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CERTAIN CONTRIBUTIONS OF THE ROMANIAN SCHOOL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY

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To Professor Dan Pascali, at his 70's anniversary

Abstract

We consider from a historical point of view the principal contributions of the Romanian School of Numerical Analysis and Approximation Theory of Cluj Napoca in Approximation Theory of the functions.

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1. Introduction.

Making a hommage to the Romanian School of Mathematics and in particular to the School of Nonlinear Analysis, in which Professor *Dan Pascali* has a very important role, we will emphasize here another Romanian Mathematical School, namely that of Numerical Analysis and Approximation Theory from Cluj Napoca.

If we can consider an important School of Mathematical Analysis, Complex Analysis and Functional Analysis in Bucharest, formed by *Miron Nicolescu*, *Simion Stoilow* and A*lexandru Ghika*, it is now established that the Numerical Analysis and the Approximation Theory has his Romanian Center of development in Cluj. (But we don't forget the famous books of numerical analysis written in Bucharest by Professor *Ion Cuculescu* and the regretted Professor *Gheorghe Marinescu*.) Of course, the Approximation Theory can be

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considered a special part of the Functional Analysis, but it has a particular character being indicated only to a single big problem and working more in certain (important)particular spaces.

The School of Cluj was formed in the years 1930 by the work of two romanian mathematicians and professors of the University of Cluj: *Tiberiu Popoviciu* (1906-1975), a former member of Romanian Academy, and *Dumitru V. Ionescu* (1901-1985). They have formed initially a strong mathematical Center, with the mentioned preoccupations and this was after few years the Romanian School of Numerical Analysis and Approximation Theory of Cluj Napoca. Now this School is leaded by Professor *Dimitrie D. Stancu* (born in 1927), Ph. D, Dr. H.C., Honorary Member of Romanian Academy.

Many domains are studied in this School. We can mention the following list of topics:

I. Interpolation theory;

II. Numerical differentiation;

III. Orthogonal polynomials;

IV. Numerical quadratures and cubatures;

V. Taylor-type expansions;

VI. Approximations by linear positive operators;

VII. Representations of remainders in linear approximation formulas in two variables;

VIII. Probabilistic methods in theory of uniform approximation of continuous functions;

IX. Use of interpolation and calculus of finite differences in probability theory.

In this report we will emphasize only some aspects connected to the point VI, namely the construction of the linear positive operators of approximation; in this domain, the Romanian School has obtained remarkable results.

But, unfortunately, two results in which *Tiberiu Popoviciu* was deeply implicated were not sufficiently emphasized by him as his original discoveries, and so a great Romanian priority was not in a correct measure emphasized. We can consider that the Romanian priority in these domains was in a certain sense lost. The two mentioned results are:

(a) The operators of *Tiberiu Popoviciu* type;

(b) The theorem (criterion) of *Popoviciu-Bohman-Korovkin.*

The problem of the construction of the approximation operators was posed, as it is well known, starting from the famous approximation theorem of *K. Weierstrass* (1885) which affirms that every continuous function defined on a compact interval $[a, b]$ on the real axis is a uniform limit of a sequence of polynomials.

In 1912, in a famous work [4] "Démonstration du théorème de Weierstrass, fondée sur le calcul des probabilités", *S. Bernstein* gave one of the first constructive proofs of the theorem of *Weierstrass.* He has introduced, for $f \in \mathcal{C}[0,1]$, the sequence of polynomials $(B_n f)_n$ defined by the equalities:

$$(B_n f)(x) = \sum_{k=0}^{n} \mathtt{C}_n^k x^k (1-x) f(kn) \quad (n = 0, 1, 2, \ldots).$$

These are called today the polynomials of *Bernstein.* We have so a sequence of operators B_n:

$$B_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

and:

$$B_n f \xrightarrow[(n\to\infty)]{} f \quad (uniformly).$$

2. The operators of Tiberiu Popoviciu type

The discovery of *S. Bernstein* was very important. His operator has conducted, as it is very known, to the idea of finding other linear positive operators $L_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$, so that $L_n f \xrightarrow[(n\to\infty)]{} f$(uniformly) and so that of generalizing the operator of Bernstein.

Initially, the results in this domain were few, but the problem became more important after the years 1930.

One of the first possibilities to obtain such operators was given by *Tiberiu Popoviciu* in his paper published in 1932, "Remarques sur polynômes binomiaux" [30]. A sequence of polynomials $(p_n)_n$ with $\deg(p_n) = n$ for each $n = 0, 1, 2, \ldots$ is called to be of *binomial type*, if we have the identities:

$$p_n(x+y) = \sum_{k=0}^{n} \mathtt{C}_n^k p_k(x) p_{n-k}(y) \quad (n = 0, 1, 2, \ldots)$$

identically satisfied.

In his paper [32], *Tiberiu Popoviciu* has required more, namely:
(i) $p_n(1) \neq 0$ for any $n = 0, 1, 2, \ldots$
(ii) $p_n(x) \geq 0$ on $[0,1]$, for any $n = 0, 1, 2, \ldots$

He has defined the operator $T_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by the equalities:

$$(T_n f)(x) = 1 p_n(1) \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f(kn).$$

In the same paper, *Tiberiu Popoviciu* gave the convergence property of his operators.

