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SET-VALUED INTEGRATION IN SEMINORM II

Anca Croitoru, Christiane Godet-Thobie

Abstract

We introduced in [5] an integral for multifunctions with respect to a multimeasure. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a locally convex algebra X, then both the multifunction and the multimeasure take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$ which satisfies certain conditions. In this paper, we continue the study of this integral by establishing convergence theorems of Vitali and Lebesgue type.

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1 Preliminaries

Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex vector space and let Q be a filtering family of seminorms which defines the topology of X. We consider $(x, y) \mapsto xy$ having the following properties for every $x, y, z \in X$, $\alpha, \beta \in \mathbb{R}$, $p \in Q$:

(i) $x(yz) = (xy)z$,

(ii) $xy = yx$,

(iii) $x(y + z) = xy + xz$,

(iv) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$,

(v) $p(xy) \leq p(x)p(y)$.

Key Words: Multimeasure; Integrable multifunction; Set-valued integral.

1.1 Examples

(a) $X = \{f \mid f : T \to \mathbb{R}$ is bounded$\}$ where T is a topological space.
Let $\mathcal{K} = \{K \subset T | K$ is compact$\}$ and $Q = \{p_K | K \in \mathcal{K}\}$ where, for every $f \in X$, $p_K(f) = \sup_{t \in K} |f(t)|$.

(b) $X = \{f \mid f : T \to \mathbb{R}\}$ where T is a nonempty set.
Let $Q = \{p_t | t \in T\}$ where $p_t(f) = |f(t)|$, for every $f \in X$.

We denote by $\mathcal{P}_k(X) = \mathcal{P}_k$ the family of all nonempty compact subsets of X. If $A, B \in \mathcal{P}_k, \alpha \in \mathbb{R}$,

$$A + B = \{x + y | x \in A, y \in B\},$$

$$\alpha A = \{\alpha x | x \in A\},$$

$$A \cdot B = \{xy | x \in A, y \in B\}.$$

For every $p \in Q$, every A and $B \in \mathcal{P}_k$, let $e_p(A, B) = \sup_{x \in A} \inf_{y \in B} p(x - y)$ and $h_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}$ - the Hausdorff - Pompeiu semimetric defined by p on $\mathcal{P}_k$. We define $\|A\|_p = h_p(A, O) = \sup_{x \in A} p(x)$, $\forall A \in \mathcal{P}_k$, where $O = \{0\}$. Then $\{h_p\}_{p \in Q}$ is a filtering family of semimetrics on $\mathcal{P}_k$ which defines a Hausdorff topology on $\mathcal{P}_k$.

Let $\widetilde{X} \subset \mathcal{P}_k$ satisfying the conditions:

(x_1) $\widetilde{X}$ is complete with respect to $\{h_p\}_{p \in Q}$,

(x_2) $O \in \widetilde{X}$,

(x_3) $A + B, A \cdot B \in \widetilde{X}$ for every $A, B \in \widetilde{X}$,

(x_4) $A \cdot (B + C) = A \cdot B + A \cdot C$ for every $A, B, C \in \widetilde{X}$.

1.2 Examples

(a) $\widetilde{X} = \{\{x\} | x \in X\}$ for X like in examples (a)and (b) of 1.1.

(b) $\widetilde{X} = \{A \mid A \subset [0, +\infty), A$ is nonempty compact convex$\}$ for $X = \mathbb{R}$.

(c) $\widetilde{X} = \{[f, g] \mid f, g \in X, 0 \le f \le g\}$ for X like in example 1.1-(b), where $[f, g] = \{u \in X \mid f \le u \le g\}$ and $[f, f] = \{f\}$.

1.3 Definition

$\varphi : \mathcal{A} \to \mathcal{P}_k$ is said to be an *additive multimeasure* if:

(i) $$\varphi(\emptyset) = 0,$$

(ii) $\varphi(A \cup B) = \varphi(A) + \varphi(B),$ for every A and $B \in \mathcal{A}$ such that $A \cap B = \emptyset$.

