An. Şt. Univ. Ovidius Constanţa Vol. 12(1), 2004, 73–80

ON SOME DIOPHANTINE EQUATIONS (III)

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Abstract

In this paper we study the Diophantine equations

$$c_k(f^4+42f^2g^2+49g^4)+28d_k(f^3g+7fg^3)=m^2,$$

where (c_k, d_k) are solutions of the Pell equation $c^2-7d^2=1$.

1. Preliminaries.

We recall a classical result in [1], page 150 and our previous results in [7].

1.1. *For the quadratic field $\mathbb{Q}(\sqrt{7})$, the ring of integers is Euclidian with respect to the norm.*

1.2. *The equation $m^4-n^4=7y^2$ has an infinity of integer solutions.*

1.3. *The equations of the form*

$$(1) \quad c_k(f^4+42f^2g^2+49g^4)+28d_k(f^3g+7fg^3)=m^2,$$

where (c_k, d_k) is a solution of the Pell equation $u^2-7v^2=1$, has an infinity of integer solutions.

2. Studying the equation (1)

Let us fix y as a component of the solution of the equation $m^4-n^4=7y^2$. Then we have the following result:

Key Words: Diophantine equations, Pell equation.

Proposition 2.1. *The only cases for which, from an integer solution (m, n, y) of the equation $m^4 - n^4 = 7y^2$, we get integer solutions for the equations* (1) *is* $k \equiv 3 \ (mod\ 4)$.

Proof. In [7] we have proved that the equation $m^4 - n^4 = 7y^2$ has an infinity of integer solutions.

We know from 1.1. that the ring of algebraic integers $A = \mathbf{Z}\left[\sqrt{7}\right]$ of the quadratic field $\mathbf{Q}\left(\sqrt{7}\right)$ is Euclidian with respect to the norm N, $N(a+b\sqrt{7}) := \left| a^2 - 7b^2 \right|$.

We study the equation $m^4 - n^4 = 7y^2$ in the ring $\mathbf{Z}\left[\sqrt{7}\right]$. This equation has at least a solution: $m = 463$, $n = 113$, $y = 80880$. But then it has an infinity of integer solutions.

Consider $m^4 - n^4 = 7y^2$ written as $(m^2 - y\sqrt{7})(m^2 + y\sqrt{7}) = n^4$.

In [7], we have proved that $m^2 + y\sqrt{7}$ and $m^2 - y\sqrt{7}$ are prime to each other in $\mathbf{Z}\left[\sqrt{7}\right]$.

This implies that there exists $f + g\sqrt{7} \in \mathbf{Z}\left[\sqrt{7}\right]$ and there exists $k \in \mathbf{Z}$ such that

$$\begin{gathered} m^2 + y\sqrt{7} = (c_k + d_k\sqrt{7}) \cdot \left(f + g\sqrt{7}\right)^4, \text{with} \\ c_k + d_k\sqrt{7} \in \left\{ \pm \left(8 + 3\sqrt{7}\right)^{k+1} / \ k \in \mathbf{Z} \right\}, \end{gathered}$$

$(8, 3)$ being the fundamental solution of the Pell equation $u^2 - 7v^2 = 1$.
We obtain the equation:

$$m^2 + y\sqrt{7} = (c_k + d_k\sqrt{7}) \cdot (f^4 + 4f^3g\sqrt{7} + 42f^2g^2 + 28fg^3\sqrt{7} + 49g^4),$$

which is equivalent to the system:

$$\begin{cases} m^2 = c_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k\left(f^3g + 7fg^3\right) \\ y = d_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 4c_k\left(f^3g + 7fg^3\right). \end{cases}$$

By 1.2., the equation $m^4 - n^4 = 7y^2$ has an infinity of integer solutions.
Hence, the system

$$\begin{cases} m^2 = c_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k\left(f^3g + 7fg^3\right) \\ y = d_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 4c_k\left(f^3g + 7fg^3\right) \end{cases}$$

has an infinity of integer solutions. Then, the equation

$$m^2 = c_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k\left(f^3g + 7fg^3\right)$$

has an infinity of integer solutions.

We want to find those integers k, such that, from a solution of the equation $m^4 - n^4 = 7y^2$, we can get solutions for the system:

$$\begin{cases} m^2 = c_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k\left(f^3g + 7fg^3\right) \\ y = d_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 4c_k\left(f^3g + 7fg^3\right). \end{cases}$$

The system has been obtained from: $m^2 + y\sqrt{7} = (c_k + d_k\sqrt{7})\cdot\left(f + g\sqrt{7}\right)^4$, which is equivalent to the equation: $m^2 + y\sqrt{7} = (c_0 + d_0\sqrt{7})^{k+1}\cdot\left(f + g\sqrt{7}\right)^4$, $k \in \mathbf{Z}$.

