of the system (14), in the neighborhood of the point $0 = (0,0)^T$, for $p = p_0 = 2\zeta(\zeta + 1)$.

$$\tau = \tau_0 = \frac{2\left(\pi j - a\arctan\dfrac{1}{\sqrt{1+2\zeta}}\right)}{\sqrt{1+2\zeta}}, \ \omega = \omega_0 = \sqrt{1+2\zeta},$$

with $j \in \{1, 2, ...\}$ fixed.

Step 3. The generalized eingensubspaces associated to the system (14) in the Hopf bifurcation point $p = p_0$.

The PDE system (14) with $p = p_0$ becomes:

$$X(t) = L(p_0)X(t) + R(p_0)X(t - \tau_0), \ t \geq 0. \tag{27}$$

The linear operator A and its adjoint A^* associated to (27) are

$$A\Phi(\theta) = \begin{cases} \dfrac{d\Phi(\theta)}{d\theta}, \ \theta \in [-\tau_0, 0) \\ \\ L(p_0)\Phi(0) + R(p_0)\Phi(-\tau_0), \ \theta = 0 \end{cases} \tag{28}$$

$$A^*\Phi^*(\theta) = \begin{cases} \dfrac{-d\Phi^*(s)}{ds}, \ s \in [0, \tau_0) \\ \\ L(p_0)^T\Phi^*(0) + R(p_0)^T\Phi(\tau_0), \ s = 0 \end{cases} \tag{29}$$

where $\Phi : [-\tau_0, 0] \to C^2, \ \Phi^* : [0, \tau_0] \to R^2$.

The generalized eigensubspaces of (28) and (29) are given by

$$A\Phi(\theta) = \pm i\omega_0\Phi(\theta), \ \theta \in [-\tau_0, 0], \tag{30}$$

$$A^*\Phi^*(s) = \pm i\omega_0\Phi^*(s), \ s \in [0, \tau_0]. \tag{31}$$

From (28), (29) and (30), (31), it results that $\Phi(\theta)$ and $\Phi^*(s)$ are given by $\Phi(\theta) = e^{i\omega_0\theta}C, \ \Phi^*(s) = e^{i\omega_0\theta}C^*$, where $C, C \in C^2$ are the solutions of the linear equation systems:

$$A_1C = 0, \ A_2C^* = 0, \tag{32}$$

where

$$A_1 = \begin{pmatrix} i\omega_0 & -1 \\ 1 + p_0 - p_0e^{-i\omega_0\tau_0} & i\omega_0 + 2\zeta \end{pmatrix}, \ A_2 = \begin{pmatrix} i\omega_0 & 1 + p_0 - p_0e^{-i\omega_0\tau_0} \\ -1 & i\omega_0 + 2\zeta \end{pmatrix} \tag{33}$$

From (32), (33), it results that the generalized eigensubspaces of A and A^* corresponding to the eigenvalues $\pm i\omega_0$ are:

$$\Phi(\theta) = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} e^{i\omega_0\theta}, \ \Phi(\theta) = \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix} e^{i\omega_0\theta}, \ \theta \in [-\tau_0, 0], \tag{34}$$

$$\Phi(s) = \begin{pmatrix} i\omega_0 + 2\zeta \\ 1 \end{pmatrix} e^{i\omega_0 s}, \ \Phi^*(s) = \begin{pmatrix} -i\omega_0 + 2\zeta \\ 1 \end{pmatrix} e^{-i\omega_0 s}, \ s \in [0, \tau_0]. \tag{35}$$

Let $\mathcal{H} = \{\varphi : [-\tau, 0] \to \mathbf{C}^2\}$ and $\mathcal{H}^* = \{\varphi : [0, \tau_0] \to C^2\}$ be the spaces of the differentiable functions on $[-\tau_0, 0]$ and $[0, \tau_0]$ with values in C^2, endowed with the uniform convergence topology.

