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MODULES WITH SLIDING DEPTH

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Abstract

Several bounds for the depth of quotients of the symmetric algebra of a finitely generated module over a local C.M. ring are obtained, when its maximal irrelevant ideal is generated by a proper sequence of 1-forms and modulo conditions on the depth of the homology modules of the Koszul complex associated to this ideal (sliding depth conditions).

1 INTRODUCTION

Let R be a commutative noetherian ring, and let E be a finitely generated R-module with rank.

We denote by $Sym_R(E)$ or $S(E)$, the symmetric algebra of E over R, that is the graded algebra over R:

$$S(E) = \bigoplus_{t \geq 0} Sym_t(E)$$

and with S_+ the maximal irrelevant ideal of $S(E)$.

In [2], J.Herzog, A.Simis, W.V.Vasconcelos, introduced the sliding depth condition for a module that is precisely the following:

Let (R, m) be a C.M. local ring of dimension d. Let E be a finitely generated R-module, $S(E)$ the symmetric algebra of E and $S_+ = (x_1, \dots, x_n) = (\mathbf{x})$ its maximal irrelevant ideal generated by the linear forms x_i .

We say that E satisfies the sliding depth condition SD_k, with k integer, if

$$depth_{(m)} H_i(\mathbf{x}, S(E))_i \geq d - n + i + k \qquad 0 \leq i \leq n - k,$$

where $H_i(\mathbf{x}, S(E))_i$ is the i-th graded component of the Koszul homology module $H_i(\mathbf{x}, S(E))$, and the elements of the ring R have degree 0.

Key Words: Symmetric algebra, sliding depth.

If $k = rank(E)$ we shall say that E satisfies the sliding depth condition SD.

When the R-module E satisfies the SD condition and E verifies $\mathcal{F}_0$ (for every k, $\mathcal{F}_k$ is a condition on the Fitting ideals of a presentation of E, that is a condition on the height of the ideal generated by the minors of the presentation of E [see [2], Section 2]), this implies that $S(E)$ is Cohen Macaulay and the approximation complex $\mathcal{Z}(E)$ associated to E is acyclic (see [2], theorem 6.2).

The aciclicity of the $\mathcal{Z}(E)$-complex can be obtained for modules for which the ideal S_+ of the symmetric algebra $S(E)$ is generated by a proper sequence (see [7]) or a proper sequence in E (see [6]).

Then an important way to obtain informations about theoretic properties of $S(E)$ is that S_+ is generated by a proper sequence in E.

If E satisfies SD_k and S_+ is generated by a proper sequence in E, we are able to obtain bounds for the depth of quotients of $S(E)$ by ideals generated by a subsequence of a system of generators of S_+.

The idea arises from some results about ideals of J. Herzog, W.V. Vasconcelos, R. Villarreal in [3].

Our results concern modules E finitely generated over a local $C.M.$ ring R and that are not necessarily ideals of R. Moreover, in the case $E = I =$ faithful ideal of R, we obtain the classical results on ideals.

More precisely, in section 2, we define the depth condition SD_k for a module E and we prove some properties related to it.

In section 3, we study the Koszul complex associated to a sequence $(x_1, \ldots, x_n)$ generating the ideal $S_+ = \bigoplus_{t>0} S_t(E)$ of the symmetric algebra $S(E)$ and when this sequence is a proper sequence in E, we investigate the link between the SD_k condition and the existence of bounds for quotients of $S(E)$ by a subsequence $(x_1, \ldots, x_i)$, $i = 0, \ldots, n$ of $(x_1, \ldots, x_n)$ (that is a proper sequence again) and special quotients of ideals of $S(E)/(x_1, \ldots, x_i)$ constructed starting from $(x_1, \ldots, x_n)$.

The main theorem is the following: Let (R, m) be a C.M. local ring of dimension d. Let $x_1, \ldots, x_n$ a proper sequence of the module E, under a condition on the depth of the homology module of the Koszul complex (see theorem 3.3), the following conditions are equivalent:

i) E satisfies SD_k;

ii) $depth_{(m,S_+)} S(E)/(x_1, \ldots, x_i) \geq d - i + k$, $i = 0, \ldots, n$;

iii) $depth_{(m,S_+)} (x_1, \ldots, x_{i+1})/(x_1, \ldots, x_i) \geq d - i + k$, $i = 0, \ldots, n-1$.

Finally we remark that in the case $E = I$, our results complete those contained in [3].

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2 PRELIMINARIES

Let R be a commutative noetherian ring and let E be a finitely generated R-module.