Of course, in the particular case $p_n(x) = 1$, for any $n \in \mathbb{N}$, he finds again the operator of *Bernstein*. But, unfortunately, *Popoviciu* has not more emphasized these ideas also in other publications (possibly in foreign journals) and so his work and his priority in the construction of the operators of binomial type were not very well known in the international mathematic community.

3. The Theorem of *Popoviciu-Bohman-Korovkin*

In a modern form, the theorem is the following:

Let $(L_n)_n$ *be a sequence of linear operators* $L_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$. If $L_n e_0 \xrightarrow[(n\to\infty)]{} e_0$, $L_n e_1 \xrightarrow[(n\to\infty)]{} e_1$, $L_n e_2 \xrightarrow[(n\to\infty)]{} e_2$ *uniformly (where* $e_k(x) = x^k$, $x \in [0,1]$, $k \in \mathbb{N}$*), then* $L_n f \xrightarrow[(n\to\infty)]{} f$ *(also uniformly), for every* $f \in \mathcal{C}[0,1]$.

Tiberiu Popoviciu has obtained a conclusion very close to it in this paper [32] of 1950. After *Tiberiu Popoviciu*, the problem has been revisited once again by the Danish mathematician *H. Bohman* in [5]. But the theorem appears very clear emphasized only in 1959 in the book of *P.P.Korovkin*[13].

So, a new Romanian priority was lost!

The theorem is usually called "'*Bohman-Korovkin* theorem", but in the light of the above comments could be called "'*Popoviciu-Bohman-Korovkin*" theorem.

4. The modern works in approximation operators of the Romanian School leaded by *D.D.Stancu*

In 1968, in a fundamental paper [38], *D.D.Stancu* has defined in a probabilistic way the operators $S_n^{<\alpha>} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$,

$$S_n^{<\alpha>} = 1p_n^{<\alpha>}(1) \sum_{k=0}^{n} \binom{n}{k} p_k^{<\alpha>}(x) p_{n-k}^{<\alpha>}(1-x) f\left(kn\right),$$

where the polynomials $p_k^{<\alpha>}$ of *D.D.Stancu* are defined by the following equalities $p_k^{<\alpha>}(x) = x(x+\alpha)(x+2\alpha)\cdot\ldots\cdot(x+(k-1)\alpha)$ and $p_0^{<\alpha>}(x) = 1$.

The sequence $(S_n^{<\alpha>})_n$ converges uniformly to $f \in \mathcal{C}[0,1]$, for each f. It is one of the most important results of the Romanian School of Numerical Analysis and Approximation Theory. In the case $\alpha = 0$ we find again the operator of *Bernstein*. Other important papers of *D.D.Stancu* in this direction were [39] and [40].

Another class of approximation operators was obtained by *Grigor Moldovan*. In 1984, in [23], *Constantin Manole* has examined again the operators of *Tiberiu Popoviciu* type using the finite operatorial calculus of *Gian-Carlo Rota* and his colaborators (*Ronald Mullin*, *Steven Roman*, *Andrei Odlyzko*, *David Kahaner* and others), also called *Umbral Calculus.*

Luciana Lupaş and Alexandru Lupaş in [20] have also defined by *Umbral Calculus* an approximation operator, namely:

$$\left(L_n^Q f\right)(x) = 1p_n(1) \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n-nx) f\left(kn\right),$$

where Q is a delta-operator (i.e. shift invariant: $E^aQ = QE^a$ for any E^a, where $(E^a p)(x) = p(x+a)$, and $Qe_1 = c \neq 0$) and $(p_n)_n$ is its basic sequence ($p_0 = 1$, $p_n(0) = 0$, $n \geq 1$ and $Qp_n = np_{n-1}$). A small generalization is given in my work [50], published in "Analele Ştiinţifice ale Universităţii Ovidius – Constanţa". In the work [45], *D.D.Stancu* and myself have given a bidimensional approximation of *Tiberiu Popviciu* type, also using the *Umbral Calculus.*

Recently, a young researcher of Cluj, *Maria Crăciun,* has constructed in her thesis, elaborated under the guidance of *D.D.Stancu*, some approximation operators using the sequences of polynomials of *Sheffer* type.

An extensive series of important results was obtained by the continuators of the works of *Tiberiu Popoviciu*, *Dumitru V. Ionescu* and *Dimitrie D. Stancu*: *Gh. Coman*, *A. Lupaş*, *W. Breckner*, *Şt. Cobzaş*, *Gh. Micula*. Other important contributions in the domain of positive linear operators of approximation were obtained by *C. Manole*, *P. Blaga*, *I. Gavrea*, *F. Stancu*, *I. Raşa*, *M. Ivan*, *G. Moldovan*, *O. Agratini*, *O. Dogaru*, *I. Gânscă*, *Şt. Măruşter*, *L. Lupaş*, *V. Miheşan*, *D. Acu*, *I. Chiorean*, *C. Cismaşiu*, *A. Ciupa*, *D. Kácso*, *R. Trâmbiţaş*, *R. Păltănea*, *D. Bărbosu*, *A. Vernescu*, *F. Sofonea*, *M. Crăciun* and others.

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