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FROM VARIATIONAL TO HEMIVARIATIONAL INEQUALITIES

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To Professor Dan Pascali, at his 70's anniversary

Abstract

A general passage connecting smooth and nonsmooth, convexity and nonconvexity, variational and hemivariational inequalities is sketched here. The last equations are important for engineering problems because they concentrate in a single inequality all intrinsic features of a phenomenon: the governing equations, the boundary conditions and the constraints. We used the detailed treatment in [4].

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We are concerned with differential inclusions of hemivarational inequalities type. For the simplicity, let X be a real reflexive Banach space, X^* its dual, $< \cdot, \cdot >$ the duality pairing, and let $A : X \to X^*$ be a monotone-like (generally, nonlinear) operator. In a concise form, for a given element, we look for a solution $u \in X$ of the hemivariational inequality

$$< Au - f,\ v - u > + J^o\left(u; v - u\right) \geq 0\ ,$$

for all $v \in X$, where $J^o\left(u; v\right)$ is the generalized directional derivative in the sense of Clarke of a locally Lipschitz function $J : X \to \mathbb{R}$. An equivalent multivalued formulation is given by

$$Au + \partial J\left(u\right) \ni f \quad in \quad X^*,$$

where $\partial J\left(u\right)$ denotes Clarke's generalized subdifferential. Its corresponding dynamic counterpart has the form

$$\frac{\partial u}{\partial t} + Au + \partial J\left(u\right) \ni f,$$

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where is assumed to be a quasilinear elliptic operator of Leray-Lions type.

It is well-known that the monotone operator theory started from the monotonicity of the derivative of convex functions. More general, for a proper convex lower semicontinuos l.s.c. function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, it was introduced the *subdifferential* $\partial\varphi : X \to 2^{X^*}$ by

$$\partial\varphi(x) = \{h \in X^* \mid < h, y - x > \leq \varphi(y) - \varphi(x), \forall y \in X\} \tag{1}$$

which is a simple nice pattern of the maximal monotone (multivalued) operator. In particular, if φ_C is the indicator function of a convex set C of X, then

$$N_C(x) = \partial\varphi_C(x) = \{g \in X^* \mid < g, y - x > \leq 0, \forall y \in C\}$$

is the *normal cone* of C at x. We mention also that $\overline{D(\partial\varphi)} = D(\varphi)$ holds.

Let C be a closed convex set of X, f be a given element in X^* and $A : C \to X^*$ be an operator, nonlinear in general. The problem of finding $u \in C$ such that

$$< Au - f, x - u \geq 0, \forall x \in C \tag{2}$$

is called a *variational inequality* (V.I.). Clearly, when $C = X$, then x' range over a neighborhood of u and the variational inequality reduces to the equation $Au = f$.

More general, let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. convex function, with $D(\varphi) = \{x \in X \mid \varphi(x) < \infty\}$ .Finding an element $u \in D(\varphi)$ such that

$$< Au - f, x - u > +\varphi(x) - \varphi(u) \geq 0, \forall x \in D(\varphi) \tag{3}$$

is also a variational inequality. We note that (3) reduces to (2), when $\varphi = \phi_C$.

According to the subgradient inequality (1), the V.I. (3) is equivalent to

$$f \in Au + \partial\varphi(u) \tag{4}$$

and, in particular, the V.I.(2) is equivalent to

$$f \in Au + N_C(u). \tag{5}$$

Making use of the forms (4) and (5), the theory of variational inequalities is extended in connection with various generalizations of the concept of subdifferential to broader classes of non-convex and nonsmooth functions.

We outline some topological methods for variational inequalities, defining a Leray-Schauder type degree and extending Szulkin's solution index method [7].

Consider first the simpler case of a hemicontinuous strongly monotone operator $A : X \to X^*$, i.e. there are $a > 0$ and $p > 1$ such that

$$< Ax - Ay, x - y \geq a \|x - y\|^p, \forall x, y \in X.$$

Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex l.s.c. function and define the mapping

$$\Pi_{A,\Phi} : X^* \to D(\Phi),$$

which associates to $f \in X^*$ the unique solution $u \in D(\Phi)$ of the variational inequality

$$< Au - f, x - u > +\Phi(x) - \Phi(u) \geq 0, \ \forall x \in D(\Phi).$$

Theorem 1. *The mapping* $\Pi_{A,\Phi} : X^* \to D(\Phi)$ *is single-valued continuous and satisfies* $\|\Pi_{A,\Phi}(f) - \Pi_{A,\Phi}(f')\| \leq \frac{1}{a}\|f - f'\|_*, \ \forall f, f' \in X^*$.