We denote with $Sym_R(E)$ or with $S(E)$, the symmetric algebra of E over R, that is the graded algebra over R:

$$S(E) = \bigoplus_{t \geq 0} Sym_t(E)$$

and with S_+ the maximal irrelevant ideal of $S(E)$.

$$S_+ = \bigoplus_{t > 0} Sym_t(E).$$

Let $S_+ = (x_1, \ldots, x_n) = (\mathbf{x})$, where x_i are elements of degree 1. We can consider the Koszul complex on the generating set $\{x_1, \ldots, x_n\}$ of S_+

$$K.(\mathbf{x}; S(E)) : 0 \to K_n \overset{d_n}{\to} \cdots \overset{d_2}{\to} K_1 \overset{d_1}{\to} K_0 \to 0$$

where

$$K_p(\mathbf{x}; S(E)) = \bigwedge^p (S(E))^n \cong \bigwedge^p R^n \otimes S(E)$$

K is a graded complex and in degree $t > 0$ we have

$$0 \to \bigwedge^n R^n \otimes S_{t-n}(E) \overset{d_n}{\to} \bigwedge^{n-1} R^n \otimes S_{t-n+1}(E) \overset{d_{n-1}}{\to} \cdots$$
$$\cdots \bigwedge^2 R^n \otimes S_{t-2}(E) \overset{d_2}{\to} R^n \otimes S_{t-1}(E) \overset{d_1}{\to} S_t(E) \to 0$$

with differential d_p defined as follows:

$$d_p(e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes f(\mathbf{x})) = \sum_{j=1}^{p} (-1)^{p-j} e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_p} \otimes x_{i_j} f(\mathbf{x}).$$

where $e_1, \ldots, e_n$ is a standard basis of R^n, $f(\mathbf{x}) \in S_{t-p}(E)$.

We also denote by $Z_p(\mathbf{x}; S(E))$ and by $B_p(\mathbf{x}; S(E))$ the cycles and boundaries of this complex, i.e.

$$Z_p = ker(K_p(\mathbf{x}; S(E)) \to K_{p-1}(\mathbf{x}; S(E))$$

$$B_p = im(K_{p+1}(\mathbf{x}; S(E)) \to K_p(\mathbf{x}; S(E)).$$

Finally we denote by $H_p(\mathbf{x}; S(E))_j$, $j \geq p$, the $j-$th graded component of the Koszul homology $H_p(\mathbf{x}; S(E)) = Z_p/B_p$.

We observe that:

$$H_p(\mathbf{x}; S(E)) = \bigoplus_{j \geq p} H_p(\mathbf{x}; S(E))_j$$

Since the ring $S(E)$ is positively graded, if (R, m) is a local ring we can consider the ring $S(E)$ as a *local ring with *maximal ideal $m_0 = m \oplus S_+$ (see [1] Chapter 1.5 for details).

Consequently, for any finitely generated $S(E)$-graded module we will calculate the depth of its graded components, that are R-modules, with respect to the maximal ideal m, and for any $S(E)$-module its depth with respect to the *maximal ideal $m_0 = m \oplus S_+$.

Definition 2.1 ([2], section 6) *Let (R, m) be a C.M. local ring of dimension d. Let E be a finitely generated R-module, $S(E)$ the symmetric algebra of E, $S_+ = (x_1, \ldots, x_n) = (\boldsymbol{x})$ its maximal irrelevant ideal generated by the linear forms x_i .*

We say that E satisfies the sliding depth condition SD_k, with k integer, if

$$depth_{(m)} H_i(\boldsymbol{x}, S(E))_i \geq d - n + i + k \qquad 0 \leq i \leq n - k$$

If $k = rank(E)$ we shall say that E satisfies the sliding depth condition SD.

Remark 2.2 *By definition $\mathcal{Z}_i(E) = H_i(\boldsymbol{x}, S(E))_i$ and since that $(B_i)_i = 0$ we have*

$$\mathcal{Z}_i = (Z_i)_i = ker(K_i(\boldsymbol{x}; S(E)) \to K_{i-1}(\boldsymbol{x}; S(E))_i$$

Therefore

$$\mathcal{Z}_i = ker(\bigwedge^{i} R^n \otimes S_0(E) \cong R \to \bigwedge^{i-1} R^n \otimes S_1(E) \cong E)$$

and

$$\mathcal{Z}_1 = ker(R^n \to E)$$

that is the first syzygy module of E.

