*

An. Şt. Univ. Ovidius Constanţa Vol. 12(2), 2004, 135–146

A KACZMAREZ-KOVARIK ALGORITHM FOR SYMMETRIC ILL-CONDITIONED MATRICES*

Constantin Popa

To Professor Dan Pascali, at his 70's anniversary

Abstract

In this paper we describe an iterative algorithm for numerical solution of ill-conditioned inconsistent symmetric linear least-squares problems arising from collocation discretization of first kind integral equations. It is constructed by successive application of Kaczmarz Extended method and an appropriate version of Kovarik's approximate orthogonalization algorithm. In this way we obtain a preconditioned version of Kaczmarz algorithm for which we prove convergence and make an analysis concerning the computational effort per iteration. Numerical experiments are also presented.

AMS Subject Classification : 65F10 , 65F20.

1 Kaczmarz extended and Kovarik algorithms

Beside many papers and books concerned with the qualitative analysis of classes of linear and nonlinear operators and operatorial equations, professor Dan Pascali also analysed the possibility to approximate solutions for some of them (see e.g. [5], [6]). This paper is written in the same direction, by considering iterative methods for numerical solution of first kind integral equations

Key Words: inconsistent symmetric least-squares problems, Kaczmarz's iteration, approximate orthogonalization, preconditioning.

*The paper was supported by the PNCDI INFOSOC Grant 131/2004

of the form (see also the last section of the paper)

$$\int_0^1 k(s,t)x(t)dt \ = \ y(s), \ s \in [0,1].$$

In this respect, the rest of this introductory section will be concerned with the description of the original versions of these methods. Let A be an $n \times n$ real symmetric matrix. We shall denote by $(A)_i, \ r(A), \ R(A), \ N(A), \ b_i$ the i-th row, rank, range, null space of A and i-th component of b, respectively (all the vectors that appear being considered as column vectors). The notations $\rho(A), \sigma(A)$ will be used for the spectral radius and spectrum of A and $\parallel A \parallel = \rho(A)$ will be the spectral norm. P_S will be the orthogonal projection onto the vector subspace S, with respect to the Euclidean scalar product and the associated norm, denoted by $\langle \cdot, \cdot \rangle$ and $\parallel \cdot \parallel$, respectively. We shall consider a vector $b \in I\!R^n$ and the linear least-squares problem : find $x^* \in I\!R^n$ such that

$$\parallel Ax^* - b \parallel = min! \tag{1}$$

It is well known (see e.g. [1]) that the set of all (least-squares) solutions of (1), denoted by $LSS(A;b)$ is a nonempty closed convex subset of $I\!R^n$ containing a unique solution with minimal norm, denoted by x_{LS}. Moreover, if $b_A = P_{R(A)}(b)$ we have

$$x^* \in LSS(A;b) \Leftrightarrow Ax = b_A. \tag{2}$$

If A has nonzero rows, i.e.

$$(A)_i \neq 0, \ i = 1, \dots, n, \tag{3}$$

we define the applications (matrices)

$$f_i(A;b;x) = x - \frac{< x, (A)_i > - b_i}{\parallel (A)_i \parallel^2}(A)_i, \ P_i(A;y) = y - \frac{< y, (A)_i >}{\parallel (A)_i \parallel^2}(A)_i, \tag{4}$$

$$K(A;b;x) = (f_1 \circ \cdots \circ f_n)(A;b;x), \ \Phi(A;y) = (P_1 \circ \cdots \circ P_n)(A;y), \tag{5}$$

for $x, y \in I\!R^n$ and R the real $n \times n$ matrix of which i-th column $(R)^i$ is given by

$$(R)^i = \frac{1}{\parallel (A)_i \parallel^2} P_1 P_2 \dots P_{i-1}((A)_i), \tag{6}$$

with $P_0 = I$ (the unit matrix). According to [11] (for symmetric matrices) we have the following results.

