An. Şt. Univ. Ovidius Constanţa Vol. 12(2), 2004, 109–114

A RIEMANN RESTRICTED CHARACTERIZATION OF THE QUASILINEARITY HIERARCHY: SOME REMARKS

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To Professor Dan Pascali, at his 70's anniversary

Abstract

A Riemann restricted version of the details connected with Lax's genuine nonlinearity / linear degeneracy ([3]) is considered. An example regarding the interface between weak quasilinearity and linearity is included. A case is presented for which the weak quasilinearity *degenerates* into a strong quasilinearity.

Introduction. A quasilinearity hierarchy

We consider the one-dimensional strictly hyperbolic system

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0, \tag{1}$$

and let $\overset{\mathrm{i}}{R}(u)$, $\overset{\mathrm{i}}{L}(u)$, and $\lambda_i(u)$, of indices $i = 1, ..., n,$ be, respectively, the right eigenvectors, the left eigenvectors, and the eigenvalues of the matrix $a(u)$.

TERMINOLOGY 1 ([3]). For a strictly hyperbolic system (1) we say that an index i is *genuinely nonlinear* in $\mathcal{R} \subset H$ (H is the hodograph space) if for it

$$\overset{\mathrm{i}}{R}(u) \cdot \mathrm{grad}_u \lambda_i(u) \neq 0, \quad u \in \mathcal{R} \tag{2}$$

Key Words: Riemann characterization; quasilinearity hierarchy; hyperbolic systems.

and it is *linearly degenerated* in $\mathcal{R} \subset H$ if for it

$$\overset{\mathrm{i}}{R}(u) \cdot \mathrm{grad}_u \lambda_i(u) \equiv 0, \quad u \in \mathcal{R} \tag{3}$$

Terminology 2. The quasilinearity associated to (1) is said to be *strong* if all the indices associated to (1) are genuinely nonlinear, *medium* if only a part of the indices associated to (1) are geniunely nonlinear [the other being linearly degenerate], and *weak* if all the indices associated to (1) are linearly degenerate.

Riemann Invariants

Let us consider the $(n-1)$-dimensional hypersurface

$$v_k(u) = \text{constant} = v_k(u^*) \tag{4}$$

through $u^* \in \mathcal{R}$ at the points of which the normal direction is given by the left eigenvector $\overset{\mathrm{k}}{L}(u)$:

$$\frac{\partial v_k}{\partial u_l} = \alpha_k(u)\overset{\mathrm{k}}{L}_l(u) \qquad 1 \leq l \leq n. \tag{5}$$

It results from (5) that the reality of a characteristic hypersurface (4) depends on the integrable character of the Pfaff form

$$\sum_{l=1}^{n} \overset{\mathrm{k}}{L}_l(u)\mathrm{d}u_l. \tag{6}$$

In terms of (6), the geometrical restrictions mentioned here above appear to be integrability restrictions. A hypersurface (4), (5) is called a *Riemann invariant* of (1).

We use (5) to compute for each k:

$$\begin{aligned}\alpha_k(u)\overset{\mathrm{k}}{L}(u)\left[\frac{\partial u}{\partial t} + a(u)\,\frac{\partial u}{\partial x}\right] &= \sum_{j=1}^{n} \frac{\partial v_k}{\partial u_j} \cdot \frac{\partial u_j}{\partial t} + \lambda_k(u)\alpha_k(u)\overset{\mathrm{k}}{L}(u)\frac{\partial u}{\partial x} \\ &= \frac{\partial v_k}{\partial t} + \overline{\lambda}_k(v)\frac{\partial v_k}{\partial x}\ , \qquad 1 \leq k \leq n\end{aligned}$$

and notice that the Riemann invariants $v_1, \ldots, v_n$ satisfy a *diagonal* system

$$\frac{\partial v_k}{\partial t} + \overline{\lambda}_k(v)\frac{\partial v_k}{\partial x} = 0, \quad 1 \leq k \leq n; \quad \overline{\lambda}_k(v) \equiv \lambda_k[u(v)] \tag{7}$$

associated to (1). In fact, a Riemann invariant v_k is constant, cf. (7), on each characteristic line of index k. In (7) we used, in order to define $\overline{\lambda}_k$, the non-singular transformation character of the connection between v and u around each point of $\mathcal{R}$.