We define the bilinear form $(\cdot, \cdot) : \mathcal{H}^* \times \mathcal{H} \to C$,

$$(\varphi^*(s), \varphi(\theta)) := \bar{\varphi}^{*T}(0)\varphi(0) - \int_{-\tau_0}^{0} \left(\int_0^\theta \bar{\varphi}^* (\xi - \theta) R(p_0)\varphi(\xi)d\xi \right) d\theta, \tag{36}$$

$$s \in [0, \tau_0],\ \theta \in [-\tau_0, 0].$$

Using this bilinear form for $(\Phi(\theta), \Phi(\theta))$ and $(\Phi^*(s), \Phi^*(s))$given by (34) and (35), we get:

$$(\Phi^*(s), \Phi(\theta)) = (2\zeta - i\omega_0, 1)\begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} -$$

$$-\int_{-\tau_0}^{0}\left(\int_0^{\theta} e^{-i\omega_0(\xi-\theta)}(2\zeta - i\omega_0, 1)\begin{pmatrix} 0 & 0 \\ p_0 & 0 \end{pmatrix} e^{i\omega_0\xi} d\xi\right) d\theta =$$

$$= 2\zeta + \tau_0 e^{-i\omega_0\tau_0} p_0 = e_{11}.$$

$$\left(\overline{\Phi}^*(s), \Phi(\theta)\right) = (2\zeta - i\omega_0, 1)\begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} -$$

$$-\int_{-\tau_0}^{0}\left(\int_0^{\theta} e^{i\omega_0(\xi-\theta)}(2\zeta - i\omega_0, 1)\begin{pmatrix} 0 & 0 \\ p_0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} e^{i\omega_0\xi} d\xi\right) d\theta =$$

$$= 2\zeta + 2i\omega_0 - \frac{p_0}{2i\omega_0}\left(e^{-i\omega_0\tau_0} - e^{i\omega_0\tau_0}\right) = e_{12}. \tag{37}$$

$$\left(\Phi^*(s), \overline{\Phi}(\theta)\right) = \overline{e}_{12},\ \ \left(\overline{\Phi}^*(s), \overline{\Phi}(\theta)\right) = \overline{e}_{11}.$$

Let $E = (e_{ij}),\ i, j = 1, 2$, be the matrix formed with the coefficients in (37) and $F = (f_{ij})$, the inverse matrix of E. The vectors

$$\Psi^*(s) = f_{11}\Phi^*(s) + f_{12}\overline{\Psi}^*(s), \overline{\Psi}^*(s) = \overline{f}_{12}\Phi^*(s) + \overline{f}_{11}\overline{\Psi}^*(s),\ s \in [0, \tau_0] \tag{38}$$

are eigenvectors of the operators A and and A^*, by introducing them into the bilinear form (36), we obtain:

$$(\Psi^*(s), \Phi(\theta)) = 1,\ \left(\overline{\Psi}^*(s), \Phi(\theta)\right) = 0, \left(\Psi^*(s), \overline{\Phi}(\theta)\right) =$$

$$\left(\overline{\Psi}^*(s), \overline{\Phi}(\theta)\right) = 1,\ s \in [0, \tau_0],\ \theta \in [-\tau_0, 0], \tag{39}$$

hence the relations are satisfied.

Step 4. The central variety in bifurcation point and the limit cycle associated to it.

The orbit of the PDE system (14) can be described by using the usual Poincare form associated to the system (14) on the central variety in the point $p = p_0$. The central variety in $p = p_0$ is a subvariety in the space $\mathcal{H} = \{u : [-\tau_0, 0] \to C^2\}$, which is tangent to the eigensubspace generated by $\{\Phi(\theta), \overline{\Phi}(\theta)\}$ in (34). The central variety has the form

$$W^C(p_0) = \left\{z\Phi(\theta) + \overline{z}\overline{\Phi}(\theta) + W(\theta, z, \overline{z}), z = x + iy,\ (x, y)V(0),\ \theta \in [-\tau_0, 0]\right\}, \tag{40}$$

where $V(0) \subset R^2$ is a neighborhood of the origin in R^2 and

$$w(\theta, z, \overline{z}) = \frac{1}{2}w_{20}(\theta)z^2 + w_{11}(\theta)z\overline{z} + \frac{1}{2}w_{02}(\theta)\overline{z}^2,$$

with

$$w_{20}, w_{11} : [-\tau_0, 0] \to C^2,\ w_{02}(\theta) = \overline{w}_{20}(\theta),\ w_{11}(\theta) \in R,\ \theta \in [-\tau_0, 0].$$

