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AN EXISTENCE AND UNIQUENESS RESULT FOR SEMILINEAR EQUATIONS WITH LIPSCHITZ NONLINEARITY

Dinu Teodorescu

To Professor Dan Pascali, at his 70's anniversary

Abstract

In this note, it is presented an existence and uniqueness result for the semilinear equation $Au + F(u) = f$, where the nonlinearity F is a Lipschitz operator.

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1. Introduction

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$.

In Mortici [2], the semilinear equation $Au + F(u) = 0$ is considered, where $A : D(A) \subseteq H \longrightarrow H$ is a linear maximal monotone operator and the nonlinear operator $F : H \longrightarrow H$ is a strongly monotone Lipschitz operator. It is proved that, under these assumptions, the equation $Au + F(u) = 0$ has a unique solution.

In this paper, we prove an existence and uniqueness result for the semilinear equation

$$Au + F(u) = f, \tag{1}$$

where the nonlinearity F is a Lipschitz operator.

So, we show that the supposition *"F is a Lipschitz operator"* is sufficient for obtaining a unique solution for the equation (1).

Key Words: Semilinear equations; Lipschitz operators; Strongly positive operators.

2. The result

Theorem. *Let $A : D(A) \subseteq H \longrightarrow H$ be linear and maximal monotone, $F : H \longrightarrow H$ be nonlinear and assume that for some positive real $c > M$, we have:*

i) A is a strongly positive operator with the constant c, namely

$$\langle Ax, x\rangle \geq c\,\|x\|^2, \qquad \text{for all } x \in D(A);$$

ii) F is a Lipschitz operator with the constant M,

$$\|F(x) - F(y)\| \leq M\,\|x - y\|, \qquad \text{for all } x, y \in H.$$

Then the equation (1) has a unique solution for all $f \in H$.

Proof. The equation (1) can be equivalently written as

$$Lu + N(u) = f, \tag{2}$$

where $L = I + A$ and $N = -I + F$ (I is the identity of H).

It's clear that L is a strongly positive linear operator with constant $c_1 = c + 1$, N is a Lipschitz operator with the constant $M_1 = M + 1$ and $c_1 > M_1$.

We have $Rg(L) = \{Lx | x \in D(L) = D(A)\} = H$ because A is maximal monotone. Also, from $\langle Lx, x\rangle \geq c_1\,\|x\|^2$ for all $x \in D(L)$, we obtain that

$$\|Lx\| \geq c_1\,\|x\|, \qquad \text{for all } x \in D(L).$$

Consequently there exists $L^{-1} : H \longrightarrow D(L) \subseteq H$ which is linear and continuous, $L^{-1} \in L(H)$, the Banach space of all linear and continuous operators from H to H. Moreover,

$$\left\|L^{-1}\right\|_{L(H)} \leq \frac{1}{c_1},$$

where $\left\|L^{-1}\right\|_{L(H)} = \sup\left\{\left\|L^{-1}v\right\| | v \in H, \|v\| \leq 1\right\}$.

Now, the equation (2) can be equivalently written as

$$(I + L^{-1}N)(u) = L^{-1}f. \tag{3}$$

With the notations $V = I + L^{-1}N$ and $g = L^{-1}f$, the equation (3) becomes

$$Vu = g \qquad (V : H \longrightarrow H) \tag{4}$$

Using the Cauchy-Schwarz inequality, we obtain:

$$-\left\langle L^{-1}Nx - L^{-1}Ny, x - y\right\rangle \leq \left|\left\langle L^{-1}\left(Nx - Ny\right), x - y\right\rangle\right| \leq$$

$$\leq \left\| L^{-1}\left(Nx - Ny\right) \right\| \cdot \|x - y\| \leq$$

$$\leq \left\| L^{-1} \right\|_{L(H)} \cdot \|Nx - Ny\| \cdot \|x - y\| \leq \frac{M_1}{c_1} \|x - y\|^2$$

($|s|$ denote the absolute value of the real number s)
and then

$$\langle Vx - Vy, x - y \rangle = \left\langle x + L^{-1}Nx - y - L^{-1}Ny, x - y \right\rangle =$$

$$= \|x - y\|^2 + \left\langle L^{-1}Nx - L^{-1}Ny, x - y \right\rangle \geq$$

$$\geq \|x - y\|^2 - \frac{M_1}{c_1} \|x - y\|^2 = \left(1 - \frac{M_1}{c_1}\right) \|x - y\|^2,$$

for all $x, y \in H$.

It follows that V is a strongly monotone operator with the constant $\alpha = 1 - \frac{M_1}{c_1} > 0$, because $c_1 > M_1$. It is clear that the operator V is continuous. Also V is coercive $\left(\text{i.e.} \frac{\langle Vx, x \rangle}{\|x\|} \longrightarrow \infty \text{ when } \|x\| \longrightarrow \infty\right)$ and strictly monotone (i.e.$\langle Vx - Vy, x - y \rangle > 0$, for all $x, y \in H$ with $x \neq y$), because V is strongly monotone.

By the Minty-Browder theorem (see Brezis [1], p.88), we obtain that the equation (4) has a unique solution. It follows that the equation (1) has a unique solution. □

3. An application

Let $D \subset R^n$ be a bounded domain and $f \in L^2(D)$. We consider the Dirichlet problem

$$\begin{cases} -\Delta u(x) + au(x) + g(x, u(x)) = f(x), & x \in D \\ u(x) = 0, \quad x \in \partial D. \end{cases} \tag{5}$$

We suppose that $g : D \times R \to R$ has partial derivative in u of the first order and

$$\left| \frac{\partial g}{\partial u} \right| \leq M \text{ in } D \ (M > 0).$$

Also, we suppose that $a \in R, a > M$.

We study now the problem (5), in the following functional background:

$$H = L^2(D), \quad Au = -\Delta u + au, \quad D(A) = H^2(D) \cap H_0^1(D), \quad F(u) = g(\cdot, u).$$

Let $Bu = -\Delta u$ from H to H, defined on $D(B) = H^2(D) \cap H_0^1(D)$. It is well-known the fact that B is a maximal monotone operator. It results that the operator $A = B + aI$ is strongly positive with the constant a and maximal monotone. Also F is a Lipschitz operator with the constant M.

From the result we obtain that the problem (5) has a unique solution in $H^2(D) \cap H_0^1(D)$.

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Valahia University of Targoviste
Department of Mathematics,
Bd. Unirii 18, Targoviste,
Romania
e-mail: dteodorescu2003@yahoo.com