If $\Phi = \phi_C$, we denote $\Pi_{A,\phi_C} \equiv \Pi_{A,C}$.

Moreover, let $F : X \to X^*$ be a nonlinear compact *operator and consider the variational inequality*: Find $u \in D(\Phi)$ such that

$$< Au - F(u), v - u > +\Phi(v) - \Phi(u) \geq 0, \ \forall v \in D(\Phi). \tag{7}$$

This inequality is equivalent to the fixed point problem: Find $u \in D(\Phi)$ such that .

$$u = \Pi_{A,\Phi}(Fu). \tag{8}$$

Provided that the inequality (7) does not admits solution such that $\|u\| = R$, for some $R > 0$, the integer $\deg(I - \Pi_{A,\Phi}(F(g)), B_R, 0)$, in the Leray-Schauder sense, is well-defined. Here B_R is the ball in the origin of the radius R.

This kind of degree is of particular interest for the study of unilateral eigenvalue problems. For a compact operator $F : \mathbb{R} \times X \to X^*$, we look for eigensolutions $(\lambda, u) \in \mathbb{R} \times X$ such that

$$< Au - F(\lambda, u), v - u > +\Phi(v) - \Phi(u) \geq 0, \ \forall v \in D(\Phi). \tag{9}$$

For $\lambda \in \mathbb{R}$, this variational inequality is equivalent to the fixed point problem

$$u(\lambda) = \Pi_{A,\Phi} F(u(\lambda))$$

and the integer

$$\deg(I - \Pi_{A,\Phi}(F(\lambda, g)), B_R, 0)$$

is well-defined with respect to the parameter λ if there are no solutions u, with $\|u(\lambda)\| = R$.

We can remove the strong monotonicity of A by considering maximal monotone operators in Hilbert spaces. In this case, we use the hypothesis

$$int\{D(\partial\Phi)\} \cap D(A) \neq \emptyset \tag{10}$$

which assures the maximality of the monotone sum $A + \partial\Phi$.

Proposition 2. *Let H be a real Hilbert space, $\Phi : H \to \mathbb{R}\{+\infty\}$ be a proper convex l.s.c. function and $A : D(A) \subseteq H \to H$ a maximal monotone operator, such that the condition (10) holds. Then for each $\varepsilon > 0$ and $g \in H$, there exists a unique solution $u_\varepsilon \in D(A)$ of the variational inequality*

$$< \varepsilon u_\varepsilon + Au_\varepsilon - g, v - u_\varepsilon > + \Phi(v) - \Phi(u_\varepsilon) \geq 0,\ \forall v \in H. \tag{11}$$

Moreover, the map $P^\varepsilon_{A,\Phi} : H \to D(A) : g \to P^\varepsilon_{A,\Phi}(g)$is continuous, where $P^\varepsilon_{A,\Phi}(g)$ denotes a unique solution of (11).

Of course, the inequality (10) is equivalent to finding

$$u_\varepsilon \in D(A) \cap D(\partial\Phi),$$

such that $g \in \varepsilon u_\varepsilon + Au_\varepsilon + \partial\Phi(u_\varepsilon)$.

As $(\varepsilon I_H + A + \partial\Phi)^{-1}$is single-valued, the above set-valued problem has a unique solution $u_\varepsilon = P^\varepsilon_{A,\Phi}(g)$,whose continuity follows from the estimate (easily to show)

$$\|u_n - u\| \leq \frac{1}{\varepsilon}\|g_n - g\|.$$

Under some additional conditions on A and φ, we may prove the compactness of $P^\varepsilon_{A,\Phi}$. For instance, we have:

Proposition 3. *Assume the hypotheses of the above proposition are satisfied. Suppose further that is Lipschitz continuous on bounded sets and the sets of the form*

$$\{u \in D(A)\ \ \|u\| \leq r\} \text{ and } \|Au\| \leq r$$

are compact for each $r > 0$. The map $P^\varepsilon_{A,\Phi}$ is compact.

We remark that the condition on A appears usually in parabolic variational problems.