The orbit of the PDE system (14) is given by the mapping $X_t : [-\tau_0, 0] \to R$, where $X_t(\theta) = X(t + \theta), \theta \in [-\tau_0, 0]$, and defines a pseudogroup of operators $T(t)$, namely $T(t) : \mathcal{H} \to \mathcal{H}$, given by $X_t(\theta) = T(t)X(\theta)$. The central variety $W^C(p_0)$ associated to the differential system with (14) has the property of remaining unchanged with respect to the pseudogroup of operators $T(t)$. From the invariance, it results that $w_{20}(\theta), w_{11}(\theta), w_{02}(\theta)$ satisfy a system of differential equations. The complete description of the central variety $W^C(p_0)$ utilizes the method from [3]. In order to obtain the formula which are used to describing the orbit of the differential system, we use the nonlinear part $\mathcal{F}(X(t), X(t-\tau), p_0)$ given by (15) of the system (14), where we replace

$X_1(t)$ with $z(t)+\overline{z}(t)+\frac{1}{2}w^1_{20}(0)z^2(t)+w^1_{11}(0)z\overline{z}+\frac{1}{2}w^1_{02}(0)\overline{z}^2$ and $x_1(t-\tau)$ with

$$z(t)e^{-\omega_0\tau_0}+\overline{z}(t)e^{i\omega_0\tau_0}+\frac{1}{2}w^1_{20}(-\tau_0)z^2(t)+w^1_{11}(-\tau_0)z(t)\overline{z}(t)+\frac{1}{2}w^1_{02}(-\tau_0)\overline{z}^2(t).$$

We obtain

$$\mathcal{F}(z(t), \overline{z}(t), p_0)=\frac{1}{2}F_{20}z(t)^2+F_{11}z(t)\overline{z}(t)+\frac{1}{2}F_{02}z(t)^2+\frac{1}{2}F_{21}z(t)^2\overline{z}(t)+O(|z(t)|^3), \tag{41}$$

where

$$F_{20} = (0, F^2_{20})^T, F_{11} = (0, F^2_{11})^T, F_{02} = (0, F^2_{02})^T, F_{21} = (0, F^2_{21})^T, \tag{42}$$

$$F^2_{20} = \frac{3p_0}{5}(1-e^{-\omega_0\tau_0})^2,\ F^2_{11} = \frac{3p_0}{5}(1-e^{-i\omega_0\tau_0})(1-e^{i\omega_0\tau_0}), F^2_{02} = \frac{3p_0}{5}(1-e^{i\omega_0\tau_0})^2$$

and

$$F_{21}^2 = \frac{3p_0}{5}[(1-e^{i\omega_0\tau_0})(w_{20}^1(o)-w_{20}^1(-\tau_0)+2(1-e^{-i\omega_0\tau_0})(w_{11}^1(o)-$$

$$-w_{11}^1(-\tau_0)-3(1-e^{-i\omega_0\tau_0})^{2(1-e^i\omega_0\tau_0)})], \tag{43}$$

$$w_{20}^1(\theta) = -\frac{g_{20}}{i\omega_0}e^{i\omega_0\theta} - \frac{\overline{g}_{02}}{3i\omega_0}e^{-i\omega_0\theta} + E_1 e^{zi\omega_0\theta}, \quad \theta \in [-\tau_0, 0] \tag{44}$$

$$w_{11}^1(\theta) = \frac{g_{11}}{i\omega_0}e^{i\omega_0\theta} - \frac{\overline{g}_{11}}{i\omega_0}e^{-i\omega_0\theta} + E_2,$$

$$E_1 = -\left(L(p_0) + e^{-i\omega_0\tau_0}R(p_0) - 2i\omega_0 I\right)^{-1} F_{20},$$

$$E_2 = -(L(p_0) + R(p_0)^{-1}F_{11}, \tag{45}$$

and

$$g_{20} = \overline{\Psi}(0)F_{20}, g_{11} = \overline{\Psi}(0)F_{11}, g_{02} = \overline{\Psi}(0)F_{02}, g_{21} = \overline{\Psi}(0)F_{21}. \tag{46}$$

The limit cycle is the orbit on the central variety $W^C(p_0)$ of the solution of the differential equation:

$$z(t) = i\omega_0 z(t) + \frac{1}{2}g_{20}z(t)^2 + g_{11}z(t)\overline{z}(t) + \frac{1}{2}g_{02}\overline{z}(t)^2 + \frac{1}{2}g_{21}z(t)^2\overline{z}(t),$$

with the initial condition:

$$z(0) = (\Psi^*(s), \varphi(\theta)), \ s \in [0, \tau_0], \ \theta \in [-\tau_0, 0],$$

where $\Psi^*(s)$ is given by (38) and $\varphi(\theta)$ is the initial condition for the system (14).

