An. Şt. Univ. Ovidius Constanţa Vol. 12(2), 2004, 127–134

# ON ROTHE'S FIXED POINT THEOREM IN A GENERAL TOPOLOGICAL VECTOR SPACE

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## 1. Introduction

The generalization of Rothe's Fixed Point Theorem to general topological vector spaces, presented in this paper is related to the recent solution of the well-known conjecture defined in 1930 by J. Schauder.

The following conjecture was well known in the Fixed Point Theory.

**Conjecture [Schauder]** *For every non-empty convex subset $C$ of a topological vector space $E$, a compact continuous mapping $f : C \to C$ has a fixed point, i.e., a point $x^* \in C$ such that $f(x^*) = x^*$.* (See [16], problem 54).

We recall that a mapping $f : C \to C$ is said to be compact if $f(C)$ is contained in a compact subset of $C$.

Schauder proved in 1930 that his conjecture holds for normed vector spaces and Hukuhara proved that Schauder's conjecture is true for locally convex spaces.

In 2001, Schauder's conjecture was resolved affirmatively by R. Cauty [2].

**THEOREM 1 [CAUTY]**

*Let $E(\tau)$ be a Hausdorff topological vector space, $C$ a convex subset of $E$ and $f : C \to$ a continuous mapping.*

*If $f(C)$ is contained in a compact subset of $C$, then $f$ has a fixed point.*

As a consequence of *Theorem 1* we will present in this paper an extension of Rothe's Fixed Point Theorem to general Hausdorff topological vector spaces. Rothe's Fixed Point Theorem is a classical result. [15], [5].

From Rothe's Theorem we will deduce an Implicit Leray-Schauder Type Alternative for general topological vector spaces, which contains as a particular

Key Words: fixed point theorem; Hausdorff vector space.

case the Leray-Schauder Alternative [7]. It is well known this Alternative has many applications [6], [9], [10] and it is a fundamental result in Nonlinear Analysis.

We used recently the Implicit Leray-Schauder Alternative in the study of Complementarity Problems and also in the study of Variational Inequalities [6].

Initially, the Leray-Schauder Theorem was probed by using the topological degree [7] but now, several kinds of proofs without topological degree are known [1], [3], [8], [13].

The Implicit Leray-Schauder Type Alternative presented in this paper is a generalization to arbitrary Hausdorff topological vector spaces of a similar result proved by A. J. B. Potter in 1972 [11]. We note that the proof given by Potter has some obscure parts. We hope that the reader will find clearer our proof. In the proofs presented in this paper some details are inspired by [11].

## 2. Preliminaries

We denote by $E(\tau)$ a Hausdorff topological vector space and by $(X,\tau)$ a general Hausdorff topological vector space.

We recall that a topological space $(X,\tau)$ is *countable compact* if and only if any countable open cover of $X$, has a finite subcover, [4]. Any compact topological space is countable compact.

It is known that a topological space $(X,\tau)$ is countable compact if and only if every countable infinite subset of $X$ has at least one accumulation point. (See [4], Proposition 13, pg. 179).

From this result we deduce that if $\{y_n\}_{n\in N}$ is a sequence in a relative compact set $M$, then $\{\mathrm{y}_n\}$ has an accumulation point in $\overline{M}$.

If $B$ is a subset of a topological space $(X,\tau)$, we denote by $\partial B$ its boundary and by $int(B)$ the interior of $B$. Let $E(\tau)$ be a topological vector space and let $A, B$ be subsets of $E$.

We say that $A$ *absorb* $B$ if there exists $\lambda* \in P$ (the real field) such that $B \subset \lambda A$, whenever $|\lambda| \geq \lambda^*$. A subset $U$ of $E$ is called *radial* (absorbing) if $U$ absorbs every finite subset of $E$. We say that $U$ is *circled* if $\lambda U \subseteq U$ whenever $|\lambda| \leq 1$. For other notions and results the reader is referred to [14].

## 3. A generalization of Rothe's theorem

The following result is an extension to a general topological Hausdorff space of the classical Rothe's theorem.

**THEOREM 2 [A Rothe's type theorem]**

*Let $E(\tau)$ be a Hausdorff topological vector space. Let $B \subset E$ be a closed convex subset such that the zero of $E$ is contained in the interior of $B$.*

*Let $\Phi : B \to E$ be a continuous mapping with $\Phi(B)$ relatively compact in $E$ and $\Phi(\partial B) \subset B$.*

*Then there is a point $x^* \in B$ such that $\Phi(x^*) = x^*$.*

**PROOF.** We denote that $int(B)$ is non-empty since $0 \in int(B)$. We recall that because $E(\tau)$ is a topological vector space, then the topology $\tau$ possess a 0-neighborhood base $Y$ such that any $V \in Y$ is *radial* and *circled*.

Then because $int(B) \subset B$ we have that $B$ is a radial (absorbing) set.