Suppose now the hypotheses of Proposition 3 are satisfied and consider the problem of finding the solutions $u_\varepsilon \in D(A) \cap D(\partial\Phi)$ of the variational inequality

$$< \varepsilon u_\varepsilon + Au_\varepsilon - F(u_\varepsilon), v - u_\varepsilon > \Phi(u_\varepsilon) - \Phi(u_\varepsilon) \geq 0,\ \forall v \in H, \tag{11'}$$

where $F : H \to H$ is a (nonlinear) continuous operator. As above, this inequality is equivalent to the fixed point problem

$$u_\varepsilon = P^\varepsilon_{A,\Phi}(F(u_\varepsilon)),\ u_\varepsilon \in H.$$

Similarly, provided the variational inequality (12) does not admit solutions $\|u\| = R > 0$, the degree of Leray-Schauder type

$$\deg\left(I_H - P^{\varepsilon}_{A,\Phi}\left(F\left(g\right)\right), B_R, 0\right)$$

is well-defined.

Now, we pass from variational to hemivariational inequalities, coming back to the inclusions (4)-(5). Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be *Lipschitz of rank* $K > 0$ near a point $x \in X$ i.e., for some $\varepsilon > 0$, we have

$$|F(y) - F(z)| \leq K \|y - z\|, \ \forall y, z \in B(x, \varepsilon).$$

The generalized directional derivative of F at x in the direction y, denoted $F^o(x; y)$, is defined by

$$F^o(x; y) = \lim_{v \to x, t]} \sup_{0} \frac{F(v + ty) - F(v)}{t},$$

where v is a vector in X and t is a positive scalar. This definition involves an upper limit only.

Likewise,

$$\overline{\partial} F(x) = \{f \in X^* \mid F^o(x; y) \geq < f, y > \ \forall y \in X\}$$

is a Clarke's generalized subgradient of F *at* x. We have the following basic properties [2]:

1) $\overline{\partial} F(x)$ is a nonempty, convex, and weak*-compact subset of X^*, for each $x \in X$;

2) $\|f\| \leq K$, for each $f \in \overline{\partial} F(x)$;

3) $\overline{\partial} F : X \to 2^{X^*}$ is weak*-closed and upper semi-continuous;

4) For all $x \in X, F^o(x, g)$ is the support function of $\overline{\partial} F(x)$, i.e.,

$$F^o(x, g) = \max\left\{< f, y > | f \in \overline{\partial} F(x)\right\}, \ \forall y \in X.$$

In the case of lack of convexity of the underlying stress-strain or reaction-displacement conditions, the weak formulations like

$$-S \in \overline{\partial} J(u)$$

are called roughly *hemivariational inequalities*. For instance, Ω is a bounded domain in $\mathbb{R}^3$ occupied by a deformable body, Γ is its boundary, S is the locally Lipschitz stress vector on Γ and J is a so called "*nonconvex superpotential*". The last concept has been introduced by P.D. Panagiotopoulos (1985) to study nonmonotone semipermeability problems, composite structures, etc. For recent applications of variational and hemivariational inequalities, we mention [5] and [7].

Later, we specify the subdifferentiation of integral functionals called also the generalized gradients of type Chang [1]. For a function

$$\beta \in L^{\infty}_{loc}(\mathbb{R}),$$

we set

$$j(t) = \int_0^1 \beta(s)\, ds,\ t \in \mathbb{R},$$

which is clearly locally Lipschitz. Let us calculate its generalized gradients.

For any $\delta > 0$ and $t \in \mathbb{R}$ we put

$$\underline{\beta}_{\delta}(t) = \underset{|\tau - t| < \delta}{ess} \inf \beta(\tau)$$

and

$$\overline{\beta}_{\delta}(t) = \underset{|\tau - t| < \delta}{ess} \sup \beta(\tau)$$

For t fixed, $\underline{\beta}_{\delta}$ is decreasing in δ while $\overline{\beta}_{\delta}$ is increasing in δ. Thus, the limits

$$\underline{\beta}(t) = \lim_{\delta \to 0_+} \underline{\beta}_{\delta}(t)$$

and

$$\overline{\beta}(t) = \lim_{\delta \to 0_+} \overline{\beta}_{\delta}(t)$$

exist.

Proposition 4. *With the above notations, the following relation holds* $\partial j(t) = \left[\underline{\beta}(t), \overline{\beta}(t)\right],\ \forall t \in \mathbb{R}$.

In particular, if the left limit $\beta(t-0)$ *and the right limit* $\beta(t+0)$ *exist at some* $t \in \mathbb{R}$*, then*

$$\partial j(t) = \left[\min\{\beta(t-0), \beta(t+0)\}, \max\{\beta(t-0), \beta(t+0)\}\right].$$

Roughly speaking ∂j results from the generally discontinuous function β by filling the gaps.

