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A MIXED VARIATIONAL FORMULATION FOR THE SIGNORINI FRICTIONLESS PROBLEM IN VISCOPLASTICITY

M. Sofonea[1] **and A. Matei**[2]

To Professor Dan Pascali, at his 70's anniversary

Abstract

We consider a mathematical model which describes the frictionless contact between a viscoplastic body and a rigid foundation. The process is quasistatic and the contact is modeled with Signorini's condition in the form with a zero gap function. We provide an evolutionary mixed variational formulation to the model involving a Lagrange multiplier, for which we state and prove an existence and uniqueness result. The proof is based on arguments on saddle points theory and Banach's fixed point theorem.

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1 Introduction

Unilateral problems involving Signorini's contact condition were studied by many authors, see for instance the references in [4, 10, 12]. In particular, the existence of a unique weak solution to the frictionless Signorini contact problem for rate-type viscoplastic materials was proved in [13] and the numerical analysis of the problem was considered in [3]. A convergence result in the study of the same problem was provided in [14]. There it was proved that the solution of the Signorini contact problem can be approached by the solution of the corresponding contact problem with normal compliance as the stiffness coefficient of the foundation converges to infinity. More details in the study of

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frictionless contact problems with viscoplastic materials, including the analysis of semi-discrete and fully discrete schemes, error estimates and numerical simulations, can be found in [6]. Note that in [3, 6, 13, 14] the contact process was assumed to be quasistatic and the problem was studied within the framework of variational inequalities theory.

The aim of this paper is to present a new approach in the study of the quasistatic frictionless contact problems for viscoplastic materials, based on a mixed variational formulation involving a Lagrange multiplier. We model the material behavior with the rate-type constitutive equation used in [3, 6, 13, 14] and the contact with Signorini's condition in a form with a zero gap function. We derive a new variational formulation of the problem, different from that obtained in [3, 6, 13, 14], then we obtain an existence and uniqueness result. The proof is based on arguments on the saddle point theory which can be found in [1, 2, 5, 7]. Our results in this paper lie the background necessary to the numerical analysis of the problem by using the method of Lagrange multipliers. This represents a modern method which was succesfully used in the numerical study of various contact problems, see for instance [8, 9, 11, 15, 16] and the references therein.

The rest of the paper is structured as follows. In Section 2 we present the model, set it in a variational formulation and state our main result, Theorem 2.1. It states the existence of a unique weak solution to the model. The proof of Theorem 2.1 is provided in Section 3.

2 Statement of the problem and main result

The physical setting is as follows. We consider a viscoplastic body that occupies the bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, with the boundary $\partial\Omega = \Gamma$ partitionned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that *meas* $\Gamma_1 > 0$. We assume that the boundary Γ is Lipschitz continuous and denote by $\boldsymbol{\nu}$ its unit outward normal, defined a.e. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$, surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$ and, finally, we assume that the body is in frictionless contact with a rigid foundation on $\Gamma_3 \times (0, T)$.

We denote by $\boldsymbol{u}$ the displacement vector, $\boldsymbol{\sigma}$ the stress field and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the small strain tensor. To describe the behavior of the material we use a rate-type viscoplastic constitutive law,

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, T), \tag{2.1}$$

in which $\mathcal{E}$ is a fourth order tensor and $\mathcal{G}$ is a nonlinear constitutive function. In

(2.1) and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$.

We neglect the inertial term in the equation of motion and obtain the quasistatic approximation of the process. Thus, we use the equilibrium equation,

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0,T), \tag{2.2}$$

in which $\operatorname{Div} \boldsymbol{\sigma}$ denotes the divergence of the tensor $\boldsymbol{\sigma}$. According to the physical setting, we have the following displacement-traction boundary conditions,

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{2.4}$$

in which $\boldsymbol{\sigma\nu}$ denotes the Cauchy stress vector. We assume that the contact is frictionless and it is modeled with Signorini's condition in the form with a zero gap function, that is

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3 \times (0,T). \tag{2.5}$$

Here and below the index ν and τ denote the normal and tangential components of vectors and tensors. To complete our model, we also prescribe the initial data, i.e.