Since $\mathcal{Z}_i$, $\forall i$, is an R-module, in the definition 2.1, we have to calculate $depth_{(m)} H_i(\mathbf{x}, S(E))_i$. The module $\mathcal{Z}_i(E)$ appears in the $\mathcal{Z}(E)$-complex of the module E. We can find more information about $S(E)$ in [6]. We remember that, if E is torsion-free and with rank $e > 1$, $\mathcal{Z}_{n-e}(E) \cong R$, and the complex is the following:

$$\mathcal{Z}(E) : 0 \to \mathcal{Z}_n \otimes S[-n] \stackrel{d_n}{\to} \mathcal{Z}_{n-1} \otimes S[-n+1] \stackrel{d_{n-1}}{\to} \cdots \mathcal{Z}_1 \otimes S[-1] \stackrel{d_1}{\to}$$
$$\to S = S(R^n) \stackrel{d_0}{\to} S(E) \to 0$$

with the related maps d_i induced from the Koszul complex.

$\mathcal{Z}_i(E) = 0$ for $i > n - e$, and the cokernel of d_1 is the symmetric algebra of E. When the $\mathcal{Z}(E)$-complex is acyclic, we can obtain the depth of the symmetric algebra $S(E)$ of E, by the acyclicity lemma of Peskine-Szpyro ([5]).

Proposition 2.3 *$SD_k(E)$ does not depend on the generating set of E.*

Proof: See [2] section 6, remark 1.

We recall the following

Proposition 2.4 ([7], chapter 3.3) *Let R be a C.M. local ring of dimension d with canonical module ω_R and E a finitely generated R-module. Then we have:*

1) $depth(E) = dim(R) - sup\{j | Ext^j_R(E; \omega_R) \neq 0\}$

2) $depth(E_P) = ht(P) - sup\{j | Ext^j_R(E_P; \omega_P) \neq 0\}$, where $\omega_P = \omega_R \otimes R_P$, is a canonical module for R_P for all $P \in Spec(R)$,

Proposition 2.5 *Let (R, m) be a C.M. local ring of dimension d, and let E a finitely generated R-module of rank e that verifies SD_k. Then, $\forall P \in Spec(R)$, E_P verifies SD_k.*

Proof: This is clear if R admits a canonical module ω, since ω_P is the canonical module for R_P and we can apply proposition 2.4 to the module $H_i(\mathbf{x}, S(E))_i$.

When R does not have a canonical module, we reach the same result by applying the m-adic completion of R, obtaining a quotient of a regular ring.

We recall the definition of a proper sequence for any ring

Definition 2.6 *Let $\boldsymbol{x} = x_1, \ldots, x_n$ a sequence of elements in a ring R. The sequence $\boldsymbol{x}$ is called a proper sequence if:*

$$x_{i+1} H_j(x_1, \ldots, x_i; R) = 0$$

for $i = 1, \ldots, n$, $j > 0$, where $H_j(x_1, \ldots, x_i; R)$ denotes the Koszul homology associated to the initial subsequence $x_1, \ldots, x_i$.

If we consider the symmetric algebra $S(E)$ of a finitely generated R-module E, for the ring $S(E)$, the definition of proper sequence applied to a sequence of 1-forms generating the maximal irrelevant ideal of $S(E)$, is the following

Definition 2.7 *A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of 1-forms generating the maximal irrelevant ideal S_+ of $S(E)$ is called a proper sequence if:*

$$x_{i+1}H_j(x_1, \ldots, x_i; S(E))_l = 0$$

for $i = 0, \ldots, n-1$, $j \geq 1$, $l \geq j$.

We give the following

Definition 2.8 *Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of 1-form generating the maximal irrelevant ideal S_+ of $S(E)$. Then $\boldsymbol{x}$ is called a proper sequence in E if:*

$$x_{i+1}Z_j(x_1, \ldots, x_i; S(E))_j / B_j(x_1, \ldots, x_i; S(E))_{j+1} = 0$$

for $i = 0, \ldots, n-1$, $j > 0$.

Remark 2.9 *The definition of proper sequence in E, where E is a finitely generated R-module, is introduced and discussed in [6].*

In [6] is also proved the aciclicity of the complex $\mathcal{Z}(E)$, when S_+ is generated by a proper sequence in E .

This definition will be used in the rest of this work.

Definition 2.10 *([3], Introduction) Let I be an ideal of the local ring (R, m) of dimension d. Let K be the Koszul complex on a set $\boldsymbol{x} = x_1, \ldots, x_n$ of generators of I, and let k be a positive integer.*

We say that I satisfies the sliding depth condition SD_k, if

$$depth_{(m)}H_i(\boldsymbol{x}, R) \geq d - n + i + k \qquad \forall i \geq 0$$

If $k = 0$ we shall say that I satisfies the sliding depth condition SD.