Step 5. The orbit of the system (14).

The orbit $(t, \tilde{x}(t))$ of the equation (13) is given by

$$\tilde{x}(t) = 2x_1(t) + r_{111}(x_1^2(t) + y_1(t)^2) + r_{120}(x_1(t)^2 - y_1(t)^2) - 2i_{120}x_1(t)y_1(t) \tag{47}$$

where $r_{120} = Re(w_{20}^1)$, $r_{111} = Re(w_{11}^1)$, $i_{120} = Im(w_{20}^1)$ and $(x_1(t), y_1(t))$ is the solution of the system of differential equations:

$$x_1(t) = -\omega_0 y_1(t) + \frac{1}{2}\left(R_{20} + 2R_{11} + R_{02}\right)x_1(t)^2 - \frac{1}{2}\left(R_{20} - 2R_{11} + R_{02}\right)y_1(t)^2 +$$

$$+\left(I_{02}-I_{20}\right)x_1(t)y_1(t)+\frac{1}{2}R_{21}x_1(t)(x_1(t)^2+y_1(t)^2)-\frac{1}{2}I_{21}y_1(t)(x_1(t)^2+y_1(t)^2), \quad (48)$$

$$y_1(t)=-\omega_0 x_1(t)+\frac{1}{2}\left(I_{20}+2I_{11}+I_{02}\right)y_1(t)^2-\frac{1}{2}\left(I_{20}-2I_{11}+I_{02}\right)x_1(t)^2+$$

$$+\left(R_{02}-R_{20}\right)x_1(t)y_1(t)+\frac{1}{2}R_{21}y_1(t)(x_1(t)^2+y_1(t)^2)-\frac{1}{2}I_{21}x_1(t)(x_1(t)^2+y_1(t)^2),$$

with the initial condition

$$x_1(0)=Re(\Psi,\varphi),\ y_1(0)=I_m(\Psi,\varphi) \quad (49)$$

and

$$R_{20}=Re(g_{20}),\ R_{11}=Re(g_{11}),\ R_{02}=Re(g_{02}),\ R_{21}=Re(g_{21})$$

$$I_{20}=Im(g_{20}),\ I_{11}=Im(g_{11}),\ I_{02}=Im(g_{02}),\ I_{21}=Im(g_{21}).$$

$\overline{\varphi}:[-\tau_0,0]\to\mathbf{R}$ is the initial condition of the system (13) given by $\overline{\varphi}(\theta)=\dfrac{5}{12f_0}\varphi(\theta)$, $\varphi(\theta)$ is the initial condition of the system (3.7) from [1] with $f_0=\left(\dfrac{3kW}{4m\omega_n^2}\right)^4\dfrac{1}{p_0^4}$, $p_0=2\varsigma(\varsigma+1)$.

The orbit $(t,x(t))$ of the equation (11) is given by

$$x(t)=\frac{12}{5}\left(\frac{4m\omega_n^2}{3Kw}\right)^4 p_0^4\widetilde{x}(t). \quad (50)$$

The invariants of the limit cycle given by (48) are:

$$\mu_2=-\frac{ReC_1}{ReM},\ T_2=-\frac{ImC_1+\mu_2 ImM}{\omega_0},\ \beta_2=2ReC_1,$$

where M is given by (25) and

$$C_1=\frac{i}{\omega_0}\left(g_{20}g_{11}-2\left|g_{11}\right|^2-\frac{1}{3}\left|g_{02}\right|^2\right)+\frac{g_{21}}{2}.$$