1.4 Definition

Let $\varphi : \mathcal{A} \to \mathcal{P}_k$. For every $p \in Q$, the *p-variation* of φ is the non-negative (possibly infinite) set function $v_p(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$v_p(\varphi, A) = \sup \left\{ \sum_{i=1}^{n} \|\varphi(E_i)\|_p \;\middle|\; (E_i)_{i=1}^{n} \subset \mathcal{A}, E_i \cap E_j = \emptyset \text{ for } i \neq j, \bigcup_{i=1}^{n} E_i = A, n \in \mathbb{N}^* \right\}, \forall A \in \mathcal{A}.$$

A such family $(E_i)_{i=1}^{n}$ is called an $\mathcal{A}-$partition of E.

We denote $v_p(\varphi, \cdot)$ by ν_p if there is no ambiguity.

We say that φ is with bounded $p-$variation iff ν_p is bounded for every $p \in Q$.

If φ is an additive multimeasure, then ν_p is finitely additive for every $p \in Q$.

In the sequel, $\varphi : \mathcal{A} \to \widetilde{X}$ will be an additive multimeasure with bounded $p-$variation.

2 Basic results

In the beginning, we recall some notions introduced in [5].

2.1 Definition

A multifunction $F : S \to \widetilde{X}$ is said to be a *simple multifunction* if $F = \sum_{i=1}^{n} B_i \cdot \mathcal{X}_{A_i}$, where $B_i \in \widetilde{X}$, $A_i \in \mathcal{A}, i = 1, 2, ..., n$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} A_i = S$ and $\mathcal{X}_{A_i}$ is the characteristic function of A_i.

The integral of F *over* $E \in \mathcal{A}$ *with respect to* φ is:

$$\int_E F d\varphi = \sum_{i=1}^{n} B_i \cdot \varphi(A_i \cap E) \in \widetilde{X}.$$

It is easy to see that the integral is correctly defined, thanks to conditions $(x_3), (x_4)$.

2.2 Definition

A multifunction $F : S \to \widetilde{X}$ is said to be φ*-totally measurable in seminorm* if for every $p \in Q$, there is a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$ such that:

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$ (cf. Dunford, Schwartz [7] - III.2.6).

2.3 Remarks

(a) Every simple multifunction is φ-totally measurable in seminorm.
(b) Let $F : S \to \widetilde{X}$ be a multifunction and $p \in Q$. If there is a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$ such that $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$, then, for every $n \in \mathbb{N}$, $h_p(F_n^p, F)$ and $\|F\|_p$ are ν_p-measurable (cf. Dunford, Schwartz [7] - III.2.10).

We recall the following theorem (Theorem2.2 of [5]).

2.4 Theorem

If F and $G : S \to \widetilde{X}$ are $\varphi-$totally measurable in seminorm, then, for every $p \in Q$, $h_p(F, G)$ is ν_p-measurable.

2.5 Definition

A multifunction $F : S \to \widetilde{X}$ is said to be $\varphi-$*integrable in seminorm* if for every $p \in Q$, there exists a sequence $(F_n^p)_n$ of simple multifunctions, $F_n^p : S \to \widetilde{X}$, satisfying the following conditions:

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$ (that is: F is φ-totally measurable in seminorm)

(ii) $h_p(F_n^p, F)$ is ν_p-integrable, for every $n \in \mathbb{N}$,

(iii) $\lim\limits_{n\to\infty} \int_E h_p(F_n^p, F) d\nu_p = 0$, for every $E \in \mathcal{A}$,

(iv) For every $E \in \mathcal{A}$, there exists $I_E \in \widetilde{X}$ such that, for every $p \in Q$, $\lim_{n\to\infty} h_p\left(\int_E F_n^p d\varphi, I_E\right) = 0$.

We denote $I_E = \int_E F d\varphi$ and we call it *integral of F over E with respect to φ*. The sequence $(F_n^p)_n$ is said to be a $p-$*defining sequence* for F.

Several properties of this integral are given in [5]. We close this part with two examples and a theorem of characterization of the $\varphi-$integrability in seminorm.