Let $p_B$ be the Minkowski functional of $B$, i.e., $p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}$ for any $x \in E$. The functional $p_B$ is *positive homogeneous*. Indeed, first $p_B(0) = 0$. Let $x \in E$ be arbitrary and $\lambda > 0$. We have

$$p_B(x) = \inf\{\mu > 0 : \lambda x \in \mu B\} = \inf\{\mu > 0 : x \in \lambda^{-1}\mu B\} =$$

$$= \inf\{\lambda\mu_1 : x \in \mu_1 B\} = \lambda p_B(x).$$

Now, we show that $p_B$ is a continuous mapping. The continuity of $p_B$ is a consequence of the following facts.

Let $\varepsilon > 0$ be an arbitrary real number. Form ([14], $Theorem 1.2$) there exits a radial and circled 0-neighborhood $U$ such that $0 \in U \subset int(B) \subset B$.

Let $p_U$ be the Minkowski functional of $U$. We have $p_B \leq p_U$. Because $B$ is also convex, $p_B$ is subadditive and we can show that for any $x, y \in E$ we have

$$p_B(B) - p_B(y) \leq p_B(x - y) \leq p_U(x - y)$$

and

$$p_B(y) - p_B(x) \leq p_B(y - x) \leq p_U(y - x).$$

If $x, y$ are such that $x - y \in \varepsilon U$, then we have $p_U(x - y) = p_U(y - x)$ (because $U$ is circled), which implies

$$|p_B(x) - p_B(y)| \leq \varepsilon.$$

The last relation implies that $p_B$ is continuous. We consider the mapping $\Psi : E \to E$ defined by

$$\Psi(x) = [\max\{1, p_B(x)\}]^{-1} \cdot x, \textit{ for any } x \in E.$$

The mapping $\Psi$ is continuous and $\Psi(E) \subseteq B$.

We define the mapping $f : B \to B$ by $f = \Psi \circ \Phi$. The mapping $f$ is continuous and f($B$) is relatively compact in $E$. By *Theorem 1* [Cauty] there exists an element $x^* \in B$ such that $f(x^*) = x^*$.

We have two situations:

(i) $x^* \in int(B)$ and

(ii) $x \in \partial B$.

If *(i)* holds, then we have

$$1 > p_B(x_*) = p_B(f(x_*)) = [\max\{1, p_B(\Phi(x_*))\}]^{-1} \cdot p_B\left(\Phi(x_*)\right),$$

which implies that we must have $(\Phi(x_*)) < 1$ and consequently

$$f(x_*) = \Psi(\Phi(x_*)) = \Phi(x_*).$$

Therefore $\Phi(x_*) = x_*$. Now, we suppose that *(ii)* holds. Then we have

$$x_* = f(x_*)\,[\max\{1, p_B(\Phi(x_*))\}]^{-1} \cdot \Phi(x_*),$$

and

$$1 = [\max\{1, p_B(\Phi(x_*))\}]^{-1} \cdot p_B(\Phi(x_*)).$$

If $p_B(\Phi(x_*)) < 1$ then $1 = p_B(\Phi(x_*)) < 1$ which is a contradiction.

Thus we must have $p_B(\Phi(x_*)) = 1$ (since $\Phi(\partial B) \subset B$). But $p_B(\Phi(x_*)) = 1$ implies $f(x_*) = \Phi(x_*)$ and hence we have again that $\Phi(x_*) = x_*$. Therefore there exists $x_* \in B$ such that $\Phi(x_*) = x_*$ and the proof is complete. ■

## 4.Implicit Leray-Schauder Type Alternative

The following result is a consequence of *Theorem 2*.

**THEOREM 3 [Implicit Leray-Schauder Theorem]**

*Let $E(\tau)$ be a Hausdorff topological vector space and $B \subset E$ a closed convex set such that $0 \in int(B)$.*

*Let $f : [0,1] \times B \to E$ be a continuous mapping. The set $[0,1] \times B$ is endowed with the product topology and $f([0,1] \times B)$ is relatively compact in $E$.*

*If the following assumptions are satisfied:*

(1) $f(\lambda, x) \neq x$ *for all $x \in \partial B$ and all $\lambda \in [0,1]$,*

(2) $f(\{0\} \times \partial B) \subset B$

*then, there exists an element* $x^*$ *in* $B$ *such that* $f(1, x^*) = x^*$.

**PROOF.** For any $n \in N$ we consider the mapping $f_n : B \to E$ defined by

$$F_n(x) = \begin{cases} f\left(\dfrac{1-p_B(x)}{\varepsilon_n}, \dfrac{x}{p_B(x)}\right), & if \ \ 1-\varepsilon_n \le p_B(x) \le 1, \\ f\left(1, \dfrac{x}{1-\varepsilon_n}\right), & if \ \ p_B(x) < 1-\varepsilon_n, \end{cases}$$

where $p_B$ is the Minkowski functional of the set $B$ and $\{\varepsilon_n\}_{n\in N}$ is a sequence of real numbers such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ and $0 < \varepsilon_n < \frac{1}{2}$ for any $n \in N$. For each $n \in N$, $f_n$ is continuous on $B$ and $f_n(B)$ is relatively compact in $E$.