Proposition 1 *(i) We have*

$$K(A;b;x) = Qx + Rb, \ Q + RA = I, \ Rx \in R(A), \ \forall \ x \in I\!R^n. \tag{7}$$

(ii) $N(A)$ and $R(A)$ are invariant subspaces for Φ and

$$\Phi = P_{N(A)} \oplus \tilde{\Phi}, \quad P_{N(A)}\tilde{\Phi} = \tilde{\Phi}P_{N(A)} = 0, \tag{8}$$

where $\tilde{\Phi}$ is the linear application defined by

$$\tilde{\Phi} = \Phi P_{R(A)}. \tag{9}$$

(iii) The application $\tilde{\Phi}$ satisfies

$$\parallel \tilde{\Phi} \parallel = \sqrt{\rho(\tilde{\Phi}^t\tilde{\Phi})} < 1. \tag{10}$$

The following extension of the original Kaczmarz's projections method will be considered (see [2], [7]).
Algortihm KE. Let $x^0 \in I\!R^n, y^0 = b$; for $k = 0, 1, \dots$ do

$$y^{k+1} = \Phi(A; y^k),\ \beta^{k+1} = b - y^{k+1},\ x^{k+1} = K(A; \beta^{k+1}; x^k). \tag{11}$$

Next theorem, proved in [8] explains the convergence behaviour of the algorithm KE.

Theorem 1 *Let G be the $n \times n$ matrix defined by*

$$G = (I - \tilde{\Phi})^{-1}R. \tag{12}$$

Then, for any matrix A satisfying (3) any $b \in I\!R^n$ and $x^0 \in I\!R^n$, the sequence $(x^k)_{k\geq 0}$ generated with the algorithm (11) converges,

$$\lim_{k\to\infty} x^k = P_{N(A)}(x^0) + Gb_A \tag{13}$$

and the following equalities hold

$$LSS(A; b) = \{P_{N(A)}(x^0) + Gb_A,\ x^0 \in I\!R^n\}, \quad x_{LS} = Gb_A. \tag{14}$$

Remark 1 *The first and third steps from (11) consist on succesive orthogonal projections onto the hyperplanes generated by the rows of A (see (4)-(5)). Then, faster will be the convergence of the algorithm (11) if the values of the angles between columns and rows will be closer to $90°$ (see e.g. [11]).*

According to the above Remark 1, we will consider the Inverse-free modified Kovarik algorithm from [3] (denoted in what follows by KOS). For this we shall suppose in addition that A is positive semidefinite and

$$\sigma(A) \subset [0, 1). \tag{15}$$

Let $a_j, j \geq 0$ be the coefficients of the Taylor's expansion

$$\frac{1}{\sqrt{1-x}} = a_0 + a_1 x + \ldots, x \in (-1, 1), \tag{16}$$

i.e.

$$a_0 = 1,\ a_{j+1} = \frac{2j+1}{2j+2} a_j, j \geq 0 \tag{17}$$

and, for a given integer $q \geq 1$ the truncated Taylor's series $S(A_k; q)$ defined by

$$S(A_k; q) \ = \ \sum_{i=0}^{q} a_i (-A_k)^i. \tag{18}$$

Algorithm KOS Let $A_0 = A$; for $k = 0, 1, \ldots,$ do

$$K_k = (I - A_k) S(A_k; n_k), \quad A_{k+1} = (I + K_k) A_k, \tag{19}$$

where $n_k, k \geq 0$ is a sequence of positive integers.
Next theorem (see [4]) analyses the convergence properties of the algorithm KOS.

Theorem 2 *Let A be symmetric and positive semidefinite such that (15) holds. Then the sequence of matrices $(A_k)_{k \geq 0}$ generated by the above algorithm KOS converges to $A_\infty = A^+ A$, where A^+ is the Moore-Penrose pseudoinverse. Moreover, the convergence is linear, i.e.*

$$\| A_k - A_\infty \|_2 \quad \leq \gamma^k \| A - A_\infty \|_2,\ \forall k \geq 0, \tag{20}$$

with

$$\gamma = \max\{1 - \lambda_{min}(A) + \frac{1}{2}\lambda_{min}(A)^2, 1 - \frac{\lambda_{min}(A)}{\sqrt{1 + \lambda_{min}(A)}}\}, \tag{21}$$

where by $\lambda_{min}(A)$ we denoted the minimal nonzero eigenvalue of A.