2.6 Examples

(a) If X is a real Banach algebra, then we obtain the integral defined in Croitoru [4].

(b) If $X = \mathbb{R}, \widetilde{X} = \{A \subset [0, +\infty[\mid \ A$ nonempty compact convex subset $\}$ and $\varphi = \{\mu\}$ (where μ is finitely additive), then we get the integral, defined in Sambucini [9], of the multifunction F with respect to μ.

The next theorem of characterization will be used in the next section.

2.7 Theorem

Let $F : S \to \widetilde{X}$ be a $\varphi-$totally measurable in seminorm multifunction. Then the following properties are equivalent:

(i) For every $p \in Q$, $\|F\|_p$ is ν_p-integrable.

(ii) F is $\varphi-$integrable in seminorm.

Proof.

The part (ii) $\Rightarrow$ (i) is given in theorem 2.6 of [5].

We suppose now (i): for every $p \in Q$, $\|F\|_p$ is ν_p-integrable. Since F is $\varphi-$totally measurable in seminorm, for every $p \in Q$ there is a sequence $(G_n^p)_n$ of simple multifunctions such that:

(1) $h_p(G_n^p, F) \xrightarrow{\nu_p} 0$.

From the inequality:

$$h_p(G_n^p, F) \leq \|G_n^p\|_p + \|F\|_p, \ \forall n \in \mathbb{N}$$

and the fact that $\|F\|_p$ is ν_p-integrable, it follows that, for every $n \in \mathbb{N}$, $h_p(G_n^p, F)$ is ν_p-integrable.

For every $k \in \mathbb{N}^*$, we define $A_k^{p,n} = \left\{ s \in S \,\middle|\, h_p(G_n^p(s), F(s)) > \dfrac{1}{k \cdot \nu_p(S)} \right\}$.

From (1) we have:

(2) $\lim_{n\to\infty} \nu_p(A_k^{p,n}) = 0.$

Since (2), for $\varepsilon = \frac{1}{k}$, there exists $n_k > k$ such that:

(3) $\nu_p(A_k^{p,n}) < \frac{1}{k}$, $\forall n \geq n_k$.

If $U_k^p = G_{n_k}^p \chi_{S\setminus A_k^{p,n_k}}$ for every $k \in \mathbb{N}^*$, U_k^p is a simple multifunction and from theorem 2.4, it follows:

(4) $h_p(U_k^p, F)$ is ν_p-measurable, $\forall k \in \mathbb{N}^*$.

Now, for every $\varepsilon > 0$ and $k \in \mathbb{N}^*$ we have:

$$\nu_p\left(\{s \in S | h_p(U_k^p(s), F(s)) > \varepsilon\}\right) \leq$$

$$\leq \nu_p\left(\{s \notin A_k^{p,n_k} | h_p(G_{n_k}^p(s), F(s)) > \varepsilon\}\right) + \nu_p(A_k^{p,n_k})$$

and using (1) and (2) we obtain:

(5) $h_p(U_k^p, F) \xrightarrow{\nu_p} 0.$

We have $h_p(U_k^p, F) \leq h_p(U_k^p, G_{n_k}^p) + h_p(G_{n_k}^p, F)$ and from the ν_p-integrability of $h_p(U_k^p, G_{n_k}^p) + h_p(G_{n_k}^p, F)$, we obtain:

(6) $h_p(U_k^p, F)$ is ν_p-integrable, $\forall k \in \mathbb{N}^*$.

Let $\varepsilon > 0$. Denoting $\Gamma(E) = \int_E \|F\|_p d\nu_p$, for every $E \in \mathcal{A}$, by theorem 3.7 of [5], $\Gamma \ll \nu_p$ and there exists $\delta(p, \varepsilon) = \delta > 0$, such that:

(7) $\int_E \|F\|_p d\nu_p < \varepsilon$ for every $E \in \mathcal{A}$ with $\nu_p(E) < \delta$.