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ a sequence of elements in I, where I is an ideal of R. We call $\mathbf{x}_i = \{x_1, \ldots, x_i\}$ the initial subsequence of $\mathbf{x}$.

An easy extension of lemma 3.7 in [3] is the following:

Theorem 2.11 *Let I be an ideal of local ring (R, m) of dimension d. Suppose I is generated by a proper sequence $\boldsymbol{x} = x_1, \ldots, x_n$.*

The following conditions are equivalent:

1) I satisfies SD_k;

2) $depth_{(m)}R/(\boldsymbol{x}_i) \geq d - i + k$, $i = 0, \ldots, n$;

3) $depth_{(m)}(\boldsymbol{x}_{i+1})/(\boldsymbol{x}_i) \geq d - i + k$, $i = 0, \ldots, n-1$.

Proof: The proof follows directly from lemma 3.7 [3] and by depth lemma (see [1] proposition 1.2.9) applied on the following exact sequence

$$0 \to H_j(\mathbf{x}_i, R) \to H_j(\mathbf{x}_{i+1}, R) \to H_j(\mathbf{x}_i, R)[-1] \to 0$$

$\forall j > 1$,$0 \leq i \leq n-1$;

$$0 \to Q_i \to R/(\mathbf{x}_i) \to R/(\mathbf{x}_{i+1}) \to 0,$$

with $Q_i = (\mathbf{x}_{i+1})/(\mathbf{x}_i)$, $0 \leq i \leq n-1$;

$$0 \to M_i \to R/(\mathbf{x}_i) \stackrel{x_{i+1}}{\to} Q_i \to 0,$$

with $M_i = ((\mathbf{x}_i) : x_{i+1})/(\mathbf{x}_i)$, $0 \leq i \leq n-1$;

$$0 \to H_1(\mathbf{x}_i) \to H_1(\mathbf{x}_{i+1}) \to M_i \to 0,$$

with $0 \leq i \leq n-1$.

3 THE MAIN RESULT

In this section we look at the symmetric algebra $S(E)$ of a finitely generated module E on a C.M. local ring (R, m), and a proper sequence in E, $\mathbf{x}$, generating the maximal irrelevant ideal S_+.

The depth of each $S_R(E)$-modules, is calculated with respect to the *maximal ideal $m \oplus S_+$.

In the following, for any graded R-module M, if a is an integer, $M(-a)$ is a graded R-module such that $M(-a)_i = M_{-a+i}$, $\forall i \geq a$.

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ a proper sequence in the R-module E. We call $\mathbf{x}_i = \{x_1, \ldots, x_i\}$ the initial subsequence of $\mathbf{x}$, and $H_j(\mathbf{x}_i)_l = H_j(\mathbf{x}_i; S(E))_l$. We have the following :

Lemma 3.1 *Let (R, m) be a C.M. local ring of dimension d, E a finitely generated R-module and $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ a proper sequence in E.*

The following sequences are exact for $0 \leq i \leq n-1$:

1) $0 \to H_j(\boldsymbol{x}_i)_j \to H_j(\boldsymbol{x}_{i+1})_j \to H_{j-1}(\boldsymbol{x}_i)_j[-1] \to 0$, $\forall j > 1$;

2) $0 \to Q^{(i)} \to S(E)/(\boldsymbol{x}_i) \to S(E)/(\boldsymbol{x}_{i+1}) \to 0$, *with* $Q^{(i)} = (\boldsymbol{x}_{i+1})/(\boldsymbol{x}_i)$;

3) $0 \to M^{(i)} \to S(E)/(\boldsymbol{x}_i) \stackrel{x_{i+1}}{\to} Q^{(i)} \to 0$, *with* $M^{(i)} = ((\boldsymbol{x}_i) : x_{i+1})/(\boldsymbol{x}_i)$;

4) $0 \to H_1(\boldsymbol{x}_i) \to H_1(\boldsymbol{x}_{i+1}) \to M^{(i)} \to 0$.