Remark 2 *The assumption (15) is not restrictive; it can be easy obtained be scalling the matrix coefficients in an appropriate way. Moreover, during the application of KOS an approximate orthogonalization of the rows of A occurs (see for details [10]); in this sense and according to the comments in Remark 1 before, KOS will be used as a preconditioner for KE as will be described in the next section of the paper.*

2 The preconditioned Kaczmarz algorithm

According to the results and comments from the previus section, we propose the following preconditioned Kaczmarz algorithm.
Algorithm PREKAZ. Let $x^0 \in I\!R^n$, $A_0 = A$, $b^0 = b$ and

$$K_0 = (I - A_0)S(A_0; n_0); \tag{22}$$

for $k = 0, 1, 2, \ldots$ do
Step 1. Compute A_{k+1} and b^{k+1} by

$$A_{k+1} = (I + K_k)A_k, \quad b^{k+1} = (I + K_k)b^k, \tag{23}$$

Step 2. Compute y^{k+1} and β^{k+1} by

$$y^{k+1} = \Phi^{k+1}(A_{k+1}; b^{k+1}), \tag{24}$$

$$\beta^{k+1} = b^{k+1} - y^{k+1}. \tag{25}$$

Step 3. Compute the next approximation x^{k+1} by

$$x^{k+1} = K(A_{k+1}; \beta^{k+1}; x^k) \tag{26}$$

and update K_k to K_{k+1} by

$$K_{k+1} = (I - A_{k+1})S(A_{k+1}; n_{k+1}). \tag{27}$$

Remark 3 *The step (24) means succesive application of* $\Phi(A_{k+1}; \cdot)$ $(k+1)$ *- times to the initial vector* b^{k+1}*, i.e.*

$$\Phi^{k+1}(A_{k+1}; b^{k+1}) = (\Phi(A_{k+1}; \cdot) \circ \cdots \circ \Phi(A_{k+1}; \cdot))(b^{k+1}). \tag{28}$$

This aspect will be analysed in section 3.

Remark 4 *From (23) and because the matrices* $I + K_k$ *are symmetric and positive definite* $\forall k \geq 0$*, we obtain easy that*

$$N(A_k) = N(A), \quad LSS(A_k; b^k) = LSS(A; b), \ \forall \ k \geq 0. \tag{29}$$

In what follows we shall prove convergence for the above algorithm PREKAZ. For this, let $\Phi_k, \tilde{\Phi}_k, R_k$ and G_k be the matrices defined as in (5), (9), (6), (12), respectively, but with A_k from (23) instead of A, b^k as in (23) and $b^k_{A_k}$ defined by

$$b^k_{A_k} = P_{R(A_k)}(b^k). \tag{30}$$

For proving our convergence result we need an auxiliary one which will be presented below.

Proposition 2 *(i) If $\tilde{\Phi}_\infty$ and R_∞ are the matrices defined as in (9) and (6), respectively but with A_∞ from theorem 2 instead of A, then*

$$\lim_{k\to\infty} \tilde{\Phi}_k = \tilde{\Phi}_\infty, \quad \lim_{k\to\infty} R_k = R_\infty. \tag{31}$$

(ii) The sequence $(b^k_{A_k})_{k\geq 0}$ from (30) is bounded.