Then, for every $k \in \mathbb{N}^*$ with $\frac{1}{k} < \min\{\delta, \varepsilon\}$, from (3) and (7) we have:

$$\int_S h_p(U_k^p, F) d\nu_p = \int_{S\setminus A_k^{p,n_k}} h_p(G_{n_k}^p, F) d\nu_p + \int_{A_k^{p,n_k}} \|F\|_p d\nu_p < \frac{1}{k} + \varepsilon < 2\varepsilon,$$

that is

(8) $\lim_{k\to\infty} \int_S h_p(U_k^p, F) d\nu_p = 0.$

By (5), (6) and (8), the sequence $(U_k^p)_k$ satisfies conditions (i), (ii) and (iii) of definition 2.5. It remains to prove condition (iv), so that $(U_k^p)_k$ is a $p-$defining sequence for F and F is $\varphi-$integrable in seminorm.

Since (8), $\forall p \in Q$ and $\varepsilon > 0$, $\exists k(p, \varepsilon) \in \mathbb{N}^*$ such that

(9) $\int_S h_p(U_k^p, F) d\nu_p < \varepsilon$, $\forall k \geq k(p, \varepsilon)$.

Let $J = \{(L, n) |\ L$ finite subset of Q and $n \in \mathbb{N}^*\}$ filtered by the relation: $(L_1, n_1) \leq (L_2, n_2)$ iff $L_1 \subset L_2$ and $n_1 \leq n_2$. For every finite $L \subset Q$, since Q is filtering, there is $p_L \in Q$ such that $p \leq p_L$ for every $p \in L$.

Given $(L,n) \in J$, for $\varepsilon = \dfrac{1}{n}$, by (9), there exists $k(p_L,n) \in \mathbb{N}^*$ such that:

$$\text{(10)} \qquad \int_S h_{p_L}(U_k^{p_L}, F)d\nu_{p_L} < \frac{1}{n}, \ \forall k \geq k(p_L,n).$$

For every $(L,n) \in J$, let $H_{(L,n)} = U^{p_L}_{n+k(p_L,n)}$. First, we show that the generalized sequence $\left(\int_E H_{(L,n)}d\varphi\right)_{(L,n)\in J}$ is Cauchy in $\widetilde{X}$, for every $E \in \mathcal{A}$.

For $p \in Q$ and $\varepsilon > 0$, let $(L_0,n_0) \in J$ such that $p \in L_0$ and $\dfrac{1}{n_0} < \varepsilon$. Then, for every (L_1,n_1) and $(L_2,n_2) \in J$ such that $(L_1,n_1) \geq (L_0,n_0)$ and $(L_2,n_2) \geq (L_0,n_0)$, we have:

$$\text{(11)} \qquad h_p\left(\int_E H_{(L_1,n_1)}d\varphi, \int_E H_{(L_2,n_2)}d\varphi\right) \leq \int_E h_p\left(H_{(L_1,n_1)}, H_{(L_2,n_2)}\right)d\nu_p.$$

Also, we have:

$$\text{(12)} \qquad \begin{aligned} &h_p(H_{(L_1,n_1)}, H_{(L_2,n_2)}) \leq h_p(H_{(L_1,n_1)}, F) + h_p(F, H_{(L_2,n_2)}) \leq \\ &h_{p_{L_1}}(H_{(L_1,n_1)}, F) + h_{p_{L_2}}(H_{(L_2,n_2)}, F). \end{aligned}$$

Using (10), from (11) and (12), it follows:

$$\text{(13)} \qquad \begin{aligned} &h_p\left(\int_E H_{(L_1,n_1)}d\varphi, \int_E H_{(L_2,n_2)}d\varphi\right) \leq \int_E h_{p_{L_1}}(H_{(L_1,n_1)}, F)d\nu_p + \\ &+\int_E h_{p_{L_2}}(H_{(L_2,n_2)}, F)d\nu_p \leq \int_E h_{p_{L_1}}(H_{(L_1,n_1)}, F)d\nu_{p_{L_1}} + \\ &+\int_E h_{p_{L_2}}(H_{(L_2,n_2)}, F)d\nu_{p_{L_2}} < \frac{1}{n_1} + \frac{1}{n_2} < 2\varepsilon. \end{aligned}$$

