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THE ISOMORPHIC DESCRIPTION OF TORSION UNITS IN COMMUTATIVE GROUP RINGS OVER FIELDS WITH POSITIVE CHARACTERISTIC

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Abstract

A full classification up to isomorphism of torsions in abelian group rings over an algebraically closed field of nonzero characteristic is given in the present article. Specifically, an explicit formula is established and so the problem is completely exhausted.

This characterizing strengthens classical results in that aspect due to Chatzidakis-Pappas *(J. London Math. Soc., 1991)* and to the author *(Hokkaido Math. J., 2000)* plus *(Glasgow Math. J., 2001)*.

1. Introduction and preliminaries. Throughout the text, F denotes an arbitrary field of prime characteristic p with algebraic closure F^- and G denotes an arbitrary abelian group, written multiplicatively as is customary when discussing group algebras, with maximal divisible subgroup G_d, with maximal p-divisible subgroup $G^{(p)}$ and with a reduced part G_r. For so-defined F and G, as usual, FG designates the group algebra of G with coefficients from F.

We shall also additionally use the following standard notions: $G_t = \coprod_p G_p$ is the torsion part of G with p-primary components G_p over all primes p, $U(FG)$ is the unit group of FG with subgroup of normalized invertible elements $V(FG)$ and with multiplicative group F^* of F. For any suitable set X, we recall that $F(X)$ is the algebraic extension of F by including the elements of X.

To make the current paper more nearly self-contained and friendly to the reader, we shall give a brief introducing to some basic facts on commutative

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group rings. They are necessary for our successful presentation and will be used below with the corresponding reference.

Theorem 1 ([D3]). *Let G be an abelian group with $G_p = 1$. Then*
$U(F^-G) \cong (\coprod_{|G_t|} G/G_t) \times (\coprod_{|G_t|}(F^-)^*)$.

Notice that this statement extends the corresponding one in [CP1], proved when G is torsion of infinite cardinality.

Lemma 2 ([D3]). *Let $A = A_p \times M$ be an abelian group. Then*
$V(FA) = V(FA)_p \times V(FM)$.

The following dependence is our crucial tool.

Proposition 3 ([D5]). *Let G be an abelian group. Then*
$V(FG)_t = V(FG)_p V(FG_t)_t$ *with* $V(FG)_p \cap V(FG_t)_t = V(FG_t)_p$.

To facilitate the examination, we also provide in the sequel a short history of the best principal known results as well as a resume of the goal that we pursue and which motivates this work.

Well, the more important global study of $U(FG)$ starts by Chatzidakis-Pappas in their significant, remarkable research paper ([CP1]), where they have generalized almost all previous partial results in that way. More precisely, they established two types of attainments, namely in [CP1] is obtained the complete isomorphic description of $U(FG)$ in the cases when $G_p = 1$ and at most one of the following conditions is true:

(1) G is infinite torsion and $F = F^-$;

or

(2) $G = G_t = \coprod_{i \in I} C_i$ so that $|C_i| \leq \aleph_0$;

as more generally

(3) $G \cong G_t \times G/G_t$ such that $|G_t| \geq \aleph_0$ and either $F = F^-$ or $G_t = \coprod_{i \in I} C_i$ with all C_i countable groups.

The purpose here is to continue this investigation under the presuming that $G_p \neq 1$, i.e. provided FG is arbitrary not necessarily semi-simple. Nevertheless, we will restrict our attention only on the torsion units in FG, that is $U(FG)_t$.

2. The main statement and its proof. We materialize in this section all of our aims presented. We foremost begin with a more convenient form of the foregoing Proposition 3.

Main Claim. *Suppose G is an abelian group. Then*

(a) $V(FG)_t = V(FG)_p \times \coprod_{q \neq p} V(FG_t)_q = V(FG)_p \times \coprod_{q \neq p} V(F(\coprod_{q \neq p} G_q))_q$.

Main Proposition. *Suppose G is a coproduct of countable groups or G_t is of countable length a coproduct of countables, and F is perfect. Then*

(b) $V(FG)_t \cong V(FG)_p \times \coprod_{q \neq p} \coprod_{I'_q, J'_q}((F(I_q, J_q))^*)_q \cong G_p \times V(FG)_p/G_p \times \coprod_{q \neq p} \coprod_{I'_q, J'_q}((F(I_q, J_q))^*)_q$,

where $V(FG)_p/G_p$ (whence $V(FG)_p$) being a coproduct of countables is classified by its maximal divisible subgroup described in ([D7], [D9]) jointly with the Ulm-Kaplansky functions calculated in ([D6]) and where the sets I_q and I'_q, respective J_q and J'_q, have the properties described in all details in ([D3]) which depend on F, G and $\mathbb{N}$ only.