Proof. (i) It results as in the proof of Theorem 1 from [10].
(ii) If our conclusion would be false, it would exist a subsequence of $(b^k_{A_k})_{k\geq 0}$ (which, for simplicity we shall denote in the same way) such that

$$\lim_{k\to\infty} \parallel b^k_{A_k} \parallel = +\infty. \tag{32}$$

But, from (2) and (30) we have the equivalence $x \in LSS(A_k; b^k) \Leftrightarrow A_k x = b^k_{A_k}$. Then, for any $x^* \in LSS(A; b)$ we obtain (also using (29))

$$A_k x^* = b^k_{A_k}, \quad \forall \;\; k \geq 0. \tag{33}$$

But, from Theorem 2 we have that $\lim_{k\to\infty} A_k = A_\infty$, which tells us that it exists an integer $k_0 \geq 1$ such that

$$\parallel A_k x^* \parallel \;\; \leq \;\; \parallel A_\infty x^* \parallel + 1, \; \forall \; k \geq k_0. \tag{34}$$

Now, if $k_1 \geq k_0 \geq 1$ is an integer such that (see (32))

$$\parallel b^k_{A_k} \parallel \;\; > \;\; \parallel A_\infty x^* \parallel + 1, \; \forall \;\; k \geq k_1,$$

then by also using (33) and (34) we get a contradiction which completes our proof.

Theorem 3 *For any $x^0 \in I\!R^n$ if $(x^k)_{k\geq 0}$ is the sequence generated with the algorithm (22)-(27), then*

$$\lim_{k\to\infty} x^k = P_{N(A)}(x^0) + G b_A. \tag{35}$$

Proof. Let $k \geq 0$ be arbitrary fixed and $b^k_* \in I\!R^n$ defined by

$$b^k_* = P_{N(A_k)}(b^k). \tag{36}$$

Then, we have the orthogonal decomposition of b^k (see (30))

$$b^k = b^k_{A_k} \oplus b^k_* \tag{37}$$

as in [8] we obtain

$$LSS(A_k; b^k) = \{P_{N(A_k)}(x^0) + G_k b^k_{A_k}, x^0 \in I\!R^n\}, \tag{38}$$

$$x_{LS} = G_k b^k_{A_k} = G b_A, \tag{39}$$

together with (by also using (29))

$$P_{N(A_k)}(x^k) = P_{N(A)}(x^k) = P_{N(A)}(x^0), \ \forall \ k \geq 0, \tag{40}$$

for an arbitrary fixed initial approximation $x^0 \in I\!R^n$. Using (40) together with (39), (7), (26), (8), we succesively get

$$x^{k+1} - (P_{N(A)}(x^0) + Gb_A) = x^{k+1} - (P_{N(A_{k+1})}(x^0) + G_{k+1} b^{k+1}_{A_{k+1}}) =$$

$$(P_{N(A_{k+1})}(x^k) + \tilde{\Phi}_{k+1} x^k + R_{k+1}\beta^{k+1}) - (P_{N(A_{k+1})}(x^k) + G_{k+1} b^{k+1}_{A_{k+1}}) =$$

$$\tilde{\Phi}_{k+1} x^k + R_{k+1}\beta^{k+1} - [(I - \tilde{\Phi}_{k+1}) + \tilde{\Phi}_{k+1}][(I - \tilde{\Phi}_{k+1})^{-1} R_{k+1}] b^{k+1}_{A_{k+1}} =$$

$$\tilde{\Phi}_{k+1} x^k + R_{k+1}\beta^{k+1} - R_{k+1} b^{k+1}_{A_{k+1}} - \tilde{\Phi}_{k+1} G_{k+1} b^{k+1}_{A_{k+1}} - \tilde{\Phi}_{k+1} P_{N(A_{k+1})}(x^0) =$$

$$\tilde{\Phi}_{k+1}[x^k - (P_{N(A)}(x^0) + Gb_A)] + R_{k+1}(\beta^{k+1} - b^{k+1}_{A_{k+1}}). \tag{41}$$