Riemann restricted genuine nonlinearity / linear degeneracy

We also compute from (7)

$$\frac{\partial \lambda_i}{\partial u_j} = \sum_{k=1}^{n} \frac{\partial \overline{\lambda}_i}{\partial v_k} \cdot \frac{\partial v_k}{\partial u_j} = \sum_{k=1}^{n} \alpha_k(u) \overset{\mathrm{k}}{L}_j(u) \frac{\partial \overline{\lambda}_i}{\partial v_k}$$

$$\overset{\mathrm{i}}{R}(u) \cdot \mathrm{grad}_u \lambda_i(u) = \sum_{j=1}^{n} \overset{\mathrm{i}}{R}_j(u) \sum_{k=1}^{n} \alpha_k(u) \overset{\mathrm{k}}{L}_j(u) \frac{\partial \overline{\lambda}_i}{\partial v_k} = \overline{\alpha}_i(v) \frac{\partial \overline{\lambda}_i}{\partial v_i}$$

and respectively transcribe the restrictions of genuinely nonlinearity / linear degeneracy of an index i [see (2)/(3)] by

$$\frac{\partial \overline{\lambda}_i}{\partial v_i} \neq 0, \qquad v \text{ in } \overline{\mathcal{R}} \tag{8}$$

or

$$\frac{\partial \overline{\lambda}_i}{\partial v_i} \equiv 0, \qquad v \text{ in } \overline{\mathcal{R}}. \tag{9}$$

Some remarks

At this point we shall use (8), (9) in order to characterize the quasilinearity hierarchy (Terminology 2). A complete system of Riemann invariants always exists as $n = 2$ and we notice that a representative and most suggestive characterization of the mentioned hierarchy can be done for $n = 2$. For $n = 2$, a *strong* quasilinearity means

$$\frac{\partial \overline{\lambda}_1}{\partial v_1} \neq 0 \ , \ \frac{\partial \overline{\lambda}_2}{\partial v_2} \neq 0 \quad \text{in } \overline{\mathcal{R}}$$

a *medium* quasilinearity requires

$$\frac{\partial \overline{\lambda}_1}{\partial v_1} \neq 0, \ \frac{\partial \overline{\lambda}_2}{\partial v_2} \equiv 0 \quad \text{or} \quad \frac{\partial \overline{\lambda}_1}{\partial v_1} \equiv 0, \ \frac{\partial \overline{\lambda}_2}{\partial v_2} \neq 0 \quad \text{in } \overline{\mathcal{R}}$$

and a *weak* quasilinearity has the signification

$$\frac{\partial\overline{\lambda}_1}{\partial v_1} \equiv 0,\ \frac{\partial\overline{\lambda}_2}{\partial v_2} \equiv 0 \quad \text{in } \overline{\mathcal{R}}. \tag{10}$$

A nontrivial form of (10) is complementarily characterized by

$$\frac{\partial\overline{\lambda}_1}{\partial v_2} \neq 0,\ \frac{\partial\overline{\lambda}_2}{\partial v_1} \neq 0 \quad \text{in } \overline{\mathcal{R}}. \tag{11}$$

As (10) and (11) hold we set

$$r = \overline{\lambda}_2(v_1), \quad s = \overline{\lambda}_1(v_2)$$

in order to transform the corresponding system (7) into

$$\frac{\partial r}{\partial t} + s\frac{\partial r}{\partial x} = 0, \qquad \frac{\partial s}{\partial t} + r\frac{\partial s}{\partial x} = 0. \tag{12}$$

Then, we calculate from (12)

$$\frac{\partial r}{\partial t}\frac{\partial s}{\partial x} - \frac{\partial r}{\partial x}\frac{\partial s}{\partial t} = (r-s)\frac{\partial r}{\partial x}\frac{\partial s}{\partial x} \tag{13}$$

and

$$\frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = (r-s)\frac{\partial r}{\partial x}, \qquad \frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = (s-r)\frac{\partial s}{\partial x}. \tag{14}$$

If we weaken the restriction (11), allowing for example that

$$\frac{\partial\overline{\lambda}_1}{\partial v_2} \equiv 0 \qquad \left[\text{yet } \frac{\partial\overline{\lambda}_2}{\partial v_2} \neq 0\right] \quad \text{in } \overline{\mathcal{R}},$$

then (13) takes the form

$$\frac{\partial r}{\partial t} + h\frac{\partial r}{\partial x} = 0, \qquad \frac{\partial s}{\partial t} + r\frac{\partial s}{\partial x} = 0, \qquad \text{constant } h$$

which reduces to a *linear* equation:

$$\frac{\partial s}{\partial t} + r_0(x-ht)\frac{\partial s}{\partial x} = 0.$$

It appears that the restrictions (10) and (11) characterize the *lowest* level of nonlinearity in the weak quasilinearity connected with $n = 2$.

We finally notice that a solution of (12) for which $r \equiv s$ corresponds to a *degeneration* of the weakly quasilinear system (12). In this degeneration the two equations (12) become coincident in the genuinely nonlinear equation

$$\frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0.$$

The mentioned degeneration implies a replacement of a $n = 2$ linear degeneracy by a $n = 1$ genuine nonlinearity.

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