From assumption (2) we have that $f_n(\partial B) \subset B$. The assumptions of *Theorem 2* are satisfied for any $n \in N$ and hence, for each $n \in N$ there exits an element $u_n \in B$ such that $f_n(u_n) = u_n$. Suppose that an infinite number of elements $u_n$ satisfy the relation

$$1 \ge p_B(u_n) \ge 1 - \varepsilon_n. \qquad (\alpha)$$

Because $f(B)$ is relatively compact and considering the definition of mappings $f_n$, we have that $\{u_n\}_{n\in N}$ is contained in a compact set in $E$. Hence, (see preliminaries of this paper) the sequence $\{u_n\}_{n\in N}$ or any subsequence of $\{u_n\}_{n\in N}$ has an accumulation point.

We consider the sequence $\{\lambda_n\}_{n\in N}$ defined by

$$\lambda_n \frac{1-p_B(u_n)}{\varepsilon_n}, \text{ for any } n \in N.$$

We have that $\{\lambda_n\}_{n\in N} \subset [0,1]$.

Considering eventually a subsequence we suppose that $\lim\limits_{n\to\infty} \lambda_n = \lambda_* \in [0,1]$.

The corresponding subsequence of $\{u_n\}_{n\in N}$ is denoted again by $\{u_n\}_{n\in N}$ and it satisfies the inequality $(\alpha)$.

From $(\alpha)$ we have that $\lim\limits_{n\to\infty} p_B(u_n) = 1$. Let $u^*$ be an accumulation point of $\{u_n\}_{n\in N}$. We know that $\{u_n\}_{n\in N}$ has a net converging to $u^*$. Using this fact we can show that $(\lambda_*, u_*, u_*)$ is an accumulation point of the sequence $\left\{\left(\lambda_n \dfrac{u_n}{p_B(u_n)}, u_n\right)\right\}_{n\in N}$ in $[0,1]\times E \times E$.

Considering the net convergent to $u^*$, the continuity of f, and the equation $f\left(\lambda_n, \dfrac{u_n}{p_B(u_n)}\right) = u_n$ for any $n \in N$, we obtain that $f(\lambda_*, u_*) = u_*$. This fact is a contradiction of assumption (1). Indeed, $p_B(u_*) = 1$ (since $\lim_{i \in I} p_B(y_i) = 1$, where $\{y_i\}_{i \in I}$ is the net of $\{u_n\}_{n \in N}$ convergent to $u^*$), and $u^* \in \partial B$. Then $(\alpha)$ can be satisfied only for a finite number of elements of the sequence $\{u_n\}_{n \in N}$. Hence, we can suppose that

$$p_B(u_n) < 1 - \varepsilon_n, \text{ for all } n \in N.$$

Since $\lim_{n \to \infty} (1 - \varepsilon_n) = 1$, selecting an accumulation point $u^*$ for $\{u_n\}_{n \in N}$, and using a net of $\{u_n\}_{n \in N}$ convergent to $u^*$, we obtain by continuity and considering the equation

$$f\left(1, \frac{u_n}{1 - \varepsilon_n}\right) = u_n, \text{ for all } n \in N,$$

that $f(1, u_*) = u_*$. By this conclusion the proof is complete. ■

From *Theorem 3* we deduce immediately the following alternative.

**THEOREM 4 [Implicit Leray-Schauder Alternativ].** *Let $E(\tau)$ be a Hausdorff topological vector space, $B \subset E$ a closed convex set such that $0 \in int(B)$.*

*Let $f : [0,1] \times B \to E$ be a continuous mapping such that $f([0,1] \times B)$is relatively compact in $E$. We consider on $[0,1] \times B$ the product topology. If the following assumptions are satisfied:*

(1) $f(\{0\} \times \partial B) \subset B$,

(2) $f(0, x) \neq x$ for any $x \in \partial B$,
*then at least one of the following properties is satisfied:*

(i) *there exists $x_* \in B$ such that $\phi(1, x_*) = x_*$,*

(ii) *here exists $(\lambda_*, x_*) \in [0,1] \times \partial B$ such that $f(\lambda_*, x_*) = x_*$.* ■

## 5. Comments

We presented in this paper extensions to general topological Hausdorff vector spaces of two fundamental theorems of nonlinear analysis known in Banach spaces: Rothe's Fixed Point Theorem and a Leray-Schauder Type Alternative.

Recently, we applied *Theorem 4* (in this form but on Hilbert spaces) to the study of complementarity problems and to the study of variational inequalities [6].

Certainly, these theorems in this general form can have interesting applications. *Theorem 3* contains as a particular case the classical *Leray-Schauder Theorem.*

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