Proof:

1) There exists an exact sequence of complexes:

$$0 \to K(\mathbf{x}_i; S(E)) \xrightarrow{i} K(\mathbf{x}_{i+1}; S(E)) \xrightarrow{\epsilon} K(\mathbf{x}_i; S(E))[-1] \to 0$$

where i is the natural inclusion and ϵ is defined as follows:

Given $a \in K_j(\mathbf{x}_{i+1}; S(E))_{j+\rho}$, then

$$a = b + c \wedge e_{i+1},$$

with $b \in K_j(\mathbf{x}_i; S(E))_{j+\rho}$, $c \in K_{j-1}(\mathbf{x}_i; S(E))_{j-1+\rho}$ and $\epsilon(a) = (-1)^{i+1}c$.

It is clear that

$$\epsilon(a) = \epsilon(b + c \wedge e_{i+1}) = \epsilon(b) + \epsilon(c \wedge e_{i+1}) = 0 + (-1)^{i+1}c,$$

is an epimorhism on $K_{j-1}(\mathbf{x}_i; S(E))_{j-1+\rho}$ and its kernel is $K_j(\mathbf{x}_i; S(E))_{j+\rho}$.

We obtain the long exact sequence

$$\cdots \to H_j(\mathbf{x}_i) \xrightarrow{i} H_j(\mathbf{x}_{i+1}) \xrightarrow{\epsilon} H_{j-1}(\mathbf{x}_i)[-1] \xrightarrow{\partial}$$
$$H_{j-1}(\mathbf{x}_i) \xrightarrow{i} H_{j-1}(\mathbf{x}_{i+1}) \xrightarrow{\epsilon} H_{j-2}(\mathbf{x}_i)[-1] \xrightarrow{\partial} H_{j-2}(\mathbf{x}_i) \cdots$$

We consider the degree $j + \rho$

$$\cdots \to H_j(\mathbf{x}_i)_{j+\rho}[-1] \xrightarrow{\partial} H_j(\mathbf{x}_i)_{j+\rho} \to H_j(\mathbf{x}_{i+1})_{j+\rho} \to$$
$$H_{j-1}(\mathbf{x}_i)_{j+\rho}[-1] \xrightarrow{\partial} H_{j-1}(\mathbf{x}_i)_{j+\rho} \to H_{j-1}(\mathbf{x}_{i+1})_{j+\rho} \to \cdots$$

In particular when $\rho = 0$ we have for the Koszul complex

$$a \in K_j(x_{i+1}; S(E))_j = (\bigwedge^j R^{i+1} \otimes S(E))_j \cong \bigwedge^j R^{i+1}$$

$$b \in K_j(x_i; S(E))_j = (\bigwedge^j R^i \otimes S(E))_j \cong \bigwedge^j R^i$$

$$c \in K_{j-1}(x_i; S(E))_j[-1] = K_{j-1}(x_i; S(E))_{j-1} =$$
$$(\bigwedge^{j-1} R^i \otimes S(E))_{j-1} \cong \bigwedge^{j-1} R^{i+1}$$

and in the Koszul homology

$$a \in H_j(x_{i+1}; S(E))_j = Ker(\bigwedge^j R^{i+1} \to \bigwedge^{j-1} R^{i+1} \otimes S_1(E))$$

$$b \in H_j(x_i; S(E))_j = Ker(\bigwedge^j R^i \to \bigwedge^{j-1} R^i \otimes S_1(E))$$

$$c \in H_{j-1}(x_i; S(E))_j[-1] \cong H_{j-1}(x_i; S(E))_{j-1} =$$
$$Ker(\bigwedge^{j-1} R^i \to \bigwedge^{j-2} R^i \otimes S_1(E))$$

The assertion follows from the following sequence

$H_j(\mathbf{x}_i)_j[-1] \to H_j(\mathbf{x}_i)_j \xrightarrow{i_1} H_j(\mathbf{x}_{i+1})_j \xrightarrow{\epsilon_1} H_{j-1}(\mathbf{x}_i)_j[-1] \xrightarrow{\partial}$
$H_{j-1}(\mathbf{x}_i)_j \xrightarrow{i_2} H_{j-1}(\mathbf{x}_{i+1})_j$

In fact, by consideration on the degree $H_j(\mathbf{x}_i)_j[-1] \cong (0)$ and since $\mathbf{x}$ is a proper sequence in E, the homomorphism i_2 is injective.

In fact, we have:

$$x_{i+1}H_{j-1}(\mathbf{x}_i; S(E))_j[-1] \cong x_{i+1}Z_{j-1}(\mathbf{x}_i; S(E))_{j-1}$$

$$x_{i+1}Z_{j-1}(\mathbf{x}_i; S(E))_{j-1} \subseteq B_{j-1}(\mathbf{x}_i; S(E))_j.$$

2) Obvious.