In [D8] we have found a sufficient condition for $U(FG)$ to be a coproduct of countable groups. Moreover, a criterion for $V(FG)$ to be a coproduct of countables was founded in [D1] when $G_t = G_p$. The following statement sheds a new perspective to resolve the question in all generality.

Main Corollary(Criterion). *Suppose G is an abelian group whose G_p is of countable length and F is algebraically closed. Then $V(FG)_t$ is a coproduct of countable groups $\iff$ G_p is a coproduct of countable groups.*

We emphasize that, inspired by the *Generalized Direct Factor Conjecture*, the assertion will perhaps remain true without the extra restriction on $length(G_p) < \Omega$. What suffices to be shown is that G_p being an arbitrary coproduct of countables implies the same property for $V(FG)_p/G_p$ and hence for $V(FG)_p$. However, this is in question yet.

We are now ready to proceed by proving the following.

Main Theorem(Structure). *Suppose G is an abelian group. Then*

(c) $V(F^-G)_t \cong V(F^-G)_p \times \coprod_{q\neq p} \coprod_{\varepsilon}((F^-)^*)_q$,

where $\varepsilon = |G_t/G_p| \geq \aleph_0$ or $\varepsilon = |G_t/G_p| - 1$ otherwise when $|G_t/G_p| < \aleph_0$.

In particular, if we assume that the Generalized Direct Factor Conjecture holds,

(d) $V(F^-G)_t \cong G_p \times \coprod_{max(|F^-|,|G^{(p)}|)} \mathbf{Z}(p^\infty) \times (V(F^-G)_p/G_p)_r$,

where $(V(F^-G)_p/G_p)_r$ being totally projective has isomorphism characterization by the Ulm-Kaplansky invariants computed as in [D6]. For instance, this is possible whenever G is a coproduct of countables.

We are now able to give

Proof of the Main Claim. Certainly, $V(FG_t)_t = V(FG_t)_p \times \coprod_{q\neq p} V(FG_t)_q$. Therefore, the first equality is fulfilled by a direct application of Proposition 3.

What we wish to show further is that $\coprod_{q\neq p} V(FG_t)_q = \coprod_{q\neq p} V(F(\coprod_{q\neq p} G_q))_q$. In order to doing this, we foremost observe via Lemma 2 that $V(FG_t) = V(F(G_p \times \coprod_{q\neq p} G_q)) = V(F(\coprod_{q\neq p} G_q)) \times V(FG_t)_p$. Consequently, since $V(FG_t)_p$ is a p-torsion group, by taking in both sides the q-operator, we deduce $V(FG_t)_q = V(F(\coprod_{q\neq p} G_q))_q$. That is why, we need to apply this identity to the coproduct $\coprod_{q\neq p}$.

So, our formulae really sustained, and we are done. The proof is finished.

Commentary: This decomposition is essential since it reduces the general case to the semi-simple one.

We continue with

Proof of the Main Theorem. Exploiting the latter equality from the Main Claim, we may write $V(F^-G)_t = V(F^-G)_p \times \coprod_{q\neq p} V(F(\coprod_{q\neq p} G_q))_q$. Because of this that $F(\coprod_{q\neq p} G_q)$ is already semi-simple, Theorem 1 now applies to show that the desired isomorphism relationship follows.

After this, we treat the second part-half. To this end, we have under the validity of the Generalized Direct Factor Conjecture that $V(F^-G)_p/G_p$ is simply presented. Hence, $V(F^-G)_p \cong G_p \times V(F^-G)/G_p$ because G_p is balanced in $V(F^-G)_p$ (see, for example, [D7]). What it suffices to do is to decompose $V(F^-G)/G_p$ into the divisible part $(V(F^-G)/G_p)_d$ and a reduced part $(V(F^-G)/G_p)_r$, respectively. Next, subsequently employing ([D7] and [D9]) along with ([D6]), we conclude that (d) is indeed satisfied, thus ending the proof.