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega, \tag{2.6}$$

where $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ represent the initial displacement and the initial stress, respectively.

Let $\mathcal{S}^d$ denote the space of second order tensors on $\mathbb{R}^d$. With the assumptions above, our mechanical problem may be formulated as follows.

Problem P**.** *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathcal{S}^d$ *such that* (2.1)–(2.6) *hold.*

In order to derive a variational formulation of problem P we need additional notation. Thus, we denote by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathcal{S}^d$; everywhere below the indices i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable; c will denote a positive generic constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$ and $\mathcal{G}$ but it is independent on time and input data, and whose value may change from place to place.

We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ. Moreover, we use also the spaces

$$\mathcal{H} = \{\ \boldsymbol{\sigma} = (\sigma_{ij})\ :\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \},$$
$$H_1 = \{\ \boldsymbol{u} = (u_i)\ :\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\ \},$$
$$\mathcal{H}_1 = \{\ \boldsymbol{\sigma} \in \mathcal{H}\ :\ \operatorname{Div} \boldsymbol{\sigma} \in L^2(\Omega)^d\ \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \\
(\boldsymbol{u}, \boldsymbol{v})_{H_1} &= (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_{L^2(\Omega)^d}.
\end{aligned}$$

The associated norms on the spaces $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

For every element $\boldsymbol{v} \in H_1$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = v \cdot \nu$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we note that when $\boldsymbol{\sigma}$ is a regular function then $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\,da \quad \forall\, \boldsymbol{v} \in H_1. \tag{2.7}$$

Now, let V be the closed subspace of H_1 given by

$$V = \{\ \boldsymbol{v} \in H_1\ :\ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\ \}.$$

Over the space V we consider the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

and let $\|\cdot\|_V$ be the associated norm. Since *meas* $\Gamma_1 > 0$ it follows from Korn's inequality that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.

Let M be the dual space of the space $H^{1/2}(\Gamma_3)^d$ and denote by $\langle\cdot,\cdot\rangle_{\Gamma_3}$ the duality pairing between M and $H^{1/2}(\Gamma_3)^d$. We also denote by K and Λ the sets

$$K = \{\ \boldsymbol{v} \in V\ :\ v_\nu \le 0 \quad \text{on } \Gamma_3\ \}, \tag{2.8}$$

$$\Lambda = \Big\{\ \boldsymbol{\mu} \in M\ :\ \langle \boldsymbol{\mu}, \boldsymbol{v}\rangle_{\Gamma_3} \le 0 \quad \forall\, \boldsymbol{v} \in K\ \Big\}. \tag{2.9}$$

Clearly K and Λ are closed convex cones in V and M, respectively, and contain the zero element of V and M, respectively.

For every subset Y of a real Banach space $(X, \|\cdot\|_X)$ we use the notation $C([0,T];Y)$ for the set of continuous functions from $[0,T]$ to Y; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

In the study of the mechanical problem (2.1)–(2.6) we make the following assumptions:

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathcal{S}^d \to \mathcal{S}^d.\\
\text{(b) } \mathcal{E}_{ijkl} \in L^\infty(\Omega).\\
\text{(c) } \mathcal{E}(\boldsymbol{x})\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{E}(\boldsymbol{x})\boldsymbol{\tau} \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{S}^d,\ \text{a.e. in } \Omega.\\
\text{(d) There exists } m>0 \text{ such that}\\
\qquad \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m|\boldsymbol{\tau}|^2 \quad \forall\,\boldsymbol{\tau}\in\mathcal{S}^d,\ \text{a.e. in } \Omega.
\end{array}\right. \tag{2.10}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathcal{S}^d \times \mathcal{S}^d \to \mathcal{S}^d.\\
\text{(b) There exists } L_{\mathcal{G}}>0 \text{ such that}\\
\qquad |\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)| \le L_{\mathcal{G}}\,(|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|)\\
\qquad\quad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathcal{S}^d,\ \text{a.e. in } \Omega.\\
\text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\ \forall\,\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathcal{S}^d.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{2.11}$$