3) It is sufficient to observe that

$$0 \to ((x_1, \dots, x_i) : x_{i+1}) \to S(E) \xrightarrow{x_{i+1}} (x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \to 0$$

is exact and $(\mathbf{x}_i) \subset ((x_1, \dots, x_i) : x_{i+1})$, $(\mathbf{x}_i) \subset S(E)$.

4) This sequence follows directly from the Koszul homology. In fact we have

$0 \to H_1(\mathbf{x}_i)_1 \to H_1(\mathbf{x}_{i+1})_1 \to S(E)/(\mathbf{x}_i) \xrightarrow{x_{i+1}}$
$S(E)/(\mathbf{x}_i) \to S(E)/(\mathbf{x}_{i+1}) \to 0$

and substituting $Q^{(i)}$ in the tail of the sequence by 2), we have

$$0 \to H_1(\mathbf{x}_i)_1 \to H_1(\mathbf{x}_{i+1})_1 \to S(E)/(\mathbf{x}_i) \xrightarrow{x_{i+1}} Q^{(i)} \to 0$$

At the end, replacing $M^{(i)}$ by 3) we have the assertion.

Proposition 3.2 *Let (R, m) be a C.M. local ring of dimension d. Let $\boldsymbol{x} = \{x_1, \dots, x_n\}$ a proper subsequence of the module E. We call $H_j(\boldsymbol{x}_i)_l = H_j(\boldsymbol{x}_i; S(E))_l$, and let $i \in \mathbb{N}$.*

We have the following properties:

1) If $\boldsymbol{x}$ satisfies SD_k then depth$(H_1(\boldsymbol{x}_i)_1) \geq d - i + k + 1$

2) If depth$(H_1(\boldsymbol{x}_i)_1) \geq d - i + k + 1$ then depth$(H_j(\boldsymbol{x})_j) \geq d + k - n + j$

Proof:

1) Since $\mathbf{x}$ is a proper sequence, by lemma 3.1 1) we have

$$0 \to H_j(\mathbf{x}_i)_j \to H_j(\mathbf{x}_{i+1})_j \to H_{j-1}(\mathbf{x}_i)_j[-1] \cong H_{j-1}(\mathbf{x}_i)_{j-1} \to 0$$

for all $j > 1$.

In particular for $i = n - 1$ and $j = n$, the exact sequence is

$$0 \to H_n(\mathbf{x}_{n-1})_n = 0 \to H_n(\mathbf{x}_n)_n \to H_{n-1}(\mathbf{x}_{n-1})_{n-1} \to 0$$

and $depth_m H_{n-1}(\mathbf{x}_{n-1})_{n-1} \geq d+k$.

Now it is possible to compute an estimation for the depth of $H_{n-2}(\mathbf{x}_{n-1})_{n-2}$

$$0 \to H_{n-1}(\mathbf{x}_{n-1})_{n-1} \to H_{n-1}(\mathbf{x}_n)_{n-1} \to H_{n-2}(\mathbf{x}_{n-1})_{n-2} \to 0$$

that is $depth_m H_{n-2}(\mathbf{x}_{n-1})_{n-2} \geq d+k-1$.

At the end we will have $depth_m H_1(\mathbf{x}_{n-1})_1 \geq d+k-n+2$.

We can continue with the same argument and we have the assertion.

2) Let $\mathbf{x}$ is a proper sequence in E and $depth(H_1(\mathbf{x}_i)_1) \geq d-i+k+1$, then by the exact sequence

$$0 \to H_2(x_1)_2 = (0) \to H_2(\mathbf{x}_2)_2 \to H_1(x_1)_2[-1] \to 0$$

we obtain $depth(H_2(\mathbf{x}_2)_2) \geq d+k$.

Now, considering the exact sequence

$$0 \to H_2(\mathbf{x}_2)_2 \to H_2(\mathbf{x}_3)_2 \to H_1(\mathbf{x}_2)_2[-1] \to 0$$

we obtain $depth(H_2(\mathbf{x}_3)_2) \geq d+k-1$.

At the end we will have $depth(H_2(\mathbf{x}_n)_2) \geq d+k-n+2$.

With the same technique we can calculate $depth(H_3(\mathbf{x}_n)_3) \geq d+k-n+3$, and so on. Finally we have $depth(H_j(\mathbf{x}_n)_j) \geq d+k-n+j$.