First, we give an example. A solution of the equation $m^4 - n^4 = 7y^2$ is $m = 463$, $y = 80880$, $n = 113$.Using this solution, we can get a solution for the equation: $m^2 + y\sqrt{7} = (c_0 + d_0\sqrt{7})^{k+1}\cdot\left(f + g\sqrt{7}\right)^4$, $k \in \mathbf{Z}$ (where $c_0 = 8$, $d_0 = 3$), namely $f = 15$, $g = 4$, $k = -1$.

For $k = 3$, the equation $m^2 + y\sqrt{7} = (c_0 + d_0\sqrt{7})^{k+1}\cdot\left(f + g\sqrt{7}\right)^4$ becomes:

$$m^2 + y\sqrt{7} = \left[(8f + 21g) + (8g + 3f)\sqrt{7}\right]^4.$$

We obtain: $\begin{cases} 8f + 21g = 15 \\ 8g + 3f = 4 \end{cases}$, which implies $f = 36$, $g = -13$.
Analogously, for $k = 7$, we obtain: $f = 561$, $g = -212$.
We succeed to obtain a general result.
The equation

$$m^2 + y\sqrt{7} = (c_0 + d_0\sqrt{7})^{4(k'+1)}(f + g\sqrt{7})^4$$

is equivalent to the equation:

$$m^2 + y\sqrt{7} = (c_{k'} + d_{k'}\sqrt{7})^4(f + g\sqrt{7})^4,$$

and we obtain:

$$m^2 + y\sqrt{7} = \left[(fc_{k'} + 7gd_{k'}) + (gc_{k'} + fd_{k'})\sqrt{7}\right]^4.$$

We consider the same solution $(463, 15, 4)$ and we get that the system:

$$\begin{cases} fc_{k'} + 7gd_{k'} = 15 \\ fd_{k'} + gc_{k'} = 4 \end{cases}$$

has the integer solution: $g = 4c_{k'} - 15d_{k'}$; $f = 15c_{k'} - 28d_{k'}$.

In general, for $a, b \in \mathbf{Z}$, the system:

$$\begin{cases} fc_{k^{'}} + 7gd_{k^{'}} = a \\ fd_{k^{'}} + gc_{k^{'}} = b \end{cases}$$

has the solution: $f = -7bd_{k^{'}} + ac_{k^{'}}$, $g = bc_{k^{'}} - ad_{k^{'}}$ in $\mathbf{Z}$.

In conclusion, in the case $k \equiv 3 \pmod 4$, for each solution of the equation $m^4 - n^4 = 7y^2$, we get an infinity of integer solutions for the system:

$$\begin{cases} y = 4c_k \left(f^3 g + 7fg^3\right) + d_k \left(f^4 + 42f^2g^2 + 49g^4\right) \\ m^2 = c_k \left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k \left(f^3 g + 7fg^3\right), \end{cases}$$

therefore, an infinity of integer solutions for the equation

$$m^2 = c_k \left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k \left(f^3 g + 7fg^3\right).$$

Now we consider the cases $k \neq 3 \pmod 4$.

We use the following notations: $f^4 + 42f^2g^2 + 49g^4 = u$ and $f^3g + 7fg^3 = v$.

The system:

$$\begin{cases} y = 4c_k \left(f^3 g + 7fg^3\right) + d_k \left(f^4 + 42f^2g^2 + 49g^4\right) \\ m^2 = c_k \left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k \left(f^3 g + 7fg^3\right) \end{cases}$$

is equivalent to the system:

$$\begin{cases} 4c_k v + d_k u = y \\ 28d_k v + c_k u = m^2. \end{cases}$$

Then u being an integer number, we get $u = -7d_k y + c_k m^2$ and $v = (c_k y - d_k m^2) \,/\, 4$.

When is v an integer number?