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C([0,T];L^2(\Gamma_2)^d). \tag{2.12}$$

$$\boldsymbol{u}_0 \in K, \quad \boldsymbol{\sigma}_0 \in \mathcal{H}_1. \tag{2.13}$$

Next, let $a : V\times V \to \mathbb{R}$ and $b : V\times M \to \mathbb{R}$ be the bilinear forms

$$a(\boldsymbol{u},\,\boldsymbol{v}) = \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \tag{2.14}$$

$$b(\boldsymbol{v},\,\boldsymbol{\mu}) = \langle\boldsymbol{\mu},\boldsymbol{v}\rangle_{\Gamma_3}, \tag{2.15}$$

and, using Riesz's representation theorem, define the function $\boldsymbol{f} : [0,T] \to V$ by

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v}\in V,\ t\in[0,T]. \tag{2.16}$$

It follows from (2.10) that a is symmetric, continuous and coercive, since

$$a(\boldsymbol{v},\boldsymbol{v}) \geq m\,\|\boldsymbol{v}\|_V^2 \quad \forall\,\boldsymbol{v}\in V. \tag{2.17}$$

Also, it follows from the properties of the trace operator that the bilinear form b is continuous and satisfies the following inf-sup property,

$$\text{there exists } \alpha>0 \text{ such that } \inf_{\mathbf{0}\neq\boldsymbol{\mu}\in M,}\ \sup_{\mathbf{0}\neq\boldsymbol{v}\in V} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V\|\boldsymbol{\mu}\|_M} \geq \alpha. \tag{2.18}$$

As a consequence of (2.18) we obtain

$$\sup_{\mathbf{0}\neq\boldsymbol{v}\in V} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V} \geq \alpha\,\|\boldsymbol{\mu}\|_M \quad \forall\,\boldsymbol{\mu}\in M. \tag{2.19}$$

Finally, note that assumptions (2.12) imply that

$$\boldsymbol{f}\in C([0,T];V). \tag{2.20}$$

We now derive the mixed variational formulation of Problem P. To this end we assume that $(\boldsymbol{u},\boldsymbol{\sigma})$ are regular functions which satisfy (2.1)–(2.6) and let $\boldsymbol{v}\in V$, $\boldsymbol{\mu}\in\Lambda$ and $t\in[0,T]$. Using Green's formula (2.7) and (2.2) we get

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}_0(t),\boldsymbol{v})_{L^2(\Omega)^d} + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,da$$

and, due to (2.3), (2.4) and (2.16), we obtain

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}(t),\boldsymbol{v})_V + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,da.$$

Since $\boldsymbol{\sigma}_\tau = 0$ on $\Gamma_3\times(0,T)$, it follows from the previous equality that

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}(t),\boldsymbol{v})_V + \int_{\Gamma_3} \sigma_\nu(t)v_\nu\,da. \tag{2.21}$$

Denote by $\boldsymbol{\beta}(t)$ the viscoplastic stress,

$$\boldsymbol{\beta}(t) = \boldsymbol{\sigma}(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{2.22}$$

and define the Lagrange multiplier $\boldsymbol{\lambda}(t)$,

$$\langle\boldsymbol{\lambda}(t),\boldsymbol{v}\rangle_{\Gamma_3} = -\int_{\Gamma_3} \sigma_\nu(t)v_\nu\,da \quad \forall\,\boldsymbol{v}\in V. \tag{2.23}$$

It follows from (2.14)–(2.16), (2.21)–(2.23) that

$$a(\boldsymbol{u}(t),\boldsymbol{v})+(\boldsymbol{\beta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}+b(\boldsymbol{v},\boldsymbol{\lambda}(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V. \tag{2.24}$$