Now, from (25), (37), (24), (8) and (36) we obtain

$$\beta^{k+1} - b^{k+1}_{A_{k+1}} = b^{k+1} - y^{k+1} - b^{k+1}_{A_{k+1}} = b^{k+1}_* - y^{k+1} = b^{k+1}_* - \Phi^{k+1}(A_{k+1}; b^{k+1}) =$$

$$b^{k+1}_* - [P_{N(A^t_{k+1})} \oplus \tilde{\Phi}_{k+1}]^{k+1}(b^{k+1}) = b^{k+1}_* - [P_{N(A^t_{k+1})} \oplus (\tilde{\Phi}_{k+1})^{k+1}](b^{k+1}) =$$

$$[b^{k+1}_* - P_{N(A^t_{k+1})}(b^{k+1})] - (\tilde{\Phi}_{k+1})^{k+1}(b^{k+1}) =$$

$$-(\tilde{\Phi}_{k+1})^{k+1}(b^{k+1}) = -(\tilde{\Phi}_{k+1})^{k+1}(b^{k+1}_{A_{k+1}}). \tag{42}$$

Let $x^* \in I\!R^n$ be defined by (see (35))

$$x^* = P_{N(A)}(x^0) + Gb_A. \tag{43}$$

Then, from (41) and (42) we obtain

$$x^{k+1} - x^* = \tilde{\Phi}_{k+1}(x^k - x^*) - R_{k+1}(\tilde{\Phi}_{k+1})^{k+1}(b^{k+1}_{A_{k+1}}), \quad \forall \ \ k \geq 0. \tag{44}$$

By iterating the equality (44) we get

$$x^{k+1} - x^* = \tilde{\Phi}_{k+1} \dots \tilde{\Phi}_1(x^0 - x^*) -$$

$$\sum_{j=1}^{k} \tilde{\Phi}_{k+1} \dots \tilde{\Phi}_{j+1} R_j (\tilde{\Phi}_j)^j (b^j_{A_j}) - R_{k+1}(\tilde{\Phi}_{k+1})^{k+1}(b^{k+1}_{A_{k+1}}),$$

thus, by taking norms

$$\parallel x^{k+1} - x^* \parallel \ \ \leq \ \ \parallel \tilde{\Phi}_{k+1} \parallel \dots \parallel \tilde{\Phi}_1 \parallel \parallel x^0 - x^* \parallel +$$

$$\sum_{j=1}^{k}(\parallel \tilde{\Phi}_{k+1} \parallel \ldots \parallel \tilde{\Phi}_{j+1} \parallel \parallel \tilde{\Phi}_j \parallel^j \parallel R_j \parallel \parallel b^j_{A_j} \parallel)+$$

$$\parallel R_{k+1} \parallel \parallel \tilde{\Phi}_{k+1} \parallel^{k+1} \parallel b^{k+1}_{A_{k+1}} \parallel . \tag{45}$$

From (10) we obtain that

$$\parallel \tilde{\Phi}_k \parallel < 1, \ \forall\, k \geq 0, \ \ \parallel \tilde{\Phi}_\infty \parallel < 1. \tag{46}$$

Let then $k_0 \geq 1$ and $M_0 > 0$ be such that

$$\parallel \tilde{\Phi}_k \parallel < \frac{1+ \parallel \tilde{\Phi}_\infty \parallel}{2} < 1, \tag{47}$$

$$\parallel R_k \parallel < \parallel R_\infty \parallel + 1, \quad \parallel b^{k+1}_{A_{k+1}} \parallel \leq M_0, \ \forall\, k > k_0 \tag{48}$$

(such k_0 and M_0 exist according to (31), (46) and Proposition 2(ii)). Let now $\mu \in (0, 1)$ and $M > 0$ be defined by

$$\mu = \max\{\parallel \tilde{\Phi}_1 \parallel, \ldots, \parallel \tilde{\Phi}_{k_0} \parallel, \frac{1+ \parallel \tilde{\Phi}_\infty \parallel}{2}\}, \tag{49}$$

$$M = \max\{\parallel R_1 \parallel, \ldots, \parallel R_{k_0} \parallel, \parallel R_\infty \parallel + 1, \parallel b^0_{A_0} \parallel, \ldots, \parallel b^{k_0}_{A_{k_0}} \parallel, M_0\}. \tag{50}$$

Then, from (45)-(50) we get

$$\parallel x^{k+1} - x^* \parallel \leq \mu^{k+1}(\parallel x^0 - x^* \parallel + M^2(k+1)), \ \forall\, k \geq 0, \tag{51}$$

thus $\lim_{k\to\infty} \parallel x^{k+1} - x^* \parallel = 0$ and the proof is complete.