We take $c_k + d_k\sqrt{7} = (c_0 + d_0\sqrt{7})^{k+1}, k \in \mathbf{Z}$, $c_0 = 8$, $d_0 = 3$, and we obtain the

equalities:

$$\begin{cases} c_k = \frac{1}{2}\left[(c_0 + d_0\sqrt{7})^{k+1} + (c_0 - d_0\sqrt{7})^{k+1}\right] \\ d_k = \frac{1}{2\sqrt{7}}\left[(c_0 + d_0\sqrt{7})^{k+1} - (c_0 - d_0\sqrt{7})^{k+1}\right] \end{cases}, k \in \mathbf{Z}.$$

These are equivalent to the equalities:

$$\begin{cases} c_k = 8^{k+1} + C_{k+1}^2 \cdot 9 \cdot 7 \cdot 8^{k-1} + C_{k+1}^4 \cdot 9^2 \cdot 7^2 \cdot 8^{k-3} + \ldots.. \\ d_k = (k+1) \cdot 8^k \cdot 3 + C_{k+1}^3 \cdot 8^{k-2} \cdot 3^3 \cdot 7 + C_{k+1}^5 \cdot 8^{k-4} \cdot 3^5 \cdot 7^2 + \ldots. \end{cases}$$

By computing these values, we obtain the following result:

If k is an odd number, then c_k is an odd number too ($c_k \equiv \pm 1 \pmod 8$) and d_k is an even number ($d_k \equiv 0 \pmod 8$).

If k is an even number, then c_k is an even number ($c_k \equiv 0 \pmod 8$) and d_k is an odd number ($d_k \equiv \pm 3 \pmod 8$) and knowing that m is an odd number we obtain that $c_k y - d_k m^2$ is an odd number. This implies that v is not an integer number.

If k is an odd number, $k \equiv 1 \pmod 4$, then $d_k \equiv 0 \pmod 4$, $y \equiv 0 \pmod 4$, therefore $c_k y - d_k m^2 \equiv 0 \pmod 4$. This implies $v \in \mathbf{Z}$.

Then the system:

$$\begin{cases} f^4 + 42f^2g^2 + 49g^4 = u \\ f^3g + 7fg^3 = v \end{cases}$$

is equivalent to the system:

$$\begin{cases} f^4 + 42f^2g^2 + 49g^4 = -7d_k y + c_k m^2 \\ f^3g + 7fg^3 = \frac{c_k y - d_k m^2}{4}. \end{cases}.$$

Let s be the least common divisor of u and v. We prove that $s = 1$. If $s > 1$, we take a prime divisor s_1 of s.Since s_1/u and s_1/v, we get that s_1 / $(4c_k{\cdot}v + d_k{\cdot}u)$ and s_1/ $(28d_k{\cdot}v + c_k{\cdot}u)$, hence s_1/ y and s_1/ m^2, therefore s_1/ n^4, in contradiction with the assumption $(m, n) = 1$. Therefore, $s = 1$.

We come back to the system:

$$\begin{cases} f^4 + 42f^2g^2 + 49g^4 = u \\ f^3g + 7fg^3 = v. \end{cases}$$

We have the equation

$$vf^4 - uf^3g + 42vf^2g^2 - 7ufg^3 + 49vg^4 = 0.$$

This is equivalent to:

$$v{\cdot}\left(\frac{f}{g}\right)^4 - u{\cdot}\left(\frac{f}{g}\right)^3 + 42v{\cdot}\left(\frac{f}{g}\right)^2 - 7u{\cdot}\frac{f}{g} + 49v = 0.$$

We denote $\frac{f}{g}=t$ and we get the equation $vt^4-ut^3+42vt^2-7ut+49v=0$.

Let $\varphi=vt^4-ut^3+42vt^2-7ut+49v$ be a polynomial in $\mathbf{Z}[t]$. We may take a

monic polynomial $\varphi_1$deduced from φ :

$$\varphi_1(t)=v^3\cdot\varphi\left(\frac{t}{v}\right)=v^3\cdot\left[v\cdot\left(\frac{t}{v}\right)^4-u\cdot\left(\frac{t}{v}\right)^3+42v\cdot\left(\frac{t}{v}\right)^2-7v\cdot\frac{t}{v}+49v\right],\text{hence}$$

$$\varphi_1=t^4-ut^3+42v^2t^2-7uv^2t+49v^4\in\mathbf{Z}[t].$$

We consider $\overline{\varphi_1}=t^4-\overline{u}t^3\in\mathbf{Z}_7[t]$.The only divisor of degree $1\le 2$ of $\overline{\varphi_1}\in\mathbf{Z}_7[t]$

is $\overline{g}=t-\overline{u}$.

We search for a representative of $\overline{u}$ (in $\mathbf{Z}_7$) found in the interval$\left(-\frac{7}{2};\frac{7}{2}\right]$,

therefore in $[-3,3]$.