Moreover, taking into account (2.5), (2.8) and (2.9), we deduce that

$$\boldsymbol{\lambda}(t)\in\Lambda,\quad b(\boldsymbol{u}(t),\boldsymbol{\lambda}(t))=0,\quad b(\boldsymbol{u}(t),\boldsymbol{\mu})\le 0\quad\forall\,\boldsymbol{\mu}\in\Lambda \tag{2.25}$$

and, as a consequence of (2.22), (2.1) and (2.6), we obtain

$$\boldsymbol{\beta}(t)=\int_0^t\mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))+\boldsymbol{\beta}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{2.26}$$

To conclude, from (2.24), (2.25) and (2.26) we obtain the following variational formulation of the mechanical problem P.

Problem P_V**.** *Find a displacement field* $\boldsymbol{u}:[0,T]\to V$*, a viscoplastic stress field* $\boldsymbol{\beta}:[0,T]\to\mathcal{H}$ *and a Lagrange multiplier* $\boldsymbol{\lambda}:[0,T]\to\Lambda$ *such that*

$$a(\boldsymbol{u}(t),\boldsymbol{v})+(\boldsymbol{\beta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}+b(\boldsymbol{v},\boldsymbol{\lambda}(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V, \tag{2.27}$$

$$b(\boldsymbol{u}(t),\boldsymbol{\mu}-\boldsymbol{\lambda}(t))\le 0, \tag{2.28}$$

$$\boldsymbol{\beta}(t)=\int_0^t\mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))+\boldsymbol{\beta}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{2.29}$$

for all $\boldsymbol{v}\in V$*,* $\boldsymbol{\mu}\in\Lambda$ *and* $t\in[0,T]$*.*

Our main result that we state here and prove in the next section is the following.

Theorem 2.1. *Assume that* (2.10)–(2.13) *hold. Then, there exists a unique solution* $(\boldsymbol{u},\,\boldsymbol{\beta},\boldsymbol{\lambda})$ *of Problem* P_V*. Moreover, the solution satisfies*

$$\boldsymbol{u}\in C([0,T];V),\quad \boldsymbol{\beta}\in C([0,T];\mathcal{H}),\quad \boldsymbol{\lambda}\in C([0,T];\Lambda). \tag{2.30}$$

A triplet $(\boldsymbol{u},\boldsymbol{\beta},\boldsymbol{\lambda})$ which satisfies (2.27)–(2.29) is called a *weak solution* to the contact problem P and we conclude by Theorem 2.1 that Problem P has a unique weak solution. Note that once the weak solution is know, then the stress field $\boldsymbol{\sigma}$ can be easily obtained by using (2.22). It can be shown that, under the assumption of Theorem 2.1, $\boldsymbol{\sigma}\in C([0,T];\mathcal{H}_1)$.

3 Proof of Theorem 2.1

The proof of Theorem 2.1 will be carried out in several steps and is based on arguments on saddle point theory and fixed point. Everywhere below we assume that (2.10)–(2.13) hold. We start by solving the contact problem in the particular case when the viscoplastic stress is known. To this end let $\boldsymbol{\eta}$ be an arbitrary element of the space $C([0,T];V)$ and consider the following auxiliary problem.

Problem P^1_η. *Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ and a Lagrange multiplier $\boldsymbol{\lambda}_\eta : [0,T] \to \Lambda$ such that, for all $t \in [0,T]$,*

$$a(\boldsymbol{u}_\eta(t), \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t)) = (\boldsymbol{f}(t) - \boldsymbol{\eta}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V, \tag{3.1}$$

$$b(\boldsymbol{u}_\eta(t), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t)) \le 0 \quad \forall\, \boldsymbol{\mu} \in \Lambda. \tag{3.2}$$

In the study of Problem P^1_η we have the following result.