Theorem 3.3 *Let (R,m) be a C.M. local ring of dimension d. Let $x_1,\dots,x_n$ a proper sequence of the module E, and let $M^{(i)}$ the $S(E)$-module $((\boldsymbol{x}_i) : x_{i+1})/(\boldsymbol{x}_i)$, $M_1^{(i)}$ the component of degree 1 of $M^{(i)}$. Suppose that for every $0 \leq i \leq n-1$*

$$depth_{(m,S_+)} M^{(i)} \geq d-i+k$$

implies

$$depth_{(m)} M_1^{(i)} \geq d-i+k.$$

Then the following conditions are equivalent:

1) E satisfies SD_k;

2) $depth_{(m,S_+)} S(E)/(x_1,\dots,x_i) \geq d-i+k$, $i=0,\dots,n$;

3) $depth_{(m,S_+)} (x_1,\dots,x_{i+1})/(x_1,\dots,x_i) \geq d-i+k$, $i=0,\dots,n-1$.

Proof:

1) $\Rightarrow$ 2). We prove by induction on i (the length of the subsequence $\mathbf{x}_i$).

For $i = 0$ we have to calculate $depth_{(m,S_+)}S(E)$. As we already observed in remark 2.2 we can use the $\mathcal{Z}(E)$-complex that is acyclic since $\mathbf{x}$ is a proper sequence ([6],theorem 2).

The complex is

$$\mathcal{Z}(E) : 0 \to \mathcal{Z}_n \otimes S[-n] \xrightarrow{d_n} \mathcal{Z}_{n-1} \otimes S[-n+1] \xrightarrow{d_{n-1}} \cdots \mathcal{Z}_1 \otimes S[-1] \xrightarrow{d_1}$$
$$\to S = S(R^n) \xrightarrow{d_0} S(E) \to 0,$$

where $S = S(R^n) = R[T_1, \ldots, T_n]$.

It is possible to calculate a depth estimation, with respect to the ideal (m, S_+) (where S_+ is the maximal irrelevant ideal of S), of every $\mathcal{Z}_j \otimes S[-j] \cong H_j(\mathbf{x}, S(E))_j \otimes S[-j]$. That is

$$depth_{(m,S_+)} H_j(\mathbf{x}, S(E))_j \otimes S[-j] \geq d - n + j + k + n$$

In particular we have

$$0 \to \ker d_n \to \mathcal{Z}_n \otimes S[-n] \to \ker d_{n-1} \to 0$$

and since $\ker d_n = 0$, $depth_{(m,S_+)} \ker d_{n-1} \geq d + n + k$, hence

$$0 \to \ker d_{n-1} \to \mathcal{Z}_{n-1} \otimes S[-n+1] \to \ker d_{n-2} \to 0$$

with $depth_{(m,S_+)} \ker d_{n-2} \geq d + n + k - 1$, and so on.

Therefore, considering the tail of the sequence

$$0 \to \ker d_0 \to S \to S(E) \to 0$$

since $depth_{(m,S_+)}\ker d_0 \geq d + k + 1$ and $depth_{(m,S_+)}S = d + n$, by depth lemma we have $depth_{(m,S_+)}S(E) \geq d + k$.

Observing that

$$depth_{(m,S_+)}S(E)/(x_1, \ldots, x_i) \geq depth_{(m,S_+)}S(E)/(x_1, \ldots, x_{i+1})$$

the assertion follows.

2) $\Rightarrow$ 3). We consider the exact sequence of $S(E)$-modules

$$0 \to Q^{(i)} \to S(E)/(x_1, \ldots, x_i) \to S(E)/(x_1, \ldots, x_{i+1}) \to 0$$

and by the depth lemma we have

$$depth_{(m,S_+)}Q^{(i)} \geq$$
$$\geq \min\{depth_{(m,S_+)}S(E)/(x_1, \ldots, x_i),$$
$$depth_{(m,S_+)}S(E)/(x_1, \ldots, x_{i+1}) + 1\} = d + k - i.$$

3) $\Rightarrow$ 2). We consider the short exact sequence

$$0 \to (x_1) \to S(E) \to S(E)/(x_1) \to 0$$

where by hypothesis $depth_{(m,S_+)}(x_1) \geq d+k$ and $depth_{(m,S_+)}S(E) \geq d+k$ then for depth lemma, $depth_{(m,S_+)}S(E)/(x_1) \geq d+k-1$. Watching the sequence

$$0 \to Q^{(i)} \to S(E)/(x_1,\dots,x_i) \to S(E)/(x_1,\dots,x_{i+1}) \to 0$$

we have the assertion by induction on i.