Since the completeness of $\widetilde{X}$, for every $E \in \mathcal{A}$, there exists $I_E \in \widetilde{X}$ such that:

$$\text{(14)} \qquad \lim_{(L,n)\in J} \int_E H_{(L,n)}d\varphi = I_E.$$

Then, given $p \in Q$, $\varepsilon > 0$ and $E \in \mathcal{A}$, there is $(L_0,n_0) \in J$ with $p \in L_0$ and $\dfrac{1}{n_0} < \varepsilon$ such that:

$$\text{(15)} \qquad h_p\left(\int_E H_{(L_0,n_0)}d\varphi, I_E\right) < \varepsilon.$$

For every $k > k(p,n_0)$, since (9), (10) and (15) we obtain:

$$h_p\left(\int_E U_k^p d\varphi, I_E\right) \leq h_p\left(\int_E U_k^p d\varphi, \int_E H_{(L_0,n_0)}d\varphi\right) + h_p\left(\int_E H_{(L_0,n_0)}d\varphi, I_E\right) <$$

$$< \int_E h_p(U_k^p, H_{(L_0,n_0)})d\nu_p + \varepsilon \leq \int_E h_p(U_k^p, F)d\nu_p + \int_E h_p(H_{(L_0,n_0)}, F)d\nu_p + \varepsilon <$$

$$< \varepsilon + \int_E h_{p_{L_0}}(H_{(L_0,n_0)}, F)d\nu_p + \varepsilon \leq$$

$$\leq 2\varepsilon + \int_E h_{p_{L_0}}(H_{(L_0,n_0)}, F)d\nu_{p_{L_0}} < 3\varepsilon$$

which assures

(16) $$\lim_{k\to\infty} h_p\left(\int_E U_k^p d\varphi, I_E\right) = 0.$$

So, for every $p \in Q$, the sequence $(U_k^p)_k$ of simple multifunctions satisfies the four conditions of definition 2.5. Consequently, $(U_k^p)_k$ is a $p-$ defining sequence for F and F is $\varphi-$integrable in seminorm.

3 Convergence theorems

We begin this section by a theorem of Vitali type.

3.1 Theorem (Vitali)

Let $F : S \to \widetilde{X}$ be a multifunction and let $F_n : S \to \widetilde{X}$ be a sequence of $\varphi-$integrable in seminorm multifunctions. We denote, for every $E \in \mathcal{A}$, $n \in \mathbb{N}^*$ and $p \in Q$, $\Gamma_n^p(E) = \int_E \|F_n\|_p d\nu_p$ and we suppose the following conditions satisfied for every $p \in Q$:

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\Gamma_n^p \ll \nu_p$, uniformly in $n \in \mathbb{N}^*$ (i.e. for every $p \in Q$ and $\varepsilon > 0$, there is $\delta(p,\varepsilon) = \delta > 0$ such that $\Gamma_n^p(E) < \varepsilon$ for all $E \in \mathcal{A}$ with $\nu_p(E) < \delta$ and for every $n \in \mathbb{N}^*$).

Then the multifunction F is $\varphi-$integrable in seminorm and, for every $E \in \mathcal{A}$, $\lim_{n\to\infty} \int_E F_n d\varphi = \int_E F d\varphi$.

Proof. We shall use the precedent theorem. First, we have to prove that $\|F\|_p$ is ν_p-integrable, $\forall p \in Q$.
For every $p \in Q, n \in \mathbb{N}^*$, since F_n is $\varphi-$integrable in seminorm, there exists a $p-$defining sequence $(G_k^{p,n})_k$ for F_n such that:

(17) $$h_p(G_k^{p,n}, F_n) \xrightarrow[k\to\infty]{\nu_p} 0$$

and

(18) $$\lim_{k\to\infty} \int_E h_p(G_k^{p,n}, F_n)d\nu_p = 0, \forall E \in \mathcal{A}.$$

From (17) and (18), for $\varepsilon = \dfrac{1}{2^n}$, there is $k(p,n) \in \mathbb{N}^*$ such that, for every $k \geq k(p,n)$,

(19) $\nu_p\left(\left\{s\in S\,\middle|\,h_p(G_k^{p,n}(s),F_n(s))>\frac{1}{2^n}\right\}\right)<\frac{1}{2^n}$ and

(20) $\int_S h_p(G_k^{p,n},F_n)d\nu_p<\frac{1}{2^n}$.