Consider now a bounded smooth domain Ω in $\mathbb{R}^N$ and $j(\cdot, y) : \Omega \to \mathbb{R}$ measurable for $y \in \mathbb{R}^m$, $j(\cdot, y) \in L^1(\Omega)$, $j(\cdot, y) : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz for all $x \in \Omega$ and satisfies thegrowth condition

$$|z| \le c\left(1 + |y|^{p-1}\right), \forall x \in \Omega,\ y \in \mathbb{R}^m, z \in \overline{\partial}_y j(x, y) \tag{12}$$

with a constant $c > 0$ and $p \in (1, +\infty)$. Here $|0|$ is the Euclidean norm in $\mathbb{R}^m$, while $\overline{\partial}_y j(x, y)$ means the generalized gradient of j with respect to the second variable $y \in \mathbb{R}^m$, i.e., $\partial j(x, g)(y)$.

We are in position to handle the integral

$$J(v) = \int_\Omega j(x, v(x))\, dx, \quad \forall v \in L^p(\Omega).$$

Theorem 5. *The functional $J : L^p(\Omega, \mathbb{R}^m) \to \mathbb{R}$ is Lipschitz continuous on bounded sets, satisfies the inequality*

$$J^o(u; v) \leq \int_\Omega j^o_y(x, u(x); v(x))\, dx, \ \forall u, v \in L^p(\Omega, \mathbb{R}^m)$$

and $\partial J(u) \subseteq \int_\Omega \partial j(x, u(x))\, dx, \ \forall u \in L^p(\Omega, \mathbb{R}^m)$ *,in the sense that, for each $z \in \partial J(u)$ and $x \in \Omega$ there is $z(x) \in \mathbb{R}^m$ such that $z(t) \in \partial_y(t, u(t))$ for a.e. $t \in \Omega$, $z(\cdot)\xi \in L^1(\Omega)$, whenever $\xi \in L^1(\Omega)$ and*

$$< z, v >= \int_\Omega z(x) v(x)\, dx, \quad \forall v \in L^p(\Omega, \mathbb{R}^m).$$

Corollary 6. *If $\beta \in L^\infty_{loc}(\mathbb{R})$, verify the growth condition*
$|\beta(t)| \leq c\left(1 + |t|^{p-1}\right)$, $\forall t \in \mathbb{R}$ *for constants $c > 0$ and $p \geq 1$, then the functional $J : L^p(\Omega) \to \mathbb{R}$ described by the integral*

$$J(v) = \int_\Omega \int_0^{v(x)} \beta(t)\, dt dx, \ \forall v \in L^p(\Omega),$$

is Lipschitz continuous on bounded sets in $L^p(\Omega)$ and satisfies at any
$u \in L^p(\Omega)$ the relation $\partial J(u)(x) \subseteq \left[\underline{\beta}(u(x)), \overline{\beta}(u(x))\right]$, for a.e. $x \in \Omega$.

Corollary 7. *For any function $j : \Omega \times \mathbb{R} \to \mathbb{R}$, consider the jump $\partial_y j(x, y) = \left[\underline{j}(x, y), \overline{j}(x, y)\right]$ for a.e. $x \in \Omega$ and $y \in \mathbb{R}$, and suppose that $\underline{j}, \overline{j}$ that are measurable. Then, for any $u \in L^p(\Omega)$, the following formula* holds: $\partial J(u)(x) \subseteq \left[\underline{j}(x, u(x)), \overline{j}(x, u(x))\right]$ *, for a.e. $x \in \Omega$..*

Regarding the subdifferentiation of composite maps and restrictions,we will the fol-lowing result due to Chang [1].

Proposition 8. *Let X and Y be two Banach spaces such that X is continuously imbedded in $Y$and X is dense in Y. Let $G : Y \to \mathbb{R}$ be a locally Lipschitz function and let $i : X \to Y$ denote the imbedding operator. The restriction $G_{|X} : X \to \mathbb{R}$ is defined by*

$$G^o_{|X}(u) = G^o i(u), \ \forall u \in X.$$

Then, for each point $u \in X$, one has the formula

$$G^o_{|X}(u; v) = G^o(i(u); i(v)), \ \forall v \in X$$

and

$$\partial \left(G^o_{|X} \right)(u) = \partial G(i(u)) \circ i = \left\{ z_{|X} \mid z \in \partial G(i(u)) \right\} = \partial G(u),$$

in the sense that each element z of $\partial \left(G_{|X} \right)(u)$ admits a unique extension to an element of $\partial G(u)$.