Lemma 3.1. *There exists a unique solution $(\boldsymbol{u}_\eta,\, \boldsymbol{\lambda}_\eta)$ of Problem P^1_η and it satisfies*

$$\boldsymbol{u}_\eta \in C([0,T];V), \quad \boldsymbol{\lambda}_\eta \in C([0,T];\Lambda). \tag{3.3}$$

Moreover, if $(\boldsymbol{u}_i,\, \boldsymbol{\lambda}_i)$ represents the solution of Problem $P^1_{\eta_i}$ for $\boldsymbol{\eta}_i \in C([0,T];V)$, $i = 1,2$, there exists $c > 0$ such that

$$\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V + \|\boldsymbol{\lambda}_{\eta_1}(t) - \boldsymbol{\lambda}_{\eta_2}(t)\|_M \le c\, \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \tag{3.4}$$

for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$ be fixed. The existence of a unique solution to (3.1)–(3.2) follows from classical results of saddle points theory, see for instance [7] p. 341. Note that the solution is the unique saddle point of the Lagrangean functional $\mathcal{L}^\eta_t : V \times \Lambda \to \mathbb{R}$ defined by

$$\mathcal{L}^\eta_t(\boldsymbol{v},\, \boldsymbol{\mu}) = \frac{1}{2}\, a(\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{f}(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\mu}) + (\boldsymbol{\eta}(t), \boldsymbol{v})_V.$$

In order to prove the regularity (3.3) of the solution, let $t_1,\, t_2 \in [0,T]$. We have

$$a(\boldsymbol{u}_\eta(t_1), \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t_1)) = (\boldsymbol{f}(t_1) - \boldsymbol{\eta}(t_1), \boldsymbol{v})_V, \tag{3.5}$$

$$b(\boldsymbol{u}_\eta(t_1), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_1)) \le 0, \tag{3.6}$$

$$a(\boldsymbol{u}_\eta(t_2), \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t_2)) = (\boldsymbol{f}(t_2) - \boldsymbol{\eta}(t_2), \boldsymbol{v})_V, \tag{3.7}$$

$$b(\boldsymbol{u}_\eta(t_2), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_2)) \le 0, \tag{3.8}$$

for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. We take $\boldsymbol{v} = \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1)$ in (3.5), $\boldsymbol{v} = \boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)$ in (3.7) and add the corresponding equalities to obtain

$$\begin{aligned} & a(\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2), \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1)) + \\ & \quad b(\boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1), \boldsymbol{\lambda}_\eta(t_1) - \boldsymbol{\lambda}_\eta(t_2)) = \\ & \qquad (\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2), \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1))_V + (\boldsymbol{\eta}(t_2) - \boldsymbol{\eta}(t_1), \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1))_V. \end{aligned} \tag{3.9}$$

We then take $\boldsymbol{\mu} = \boldsymbol{\lambda}_\eta(t_2)$ in (3.6), $\boldsymbol{\mu} = \boldsymbol{\lambda}_\eta(t_1)$ in (3.8) and add the corresponding inequalities to find

$$b(\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2), \boldsymbol{\lambda}_\eta(t_2) - \boldsymbol{\lambda}_\eta(t_1)) \leq 0. \tag{3.10}$$

We combine now (3.9) and (3.10) and use the coercivity of the form a, (2.17), to obtain

$$\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V \leq c\,(\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V + \|\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)\|_V). \tag{3.11}$$

Next, we use (3.9), the inf-sup property of the form b, (2.19), and (3.11) to deduce

$$\|\boldsymbol{\lambda}_\eta(t_1) - \boldsymbol{\lambda}_\eta(t_2)\|_M \leq c\,(\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V + \|\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)\|_V). \tag{3.12}$$

The regularity (3.3) is now a consequence of the last two inequalities, (3.11) and (3.12), combined with the regularity (2.20) of $\boldsymbol{f}$ and $\boldsymbol{\eta}$. The uniqueness of the solution follows from the unique solvability of (3.1), (3.2) at each time moment $t \in [0, T]$.