Let $G_n^p=G_{k(p,n)}^{p,n}$ for every $n\in\mathbb{N}^*$ and $p\in Q$. Now, for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}^*$ such that $\frac{1}{2^n}<\frac{\varepsilon}{2}$ and

$$\{s\in S\,|h_p(G_n^p(s),F(s))>\varepsilon\}\subset\left\{s\in S\,\middle|\,h_p(G_{k(p,n)}^{p,n}(s),F_n(s))>\frac{1}{2^n}\right\}\cup$$

$$\cup\left\{s\in S\,|h_p(F_n(s),F(s))>\varepsilon-\frac{1}{2^n}\right\}$$

for every $n\geq n_0$. Using (19) and (i), from the last inclusion, it follows that $h_p(G_n^p,F)\xrightarrow{\nu_p}0$ which together with the inequality $\left|\|G_n^p\|_p-\|F\|_p\right|\leq h_p(G_n^p,F)$ shows that

(21) $\|G_n^p\|_p\xrightarrow{\nu_p}\|F\|_p$.

Since $(G_n^p)_n$ is a sequence of simple multifunctions such that $h_p(G_n^p,F)\xrightarrow{\nu_p}0$, F is $\varphi-$totally measurable in seminorm. And according to theorem 2.4, for every $n\in\mathbb{N}^*$, $h_p(G_n^p,F)$ is ν_p-measurable.

Let $C_{n,m}^p=\{s\in S\,|h_p(F_n(s),F_m(s))>\varepsilon\}$, $\forall n,m\in\mathbb{N}^*$. It is easy to see that $C_{n,m}^p\subset\{s\in S\,\big|h_p(F_n(s),F(s))>\frac{\varepsilon}{2}\}\cup\{s\in S\,\big|h_p(F_m(s),F(s))>\frac{\varepsilon}{2}\}$ and from(i), we obtain

(22) $\lim\limits_{n,m\to\infty}\nu_p(C_{n,m}^p)=0.$

From (22) and (ii), we have:

(23) $\int_S h_p(F_n,F_m)d\nu_p=\int_{C_{n,m}^p}h_p(F_n,F_m)d\nu_p+\int_{S\setminus C_{n,m}^p}h_p(F_n,F_m)d\nu_p\leq$
$\int_{C_{n,m}^p}\|F_n\|_p d\nu_p+\int_{C_{n,m}^p}\|F_m\|_p d\nu_p+\varepsilon\cdot\nu_p(S)<\varepsilon(2+\nu_p(S)).$

Now, from (20) and (23) we obtain:

(24) $\int_S h_p(G_n^p,G_m^p)d\nu_p\leq\int_S h_p(G_n^p,F_n)d\nu_p+\int_S h_p(F_n,F_m)d\nu_p+$
$+\int_S h_p(F_m,G_m^p)d\nu_p\leq\frac{1}{2^n}+\varepsilon(2+\nu_p(S))+\frac{1}{2^m}<\varepsilon(4+\nu_p(S)),$

that is

(25) $\lim\limits_{n,m\to\infty}\int_S h_p(G_n^p,G_m^p)d\nu_p=0.$

We have $\int_S|\|G_n^p\|_p-\|G_m^p\|_p|\,d\nu_p\leq\int_S h_p(G_n^p,G_m^p)d\nu_p$ and by (25) it results:

(26) $$\lim_{n,m\to\infty}\int_S |\|G_n^p\|_p - \|G_m^p\|_p|\, d\nu_p = 0.$$

By (21) and (26), for every $p \in Q$, $\|F\|_p$ is ν_p-integrable (with defining sequence $(\|G_n^p\|_p)_n$) and according to theorem 2.7, F is $\varphi-$integrable in seminorm.