Corollary 1 *In the above hypothesis, for any $x^0 \in I\!R^n$ the sequence $(x^k)_{k\geq 0}$ generated with the algorithm PREKAZ converges to a solution of the problem (1). Moreover, it converges to the minimal norm solution x_{LS} if and only if $x^0 \in R(A)$.*

3 Some computational aspects

The step (24) of the above algorithm (in which we must apply k-times the application $\Phi\ (A_k; \cdot)$) requires a big computational effort (see also Remark 3). Indeed, if M is the number of iterations of (22) - (27) to obtain some accuracy, then the total number of applications of $\Phi\ (A_k; \cdot)$ in (24), denoted by NS, is

$$NS = \frac{M(M+1)}{2}, \tag{52}$$

which, even for small values of M can be enough big (see the last section of the paper). In order to improve this we can try to replace Φ^k in (24), by $\Phi^{f(k)}$, where $f:(0,\infty)\to(0,\infty)$ is a function such that the following assumptions are fulfiled:
(i) the algorithm (22) - (27) still converges and with "almost the same" convergence rate (see (51);
(ii) the total number of applications of $\Phi(A_k;\cdot)$ in (24), denoted by $NS(f)$ and given by

$$NS(f)=\sum_{k=1}^{M} f(k) \tag{53}$$

satisfies

$$NS(f) < NS \tag{54}$$

(in (24) we have $f(k)=k, \forall k\geq 1$). In this sense, by also taking into account (51) we formulate the following problem: for a given number $\gamma\in(0,1)$, find f as before, such that

$$\sum_{k\geq 1}\gamma^{f(k)} < +\infty \tag{55}$$

and (54) holds. The following three results give possible answers to the above request (55) (for the proof see [9]).

Theorem 4 *(i) If $a>0, a\neq 1$ the series $\sum_{k\geq 1}\gamma^{[log_a k]}$ converge if and only if $a\in(1,\frac{1}{\gamma})$;*
(ii) if $a\in(\gamma,\infty)$ then the series $\sum_{k\geq 1}\gamma^{[k^a]}$ converge;
(iii) if $a\in(\frac{1}{\gamma},\infty)$ then the series $\sum_{k\geq 1}\gamma^{[a^k]}$ converge, where by $[x]$ we denoted the integer part of the real number x.

Remark 5 *We will see in the following section of the paper that, for some values of a the choices of f as in theorem 4 before, also satisfy the assumption (55).*

4 Numerical experiments

We considered in our numerical experiments the following first kind integral equation: for a given function $y\in L^2([0,1])$, find $x\in L^2([0,1])$ such that

$$\int_0^1 k(s,t)x(t)dt \;=\; y(s),\;\; s\in[0,1]. \tag{56}$$

We discretized (56) by a collocation algorithm with the collocation points (see e.g. [10])

$$s_i = (i-1)\frac{1}{n-1}, \ i = 1, 2, \dots, n,$$

and we obtained a symmetric system

$$Ax = b, \tag{57}$$

with the $n \times n$ matrix A and $b \in I\!R^n$ given by

$$A_{ij} = \int_0^1 k(s_i, t) k(s_j, t) dt, \quad b_i = y(s_i). \tag{58}$$

We considered the following data

$$k(s,t) = \frac{1}{1 + |s - 0.5| + t}, \ y(s) = \ln\frac{2.5 - s}{1.5 - s},$$

s $\in [0, 0.5)$ ln 1.5 + s$\frac{}{0.5+s,}$s $\in [0.5, 1]$ (59)

where the right hand side y was computed such that the equation (56) has the solution $x(t) = 1, \forall t \in [0, 1]$. Then, from (58) we obtained

$$A_{ij} = \int_0^1 k(s_i, t) k(s_j, t) dt = \frac{1}{\alpha_i(1 + \alpha_i)}, \text{if}$$