2) $\Rightarrow$ 1).
By the exact sequence (lemma 3.1)

$$0 \to M^{(i)} \to S(E)/(\mathbf{x}_i) \xrightarrow{x_{i+1}} Q^{(i)} \to 0$$

we have $depth_{(m,S_+)}M^{(i)} \geq d-i+k$.

By the hypothesis $depth_{(m)}M_1^{(i)} \geq d-i+k$, and if we consider the exact sequence

$$0 \to H_1(\mathbf{x}_i)_1 \to H_1(\mathbf{x}_{i+1})_1 \to M_1^{(i)} \to 0$$

for $i=0$, then we have $depth_{(m)}H_1(\mathbf{x}_1)_1 = depth_{(m,S_+)}M_1^{(0)} \geq d+k$.

For $i=1$, $depth_{(m,S_+)}H_1(\mathbf{x}_2)_1 \geq d+k-1$, and so on, for all i, $depth_{(m)}H_1(\mathbf{x}_i)_1 \geq d+k-i+1$

$$0 \to H_1(\mathbf{x}_i)_1 \to H_1(\mathbf{x}_{i+1})_1 \to M^{(i)} \to 0.$$

By the exact sequence

$$0 \to H_2(x_1)_2 = (0) \to H_2(\mathbf{x}_2)_2 \to H_1(x_1)_1 \to 0$$

we obtain $depth(H_2(\mathbf{x}_2)_2) \geq d+k$.

Now, considering the exact sequence

$$0 \to H_2(\mathbf{x}_2)_2 \to H_2(\mathbf{x}_3)_2 \to H_1(\mathbf{x}_2)_1 \to 0$$

we obtain $depth(H_2(\mathbf{x}_3)_2) \geq d+k-1$.

At the end we will have $depth(H_2(\mathbf{x}_n)_2) \geq d+k-n+2$.

With the same technique we can calculate $depth(H_3(\mathbf{x}_n)_3) \geq d+k-n+3$, and so on. Finally we have $depth(H_j(\mathbf{x}_n)_j) \geq d+k-n+j$ and 2) $\Rightarrow$ 1) is proved.

Example 3.4 *Let $E = R^n$, then E satisfies SD (that is $k = e = n$).*

$$H_i(\boldsymbol{x}, S(E))_i = \mathcal{Z}_i = ker(\bigwedge^i R^n \to \bigwedge^{i-1} R^n \otimes R^n) = 0$$

with $S(E) = R[X_1, \dots, X_n]$, for $i > n - e = 0$.
For $i = 0$, $\mathcal{Z}_0 \cong R$, for any $i = 0, \dots, n$, we have

$$depth_{(m,S_+)} S(E)/(X_1, \dots, X_i) = depth_{(m,S_+)} R[X_{i+1}, \dots, X_n] = d + n - i$$

Then 1) and 2) of theorem 3.3 are verified.
For 3), by the exact sequence

$$0 \to (x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \to R[x_1, \dots, x_n]/(x_1, \dots, x_i) \to$$
$$\to R[x_{i+1}, \dots, x_n] \to 0$$

and by depth lemma 1, we have:

$$depth_{(m,S_+)}(x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \geq d + n - i, \qquad \forall i = 0, \dots, n-1.$$

Moreover $M_i = 0$. In fact $(x_1, \dots, x_i) : x_{i+1} = (x_1, \dots, x_i)$ and

$$(x_1, \dots, x_i)/(x_1, \dots, x_i) \cong (0).$$

Remark 3.5 *By def. 5, given in [2], ideals of rank 1 or faithful ideals (i.e. containing some regular element of R), satisfy SD_0. By definition 2.1, they satisfy SD_1. Therefore we obtain the following:*

Corollary 3.6 *Let I be an ideal of a local ring (R, m) of dimension d, containing some regular element (rank $I = 1$). Suppose I is generated by a proper sequence $\boldsymbol{x} = x_1, \dots, x_n$.*
The following conditions are equivalent:

1) I satisfies SD_0 (for ideals);

2) $depth_{(m)} R/(x_1, \dots, x_i) \geq d - i$, $i = 0, \dots, n$;

3) $depth_{(m)}(x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \geq d - i$, $i = 0, \dots, n-1$;

4) I satisfies SD_1 (for modules);

5) $depth_{(m,S_+)} S(I)/(x_1, \dots, x_i) \geq d - i + 1$, $i = 0, \dots, n$;

6) $depth_{(m,S_+)}(x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \geq d - i + 1$, $i = 0, \dots, n-1$.

Proof: We use the results of theorem 2.11 and theorem 3.3 together.

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