We reproduce here the program realized by using Maple 9 soft, and then we give the graphs that we obtained this way.

```
#Bala2 cazul neliniar masini unelte regenerative;Stepan G;
>
m:=1.;s:=2.;c:=0.02;a:=sqrt(s/m);zeta:=c/(2*m*a);j:=5.;omeg
a:=sqrt(1+2*zeta);p:=2*zeta*(1+zeta);tau:=2*(j*evalf(Pi)-
arctan(1/omega))/omega;M:=(exp(-I*omega*tau)-
1)/(2*omega*I+2*zeta+p*exp(-
omega*tau*I));A:=Re(M);Omega:=omega*evalf(Pi)/(j*evalf(Pi)-
arctan(1/omega));K:=0.2;W:=0.5;f0:=(4*m*a^2/(3*K*W))^4*p^4;
>                                       e11:=2*zeta+tau*exp(-
omega*tau*I)*p:e12:=2*zeta+2*omega*I-p*(exp(-omega*tau*I)-
exp(omega*tau*I))/(2*omega*I):e21:=conjugate(e12):e22:=conj
ugate(e11):
> with(LinearAlgebra):
E:=
<<e11,e21>|<e12,e22>>:F:=MatrixInverse(E):d11:=<1|0>.F.<1,0
>:d12:=<1|0>.F.<0,1>:
>                                d:=<omega*I*(conjugate(d12)-
conjugate(d11))+2*zeta*(conjugate(d12)+conjugate(d11))|conj
ugate(d12)+conjugate(d11)>:F20:=<0,3*p*(1-exp(-
omega*tau*I))^2/5>:F11:=<0,3*p*(1-exp(-I*omega*tau))*(1-
exp(I*omega*tau))/5>:F02:=<0,3*p*(1-
exp(omega*tau*I))^2/5>:g20:=d.F20:g11:=d.F11:g02:=d.F02:
>                                          L:=<<0,-1-p>|<1,-
2*zeta>>:R:=<<0,p>|<0,0>>:G:=<<1,0>|<0,1>>:E1:=MatrixInvers
e(-(L+exp(-I*omega*tau)*R-
2*omega*I*G)).F20:E2:=MatrixInverse(-
(L+R)).F11:E11:=<1|0>.E1:E12:=<1|0>.E2:
>                    w120:=-g20*exp(-I*omega*tau)/(I*omega)-
conjugate(g02)*exp(I*omega*tau)/(3*I*omega)+E11*exp(-
2*omega*tau*I):w111:=g11*exp(-I*omega*tau)/(I*omega)-
conjugate(g11)*exp(I*omega*tau)/(I*omega)+E12:v120:=-
g20/(I*omega)-
conjugate(g02)/(3*I*omega)+E11:v111:=g11/(I*omega)-
conjugate(g11)/(I*omega)+E12:F21:=<0,3*p*((1-
exp(omega*tau*I))*(v120-w120)+2*(1-exp(-
omega*tau*I))*(v111-w111)-3*(1-exp(-omega*tau*I))^2*(1-
exp(omega*tau*I)))/5>:g21:=d.F21:
>                                  C1:=(g20*g11-2*abs(g11)^2-
abs(g02)^2/3)*I/(2*omega)+g21/2;mu2:=-Re(C1)/Re(M);T2:=-
(Im(C1)+mu2*Im(M))/omega;beta2:=2*Re(C1);
>
>
r20:=Re(w120):i20:=Im(w120):r11:=Re(w111):i11:=Im(w111):r22
0:=Re(v120):i220:=Im(v120):r211:=Re(v111):i211:=Im(v111):
>
R20:=Re(g20):R11:=Re(g11):R02:=Re(g02):I20:=Im(g20):I11:=Im
(g11):I02:=Im(g02):R21:=Re(g21):I21:=Im(g21):
```

```
>  F1(x(t),y(t)):=-omega*y(t)+(R20/2+R11+R02/2)*x(t)^2-(I20-
```

```
>  F1(x(t),y(t)):=-omega*y(t)+(R20/2+R11+R02/2)*x(t)^2-(I20-
```


```
I02)*x(t)*y(t)-(R20/2-
R11+R02/2)*y(t)^2+R21*x(t)*(x(t)^2+y(t)^2)/2-
I21*y(t)*(x(t)^2+y(t)^2)/2:
F2(x(t),y(t)):=omega*x(t)+(I20/2+I11+I02/2)*x(t)^2-(I20/2-
I11+I02/2)*y(t)^2+(R20-
R02)*x(t)*y(t)+R21*y(t)*(x(t)^2+y(t)^2)/2+I21*x(t)*(x(t)^2+
y(t)^2)/2:        F3(x(t),y(t)):=2*x(t)+r220*(x(t)^2-y(t)^2)-
2*i220*x(t)*y(t)+r211*(x(t)^2+y(t)^2)+0.1:
F4(x(t),y(t)):=2*x(t)*cos(omega*tau)+2*y(t)*sin(omega*tau)+
r20*(x(t)^2-y(t)^2)-
2*i20*x(t)*y(t)+r11*(x(t)^2+y(t)^2):F5(x(t),y(t)):=
12*(4*m*a/(3*K*W))^4*(p^4/5)*F3(x(t),y(t))+0.1:
with(DEtools):
>                          dsys                           :=
{diff(x(t),t)=F1(x(t),y(t)),diff(y(t),t)=F2(x(t),y(t)),x(0)
=-0.001,y(0)=0.002}:
> dsol :=  dsolve(dsys,numeric):
> plots[odeplot](dsol,[
[t,F3(x(t),y(t))    ,color=red]],0..5*tau,title="Fig.3.Orbita
(t,x(t))  ");
plots[odeplot](dsol,[
[F3(x(t),y(t)),F4(x(t),y(t))
,color=red]],0..tau,title="Fig.3.Orbita      (x(t),x(t-tau))
");
plots[odeplot](dsol,[
[t,F5(x(t),y(t))    ,color=red]],0..5*tau,title="Fig.3.Orbita
(t,x(t)) a sist init  ");


> MUGqns := [
diff(x(t),t)      =-omega*y(t)+(R20/2+R11+R02/2)*x(t)^2-(I20-
I02)*x(t)*y(t)-(R20/2-
R11+R02/2)*y(t)^2+R21*x(t)*(x(t)^2+y(t)^2)/2-
I21*y(t)*(x(t)^2+y(t)^2)/2,          diff(y(t),t)          =
omega*x(t)+(I20/2+I11+I02/2)*x(t)^2-(I20/2-
I11+I02/2)*y(t)^2+(R20-
R02)*x(t)*y(t)+R21*y(t)*(x(t)^2+y(t)^2)/2+I21*x(t)*(x(t)^2+
y(t)^2)/2
 ]:
> DEplot3d(MUGqns, {x(t),y(t)}, t=0..5*tau,
          [[ x(0)=-0.05, y(0)=0.05]],
           x=-0.1..0.1,y=-0.1..0.1,scene=[x(t),y(t),t],
          stepsize=0.05, linecolour=t,
          title="Figura3.dinamica Sistemului  pe Wc(0)XR"):
>
DEs:=({diff(x(t),t)=F1(x(t),y(t)),diff(y(t),t)=F2(x(t),y(t)
```