$\alpha_i = \alpha_j$,1$\frac{}{\alpha_i - \alpha_j \ln\frac{(1+\alpha_j)\alpha_i}{(1+\alpha_i)\alpha_j},\text{if}}$$\alpha_i \neq \alpha_j$ b$_i = y(s_i)$, (60)where

$$\alpha_i = 1 + \left| s_i - \frac{1}{2} \right|, \quad i = 1, \dots, n. \tag{61}$$

For $n \geq 3$, the rank of the matrix A is given by

$$\text{rank}(A) = \frac{n+1}{2}, \text{if}$$

nisodd$\frac{n}{2}$, ifniseven.(62)First of all we have to observe that, because the problem (56) with the data (59) is consistent, it results that the system (57) is also consistent. We then applied the algorithm PREKAZ, for different values of n and different choices for the function f in (53), with the "residual" stopping rule

$$\parallel Ax^k - b \parallel \leq 10^{-6}. \tag{63}$$

The corresponding numbers of iterations are presented in Table 1 below.

Table 1. Results for the system (57)			
n	$f(k) = k$	$f(k) = [k^{0.8}]$	$f(k) = [log_{1.3}k]$
8	21	21	21
16	22	22	23
32	22	23	23
64	23	24	24
128	23	24	25

In Table 2 we computed the values $NS(f)$ from (53) for all the choices for f from Table 1.

Table 2. Total number of iterations			
n	$NS(k)$	$NS([k^{0.8}])$	$NS([log_{1.3}k])$
8	231	139	164
16	253	145	172
32	253	145	172
64	276	161	175
128	276	161	175

We may observe a reduction of the total number of iterations for reaching the accuracy requested by the stopping rule (63).
Note. All the computations were made with the Numerical Linear Algebra software package OCTAVE, freely available under the terms of the GNU General Public License, see `www.octave.org`.

References

[1] Bjork A., *Numerical methods for least squares problems*, SIAM Philadelphia, 1996.

[2] Kaczmarz S. *Angenaherte Auflosung von Systemen linearer Gleichungen*, Bull. Acad. Polonaise Sci. et Lettres **A** (1937), 355-357.

[3] Kovarik, S., *Some iterative methods for improving orthogonality*, SIAM J. Num. Anal., **7**(3)(1970), 386-389.

[4] Mohr M., Popa C., Ruede U., *A Kovarik type-algorithm without matrix inversion for the numerical solution of symmetric least-squares problems* ; Lehrstuhlbericht **05-2**, Lehrstuhl für Informatik 10 (Systemsimulation), Erlangen-Nürnberg University , Germany, 2005.

[5] Pascali, D., *On the approximation of the solution of the equations defined by potential operators*; Stud. Cerc. Math., **23(10)**(1971), 1533-1535.

[6] Pascali, D., *Approximation solvability of a semilinear wave equation*; Libertas Math., **4**(1984), 73-78.

[7] Popa C., *Extensions of block-projections methods with relaxation parameters to inconsistent and rank-defficient least-squares problems*; *B I T*, **38(1)**(1998), 151-176.

[8] Popa C., *Characterization of the solutions set of inconsistent least-squares problems by an extended Kaczmarz algorithm*; Korean Journal on Comp. and Appl. Math., vol. **6(1)**(1999), 51-64.

[9] Popa C., *Some remarks concerning a Kaczmarz-Kovarik algorithm for inconsistent least-squares problems*, in Proceedings of the 7th Conference on Nonlinear Analysis, Numerical Analysis, Applied Mathematics and Computer Science, "Ovidius" Univ. Press, Constanta 2000, 69-74.

[10] Popa, C., *A method for improving orthogonality of rows and columns of matrices*, Intern. J. Comp. Math., **77**(3)(2001), 469-480.

[11] Tanabe K., *Projection Method for Solving a Singular System of Linear Equations and its Applications*, Numer. Math., **17**(1971), 203-214.

"Ovidius" University of Constanta
Department of Mathematics and Informatics,
900527 Constanta, Bd. Mamaia 124
Romania
e-mail: cpopa@univ-ovidius.ro