Definition 9. *An element $u \in X$* is said to be a *substationary (critical) point* of a locally Lipschitz function $I : X \to \mathbb{R}$ on a Banach space if

$$0 \in \partial I(u).$$

An alternate formulation is the condition that

$$I^o(u; v) \geq 0, \ \forall v \in X.$$

By means of these preliminaries, we give a typical existence result.

Let V be a real Banach space, densely and continuously imbedded $L^2(\Omega) = L^2(\Omega, \mathbb{R}^m)$, $m \geq 1$,for a bounded domain Ω in $\mathbb{R}^N$.

Let $a : V \times V \to \mathbb{R}$be continuous, symmetric bilinear form on V, $f \in V^*$, and $j : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a measurable function in the first variable such that $j(\cdot, 0) \in L^1(\Omega)$and satisfying the condition (12).

We consider the hemivariational inequality: find $u \in V$ such that

$$a(u, v) + < f, v > + \int_\Omega j^o_y(x, u(x), v(x))\, dx \geq 0, \ \forall v \in V. \tag{13}$$

The result reveals the relationship between the above concept of critical point and the solution of the inequality (13).

Theorem 10. *Let $I : V \to \mathbb{R}$ be the locally Lipschitz functional defined by $I(v) = \frac{1}{2} a(u, v) + < f, v > + J_{|V}(v)$, $\forall v \in V$, with*

$$J(v) = \int_\Omega j(x, v(x))\, dx, \quad \forall v \in L^2(\Omega).$$

Then any substationary (critical) point $u \in V$ is a solution of the hemivariational inequality (13).

Indeed, the functional $I(\cdot)$ being locally Lipschitz, the substationary points $u \in V$ from Definition 10 make sense and

$$0 \in a(u, g) + f + \partial J_{|V}(u).$$

Using the second relation in Theorem 5, we infer that there exists
$z \in L^2(\Omega)$ such that

$$a(u,v) + < f, v > + \int_\Omega z(x)\, v(x)\, dx = 0, \quad \forall v \in V,$$

and $z(x) \in \partial_y j(x, u(x))$ for a.e. $x \in \Omega$.

According to the support property 4) of generalized subgradients, it follows that $u \in V$ is a solution of the hemivariational inequality (13).

In this setting, it is worth mentioning Palais-Smale condition variant.

Remark 11. *For a local Lipschitz function $F : X \to \mathbb{R}$, the extreme $\wedge(x) = \min\left\{\|f\|_* \mid f \in \overline{\partial} F(x)\right\}$ exists, and it is l.s.c., i.e., $\wedge(x_o) \leq \lim_{x \to x_o} \wedge(x)$. Consequently, we can apply the variational techniques to the above functional $I : V \to \mathbb{R}$ using the following Palais-Smale condition: every sequence $\{u_n\} \subset V$ for which $\{I(x_n)\}$ is bounded and $\min_{f_n \in \partial I(x_n)} \|f_n\|_{X_*} \to 0$*
as $n \to \infty$ contains a convergent subsequence in V.

On the other part, a (multivalued) operator $A : V \to V^*$ is called pseudomonotone [3] if for any sequence $\{u_n\} \subset V$ with $u_n \rightharpoonup u$, and a corresponding sequence $u_n^* \in Tu_n$ with $u_n^* \rightharpoonup u^*$ and

$$\overline{\lim} \sup < u_n^*, u_n - u > \leq 0,$$

it follows that $u^* \in Tu$ and $< u_n^*, u_n > \longrightarrow < u^*, u >$.

Proposition 12. *If the subgradient of a locally Lipschitz function is pseudomonotone, then the function is weakly l.s.c.*

The pseudomonotonicity of generalized gradients is useful in the pointwise minimum problems of a finite number of convex locally Lipschits functions.

Within this framework, we can formulate an existence result for hemivariational inequalities of the form

$$< Au - f, v - u > + J^o(u; v - u) \geq 0, \ \forall v \in V. \tag{15}$$

Assume further that the following Condition (C) holds:
For any sequence $\{u_n\} \subset V$ weakly converging to $u \in V$ with

$$\overline{\lim} < u_n^*, u_n - u > \leq 0,$$

for some $u_n^* \in \partial J(u_n)$, the corresponding sequence $\{J(u_n)\}$ possesses a subsequence converging to $J(u)$.

Theorem 13. *Let $A : V \to V^*$ be a bounded pseudomonotone operator and let $J : V \to \mathbb{R}$ satisfy the condition (C). Suppose moreover that*

$$A + \partial J$$

is coercive. Then, for any $f \in V^*$, the hemivariational inequality (15) admits at least one solution.

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