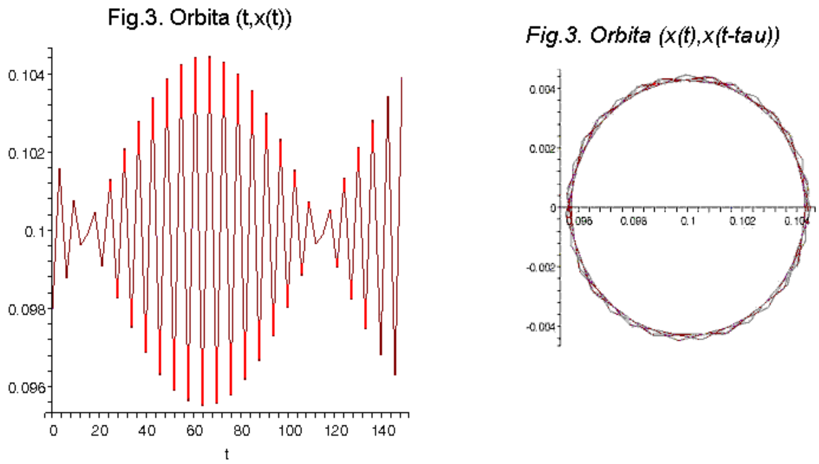

Figure 4:

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