Acting like in theorem 2.7, from the sequence $(G_n^p)_{n\in\mathbb{N}^*}$, we obtain a $p-$defining sequence $(U_k^p)_{k\in\mathbb{N}^*}$ for F. Thus, let $A_k^{p,n}=\left\{s \in S | h_p(G_n^p(s), F(s)) > \dfrac{1}{k\cdot\nu_p(S)}\right\}$, for every $k \in \mathbb{N}^*$. Since $\lim\limits_{n\to\infty}\nu_p(A_k^{p,n}) = 0$, for $\varepsilon = \dfrac{1}{k}$, there is $n_k > k$ such that:

(27) $$\nu_p(A_k^{p,n}) < \frac{1}{k},\ \forall n \geq n_k.$$

Let $U_k^p = G_{n_k}^p \cdot \mathcal{X}_{S\setminus A_k^{p,n_k}}$, for every $k \in \mathbb{N}^*$. Since $(U_k^p)_k$ is a $p-$defining sequence for F, for arbitrary $\varepsilon > 0$, there exists $n_0(p,\varepsilon) = n_0 \in \mathbb{N}^*$ such that

(28) $$h_p\left(\int_E U_k^p d\varphi, \int_E F d\varphi\right) < \varepsilon,\ \forall k \geq n_0.$$

Since (ii), for $p \in Q$ and $\varepsilon > 0$, there is $\delta(p,\varepsilon) = \delta > 0$ such that:

(29) $$\int_E \|F_n\|_p d\nu_p < \varepsilon,\ \forall E \in \mathcal{A} \text{ with } \nu_p(E) < \delta.$$

Now, choosing $k_0 > \max\left\{n_0, \dfrac{1}{\delta}\right\}$ and $n \geq n_{k_0}$, we have:

$$h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) \leq h_p\left(\int_E F_n d\varphi, \int_E G_n^p d\varphi\right) + h_p\left(\int_E G_n^p d\varphi, \int_E U_{k_0}^p d\varphi\right) +$$

$$+\underbrace{h_p\left(\int_E U_{k_0}^p d\varphi, \int_E F d\varphi\right)}_{cf.(28)<\varepsilon} < \underbrace{\int_E h_p(F_n, G_n^p) d\varphi}_{cf.(18)<\varepsilon} +$$

$$+h_p\left(\int_{E\cap cA_{k_0}^{p,n_{k_0}}} G_n^p d\varphi + \int_{E\cap A_{k_0}^{p,n_{k_0}}} G_n^p d\varphi, \int_{E\cap cA_{k_0}^{p,n_{k_0}}} G_{n_{k_0}}^p d\varphi + \right.$$

$$\left. + \int_{E\cap A_{k_0}^{p,n_{k_0}}} O d\varphi\right) + \varepsilon \leq$$

$$\leq 2\varepsilon + h_p\left(\int_{E\cap cA_{k_0}^{p,n_{k_0}}} G_n^p d\varphi, \int_{E\cap cA_{k_0}^{p,n_{k_0}}} G_{n_{k_0}}^p d\varphi\right) + \left\|\int_{E\cap A_{k_0}^{p,n_{k_0}}} G_n^p d\varphi\right\|_p \leq$$

$$\leq 2\varepsilon + \underbrace{\int_{E\cap cA_{k_0}^{p,n_{k_0}}} h_p(G_n^p, G_{n_{k_0}}^p) d\nu_p}_{cf.(25)<\varepsilon} + \int_{E\cap A_{k_0}^{p,n_{k_0}}} \|G_n^p\|_p d\nu_p < 3\varepsilon +$$

$$+\underbrace{\int_{E\cap A_{k_0}^{p,n_{k_0}}} h_p(G_n^p, F_n)d\nu_p}_{cf.(18)<\varepsilon} + \underbrace{\int_{E\cap A_{k_0}^{p,n_{k_0}}} \|F_n\|_p d\nu_p}_{cf.(29)<\varepsilon} < 5\varepsilon$$

that is $\lim\limits_{n\to\infty} h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) = 0$ and completes the proof.