Consider now $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0, T]; V)$ and denote by $(\boldsymbol{u}_i, \boldsymbol{\lambda}_i)$ the solution of Problem P_{η_i} for $i = 1, 2$. Arguments similar as those used in the proof of (3.11) and (3.12) yield to the inequalities

$$\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \leq c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \forall\, t \in [0, T], \tag{3.13}$$

$$\|\boldsymbol{\lambda}_{\eta_1}(t) - \boldsymbol{\lambda}_{\eta_2}(t)\|_M \leq c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \forall\, t \in [0, T], \tag{3.14}$$

which imply (3.4).

We now use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 3.1 to construct the following auxiliary problem for the viscoplastic stress field.

Problem P_η^2. *Find a viscoplastic stress field $\boldsymbol{\beta}_\eta : [0, T] \to \mathcal{H}$ such that*

$$\boldsymbol{\beta}_\eta(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)) + \boldsymbol{\beta}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{3.15}$$

for all $t \in [0, T]$.

In the study of Problem P_η^2 we have the following result.

Lemma 3.2. *There exists a unique solution of Problem P_η^2 and it satisfies*

$$\boldsymbol{\beta}_\eta \in C([0, T]; \mathcal{H}). \tag{3.16}$$

Moreover, if $\boldsymbol{\beta}_i$ represents the solutions of problem $P_{\eta_i}^2$ for $\boldsymbol{\eta}_i \in C([0, T]; V)$, $i = 1, 2$, there exists $c > 0$ such that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V\, ds \qquad \forall\, t \in [0, T]. \tag{3.17}$$

Proof. Let $\Theta_\eta : C([0,T];\mathcal{H}) \to C([0,T];\mathcal{H})$ be the operator given by

$$\Theta_\eta \boldsymbol{\beta}(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)) + \boldsymbol{\beta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{3.18}$$

for all $\boldsymbol{\beta} \in C([0,T];\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2 \in C([0,T];\mathcal{H})$ we use (3.18) and (2.11) to obtain

$$\|\Theta_\eta \boldsymbol{\beta}_1(t) - \Theta_\eta \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power Θ^p of the operator Θ is a contraction on the Banach space $C([0,T];V)$ and therefore there exists a unique element $\boldsymbol{\beta}_\eta \in C([0,T];V)$ such that $\Theta_\eta \boldsymbol{\beta}_\eta = \boldsymbol{\beta}_\eta$. Moreover, $\boldsymbol{\beta}_\eta$ is the unique solution of Problem P^2_η.

Consider now $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C(0,T;V)$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\beta}_{\eta_i} = \boldsymbol{\beta}_i$. Let $t \in [0,T]$; we have

$$\boldsymbol{\beta}_1(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) + \boldsymbol{\beta}_1(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),$$
$$\boldsymbol{\beta}_2(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_2(s)) + \boldsymbol{\beta}_2(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).$$

Keeping in mind (2.11) and (2.10) we deduce

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \leq c\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds + \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_{\mathcal{H}} ds\Big)$$

and, taking into account (3.4), yields

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \leq c\Big(\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V\, ds + \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_{\mathcal{H}}\, ds\Big).$$

Using now a Gronwall inequality we deduce that (3.17) holds, which concludes the proof of the lemma.

We now introduce the operator $\Theta : C([0,T];V) \to C([0,T];V)$ which maps every element $\boldsymbol{\eta} \in C([0,T];V)$ to the element $\Theta\boldsymbol{\eta} \in C([0,T];V)$ defined by

$$(\Theta\boldsymbol{\eta}(t), \boldsymbol{v})_V = (\boldsymbol{\beta}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{v} \in V,\ t \in [0,T]. \tag{3.19}$$

Recall that here $\boldsymbol{\beta}_\eta$ represents the viscoplastic stress obtained in Lemma 3.2.