But $u=-7d_ky+c_km^2$. This implies $u\equiv c_km^2(\text{mod } 7)$. As $c_k\equiv 1(\text{mod } 7)$, we have

$u\equiv m^2(\text{mod } 7)$. Knowing that, for any $m\in\mathbf{Z}$, $m^2\equiv 1,2$ or $4(\text{mod } 7)$, we obtain that $u\equiv 1,2$ or $-3(\text{mod } 7)$, hence $g=t-1$ or $g=t-2$ or $g=t+3$ is a divisor of φ_1.

Case I: $g=t-1$ implies that $\varphi_1=(t-1)\cdot\varphi_2$, with $\varphi_2\in\mathbf{Z}[t]$, hence $\varphi=\frac{1}{v^3}\cdot\varphi_1(vt)=\frac{1}{v^3}(vt-1)\cdot\varphi_2(vt)$. Therefore $\frac{1}{v}\in\mathbf{Q}$ is a root of φ.

We come back at the notation established and we get $g=vf$.

But $\begin{cases} f^4+42f^2g^2+49g^4=u \\ f^3g+7fg^3=v \end{cases}$, therefore, we obtain :

$$\begin{cases} f^4(1+42v^2+49v^4)=u \\ f^4(1+7v^2)=1. \end{cases}$$

The only integer solutions of this system are $f\in\{-1,1\}$, $v=0$, $g=0,u=1$.

Case II: $g=t-2$ implies that $\varphi_1=(t-2)\cdot\varphi_2$, with $\varphi_2\in\mathbf{Z}[t]$, hence $\varphi=\frac{1}{v^3}\cdot\varphi_1(vt)=\frac{1}{v^3}(vt-2)\cdot\varphi_2(vt)$. Therefore $\frac{2}{v}\in\mathbf{Q}$ is a root of φ.

We obtain $g=\frac{fv}{2}$.

If $g\in\mathbf{Z}$, knowing that $f^3g+7fg^3=v$, we get $f^4(4+7v^2)=8$.The equation does not have integer solutions.

Case III: $g=t+3$ implies that $\varphi_1=(t+3)\cdot\varphi_2$, with $\varphi_2\in\mathbf{Z}[t]$, hence

$\varphi=\frac{1}{v^3}\cdot\varphi_1(vt)=\frac{1}{v^3}(vt+3)\cdot\varphi_2(vt)$. Therefore $t_0=-\frac{3}{v}$ is a root of φ.

Then we get $g=-\frac{fv}{3}$.

If $g \in \mathbf{Z}$, from $f^3g + 7fg^3 = v$, we get $f^4(9 + 7v^2) = -9$. This equation
does not have integer solutions.

We come back to the cases I and II and we obtain $f \in \{-1, 1\}$, $v = 0$, $g = 0$,
$u = 1$.This implies $y = d_k$, $m^2 = c_k$, $n \in \{-1, 1\}$.

We look for $m \in \mathbf{Z}$ such that $m^2 = c_k$.

Knowing that $k \equiv 1 \pmod 4$, we obtain:
$c_k = \frac{1}{2}\left[\left(c_0 + d_0\sqrt{7}\right)^{k+1} + \left(c_0 - d_0\sqrt{7}\right)^{k+1}\right]$. This implies:
$c_k = 8^{k+1} + C_{k+1}^2 \cdot 8^{k-1} \cdot 9 \cdot 7 + C_{k+1}^4 \cdot 8^{k-3} \cdot 9^2 \cdot 7^2 + ... + (9 \cdot 7)^{\frac{k+1}{2}}$, therefore

$c_k \equiv 63^{\frac{k+1}{2}} \pmod 8$, hence $c_k \equiv 7 \pmod 8$. Then there is not an integer m such that $m^2 = c_k$.

From the previously proved, we got that $\varphi_1$does not have divisors of degree 1,
therefore $\varphi_1$does not have integer roots. This implies that φ does not have rational
roots. Hence, the system:

$$\begin{cases} f^4 + 42f^2g^2 + 49g^4 = u \\ f^3g + 7fg^3 = v \end{cases}$$

does not have nontrivial integer solutions.

In conclusion, in the case $k \equiv 1 \pmod 4$, for each solution of the equation
$m^4 - n^4 = 7y^2$, we do not get integer solutions for the equation:

$$m^2 = c_k\left(f^4 + 42f^2g^2 + 49g^4\right) + 28d_k\left(f^3g + 7fg^3\right).$$

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