As a consequence of Vitali theorem 2.8, we have a Lebesgue type theorem of dominated convergence.

3.2 Theorem (Lebesgue)

Let $F : S \to \widetilde{X}$ be a multifunction and let $F_n : S \to \widetilde{X}$ be a sequence of $\varphi-$totally measurable in seminorm multifunctions such that $h_p(F_n, F) \xrightarrow{\nu_p} 0$, for every $p \in Q$. If there exists $g : S \to [0, +\infty)$ a ν_p-integrable function such that $\|F_n\|_p \leq g$ for every $n \in \mathbb{N}^*$ and any $p \in Q$, then F is $\varphi-$integrable in seminorm and $\lim\limits_{n\to\infty} \int_E F_n d\varphi = \int_E F d\varphi$, for every $E \in \mathcal{A}$.

Proof. Since F_n is $\varphi-$totally measurable in seminorm, according to remark 2.3-(b), $\|F_n\|_p$ is ν_p-measurable for every $p \in Q$. Since g is ν_p-integrable and $\|F_n\|_p \leq g$, $\|F_n\|_p$ is ν_p-integrable for every $p \in Q$ and $n \in \mathbb{N}^*$. From theorem 2.7 it follows F_n is $\varphi-$integrable in seminorm, for every $n \in \mathbb{N}^*$.

If we denote $\mu_p(E) = \int_E g d\nu_p$, for every $E \in \mathcal{A}$ and every $p \in Q$, then

(30) $\mu_p \ll \nu_p$.

Let $\Gamma_n^p(E) = \int_E \|F_n\|_p d\nu_p$, for every $E \in \mathcal{A}$, every $p \in Q$ and every $n \in \mathbb{N}^*$. We have

(31) $\Gamma_n^p(E) \leq \mu_p(E)$, $\forall E \in \mathcal{A}$, $p \in Q$, $n \in \mathbb{N}^*$.

From (30) and (31), it results

(32) $\Gamma_n^p \ll \nu_p$, uniformly in $n \in \mathbb{N}^*$.

Using (32) and thanks to theorem 2.8, the conclusion of this theorem follows.

References

[1] Blondia, C., *Integration in locally convex spaces*, Simon Stevin 55(3), 1981, 81-102.

[2] Boboc, N., Bucur, Gh., *Măsură şi capacitate*, Ed. Stiinţifică şi Enciclopedică, Bucureşti, 1985.

[3] Bochner, S., *Integration von Functionen deren werte die Elemente eines Vectorraumes sind*, Fund.Math. 20, 1933, 262-276.

[4] Croitoru, A., *A set-valued integral*, Analele Ştiinţifice ale Universităţii "Al.I.Cuza" Iaşi, nr. 44, 1998, 101-112.

[5] Croitoru, A., Godet-Thobie, C., *Set-valued integration in seminorm. I*, to appear.

[6] Diestel, J., Uhl, J.J., *Vector measures*, Mat. Surveys 15, Amer. Math. Soc., Providence, 1977.

[7] Dunford, N., Schwartz, J., *Linear Operators I. General Theory*, Interscience, New York, 1958.

[8] Godet-Thobie, C., *Multimesures et multimesures de transition*, Thèse de Doctorat d'État de Sciences Mathématiques, Montpellier, 1975.

[9] Sambucini, A.R., *Integrazione per seminorme in spazi localmente convessi*, Riv. Mat. Univ. Parma, nr. 3, 1994, 371-381.

Anca CROITORU
Faculty of Mathematics, University "Al.I.Cuza" of Iaşi,
Carol I Street, No. 11, Iaşi 700506, ROMANIA, Tel.: 40-232201225
e-mail: croitoru@uaic.ro

Christiane GODET-THOBIE
Université de Bretagne Occidentale, Laboratoire de Mathématiques unité
CNRS UMR 6205,
6 Av. Victor Le Gorgeu, CS 93837,
29238 Brest cedex3, FRANCE,
e-mail: christiane.godet-thobie@univ-brest.fr