Proof of the Main Proposition. Invoking the second identity of point (a), we write down $V(FG)_t = V(FG)_p \times \coprod_{q\neq p} V(F(\coprod_{q\neq p} G_q))_q$. Since $F(\coprod_{q\neq p} G_q)$ is semi-simple and since G_t is a coproduct of countables, whence both G_p and $\coprod_{q\neq p} G_q$ are so, we apply ([D1], [D2]) and ([HU]) together with ([CP1]) (see [D3] too) to infer the wanted claim. The proof is over.

Proof of the Main Corollary. First of all, for each prime number q we see that $(F^*)_q$ is always divisible, hence a coproduct of countable groups. Furthermore, the statement follows immediately by appealing to point (c) combined with [D2] and a slight modification of a result from [HU] which says that $V(FG)_p$ as being a coproduct of countable groups implies that so does G_p; we also mention that the claim follows by a classical long-known theorem of P. Hill because of this that G_p is isotype in $V(FG)_p$.

This substantiates our claim, in fact. The proof is completed.

3. Concluding discussion. In closing, we realize certain detailed observations. In fact, we first bound our attention on the paramount assertion due to May ([M1]) that EG recovered G/G_t, for an arbitrary field E. The original proof is rather intrincated. Later on, W. May gives in ([M2] and [M3]) new more clear proofs. Incidentally, he established in [M3] that EG_t is the maximal algebraic subalgebra of the semi-simple algebra EG. We have assured in [D4] an independent smooth confirmation to the major result only in group terms which is based on the multiplicative groups of field extensions. In [BM] was again argued that the semi-simple group algebra EG determines EG_t. This, however, was specified by Chatzidakis-Pappas [CP2] by showing that for another group H the E-isomorphism $EG \cong EH$ insures that $EG_t \cong EH_t$ under the restriction of the whole isomorphism. We note that this follows since EG_t is the semi-simple E-algebra generated by the torsion elements of $U(EG_t)$, as claimed in [CP2], i.e. by $U(EG_t)_t = U(EG)_t$ (see [D4], too). Henceforth, $E(G/G_t) \cong E(H/H_t)$ and thereby the classical Higman's result on the trivial units (see [M1] as well) leads us to $G/G_t \cong H/H_t$, as required.

Next, we concentrate on the units in modular group rings, and more especially in FG when $G_p \neq 1$. Here the situation is quite complicated. Consulting with [M2], $F^-(G_t \times G/G_t) \cong F^-(G_p \times G_t/G_p \times G/G_t) \cong F^-(G_p \times (G/G_p)_t \times G/G_p/(G/G_p)_t) \cong F^-G_p \otimes_{F^-} F^-((G/G_p)_t \times G/G_p/(G/G_p)_t) \cong F^-G_p \otimes_{F^-} F^-(G/G_p) \cong F^-(G_p \times G/G_p) \ncong F^-G$ when $G \ncong G_p \times G/G_p$ as can be expected when the *Generalized Direct Factor Problem* does hold. By the same reason $U(F^-G) \ncong U(F^-G)_p \times U(F^-G)/U(F^-G)_p \ncong U(F^-G)_p \times U(F^-(G/G_p))$.

We conjecture that $U(F^-G) = U(F^-M) \times U(F^-G)/U(F^-M)$, where $G \geq M \cong G/G_p$ so that $M = G$ when $G_p = 1$ and $M_q = G_q$ $\forall q \neq p$ in the remaining case. This is suggested via the following arguments of utilizing of the Main Claim, namely: $[U(F^-G)/U(F^-M)]_t \cong U(F^-G)_t/U(F^-M)_t = U(F^-G)_p \times \coprod_{q\neq p} U(F^-(\coprod_{q\neq p} G_q))_q / \coprod_{q\neq p} U(F^-(\coprod_{q\neq p} M_q))_q \cong U(F^-G)_p \neq 1$ when $G_p \neq 1$. Thus the maximal torsion subgroup of the latter complementary factor is completely described, although there is no a real advantage in this for the obtaining its algebraic structure. However, this matter is a problem of some other appropriate research exploration.

Remarks: We indicate the misprint that the formula "$V(FG)_t = V(FG_t)$" everywhere in [D4] must be written and read as "$V(FG)_t = V(FG_t)_t$", since the subscript "t" was dropped involuntarily. Of course, $V(FG)_t = V(FG_t)$ when F^* is torsion $\iff$ F is an algebraic extension of a finite field, because in this situation $V(FG_t)$ is ever torsion (for example see [D8]).

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