Lemma 3.3. *The operator* Θ *has a unique fixed point.*

Proof. For $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];V)$ and $t \in [0,T]$ we have

$$(\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_2(t), \boldsymbol{v})_V = (\boldsymbol{\beta}_{\eta_1}(t) - \boldsymbol{\beta}_{\eta_2}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{v} \in V$$

which shows that

$$\|\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_2(t)\|_V \leq \|\boldsymbol{\beta}_{\eta_1}(t) - \boldsymbol{\beta}_{\eta_2}(t)\|_{\mathcal{H}}.$$

Using now (3.17) we deduce

$$\|\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_2(t)\|_V \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}}\, ds.$$

which concludes the proof of the lemma.

We have now all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\boldsymbol{\eta}^*$ be the fixed point of the operator Θ introduced in (3.19) and denote $\boldsymbol{u}^* = \boldsymbol{u}_{\eta^*}$, $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}_{\eta^*}$, $\boldsymbol{\beta}^* = \boldsymbol{\beta}_{\eta^*}$. We prove that the triple $(\boldsymbol{u}^*, \boldsymbol{\beta}^*, \boldsymbol{\lambda}^*)$ satisfies (2.27)–(2.29). To this end we use (3.1) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ to write

$$a(\boldsymbol{u}^*(t)), \boldsymbol{v})) + (\boldsymbol{\eta}^*(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}^*(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,\ t \in [0,T]$$

and, since

$$(\boldsymbol{\eta}^*(t), \boldsymbol{v})_V = (\Theta\boldsymbol{\eta}^*(t), \boldsymbol{v})_V = (\boldsymbol{\beta}^*(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{v} \in V,\ t \in [0,T],$$

we obtain

$$a(\boldsymbol{u}^*(t)), \boldsymbol{v}) + (\boldsymbol{\beta}^*(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + b(\boldsymbol{v}, \boldsymbol{\lambda}^*(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,\ t \in [0,T],$$

which shows that (2.27) holds. Taking now $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ in (3.2) we obtain (2.28) and since $\boldsymbol{\beta}^*$ is the unique solution of Problem $P^2_{\eta^*}$, we deduce that (2.29) is satisfied. Consequently, the triple $(\boldsymbol{u}^*, \boldsymbol{\beta}^*, \boldsymbol{\lambda}^*)$ is a solution of Problem P_V and, since the regularity (2.30) follows from Lemmas 3.1 and 3.2, we conclude the existence part of the theorem.

To prove the uniqueness of the solution consider two solutions $(\boldsymbol{u}_i, \boldsymbol{\beta}_i, \boldsymbol{\lambda}_i)$ of Problem P_V which satisfy (2.30) for $i = 1, 2$. Let $t \in [0,T]$; we use (2.27), (2.28) and arguments similar to those used in the proof of the inequalities (3.13) and (3.14) to obtain

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leq c\, \|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}}, \tag{3.20}$$

$$\|\boldsymbol{\lambda}_1(t) - \boldsymbol{\lambda}_2(t)\|_M \le c\,\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}}. \tag{3.21}$$

On the other hand, from (2.29) and (2.11) and (2.10) we find that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \le \tag{3.22}$$
$$c\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds + \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_{\mathcal{H}}\,ds\Big).$$

We plug now (3.20) in (3.22) to deduce

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_{\mathcal{H}} \le c\int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_V\,ds$$

and, using a Gronwall type argument, we find that $\boldsymbol{\beta}_1(t) = \boldsymbol{\beta}_2(t)$. The uniqueness part of the theorem is now a straight consequence of the inequalities (3.20) and (3.21), which concludes the proof.

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[1] Laboratoire de Mathématiques et Physique pour les Systèmes
Université de Perpignan, 52 Avenue Paul Alduy, 66860 Perpignan,
France

[2] Departement of Mathematics, University of Craiova
A.I. Cuza Street 13, 200585, Craiova, Romania