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# A NOTE ON ISOMORPHIC COMMUTATIVE GROUP ALGEBRAS OVER CERTAIN RINGS

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**Abstract**

Suppose $G$ is an abelian group and $R$ is a commutative ring with 1 of $char(R) \neq 0$. It is proved that if $G$ is $R$-favorable torsion and $RH$ and $RG$ are $R$-isomorphic group algebras for some group $H$, then $H$ is $R$-favorable torsion abelian if and only if either $inv(R) = \emptyset$ or $inv(R) \neq \emptyset$ and $R$ is an $ND$-ring. This strengthens results due to W. Ullery (Comm. Algebra, 1986), (Rocky Mtn. J. Math., 1992) and (Comment. Math. Univ. Carolinae, 1995) and shows that in some instances the condition on $H$ being a priory assumed as $R$-favorable may be removed.

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Let $G$ be a multiplicatively written abelian group with $p$-component $G_p$ for some prime number $p$ and let $R$ be a commutative ring with identity and arbitrary characteristic. Throughout the rest of this brief article, $RG$ denotes the group ring viewed as an $R$-algebra of $G$ over $R$, and $V(RG)$ is the group of all normalized invertible elements (often called normed units) in $RG$.

Before stating the main assertion motivating this paper, we need a few additional notations and definitions.

Following [Ma], we set $inv(R) = \{p|p \text{ is a unit in } R\}$, $zd(R) = \{p|p \text{ is a zero divisor in } R\}$, $supp(G) = \{p|G_p \neq 1\}$ and $G_R = \coprod_{p\in inv(R)} G_p$. Notice that the set $inv(R)$ can be equivalently restated as $inv(R) = \{p|p.1_R \text{ is a unit of } R\}$, while this is not the case for $zd(R)$ because of the following arguments: The

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fact that $p.1_R$ is a zero divisor of $R$ implies that $p$ has such a property in $R$, that is there exists $0 \neq r \in R$ with $pr = 0$, whereas the converse implication does not hold in general. For example, if $F$ is a field of $char(F) = p > 0$, we have $p.1_F = 0$ whence it is not a zero divisor, but $pf = 0$ for each $0 \neq f \in F$ and so $p$ is a prime zero divisor. Nevertheless, these two claims are equivalent for rings of characteristic distinct from $p$.

**Definition.** ([U], [Ulle]). The abelian group $G$ is called *R-favorable* when $G_R = 1$, or equivalently when $supp(G) \cap inv(R) = \emptyset$.

**Definition.** The commutative ring $R$ with identity is said to be *indecomposable* if it has no nontrivial idempotents.

**Definition.** ([U], [Ulle]). The commutative unitary ring $R$ is termed as an *ND-ring* (= nicely decomposing) if $R$ can be properly decomposed in the following manner: $R = R_1 \times \cdots \times R_n$ for some natural number $n$ such that there is an index $i$: $1 \leq i \leq n$ with $inv(R) = inv(R_i)$.

By using this definition and some other crucial facts, Ullery obtains the following necessary and sufficient condition for a commutative unitary ring to be an $ND$-ring, namely:

**Criterion.** ([U]). *The commutative ring $R$ with identity is an $ND$-ring if and only if there is a homomorphism $R \to K$ for some indecomposable commutative ring $K$ with identity so that $inv(R) = inv(K)$.*

The major purpose of the present short note is to check whether or not the property of $G$ being $R$-favorable torsion can be inherited by $RG$ and, if yes, over what rings this remains realized. We settle below this matter in the affirmative by finding a criterion for any commutative unitary ring equipped with nonzero characteristic. We terminate the exploration with a problem concerning the case of rings of zero characteristic.

Before giving the main result and its proof, a few technicalities are in order (e.g. [Ma], [May] and [U], [Ulle] for nomenclature), namely:

**Proposition.** (May, 1976). *Let $R$ be an indecomposable ring and let $supp(G) \cap inv(R) = \emptyset$. If $p \in inv(R)$, then $V_p(RG) = 1$.*

It is well-known that any ring homomorphism $R \to K$ endows $K$ with the structure of an $R$-algebra and thus it ensures the $K$-isomorphism of algebras $KG \cong RG \otimes_R K$.

According to this isomorphism property and to the preceding May's statement, Ullery establishes the following assertion.

**Theorem.** ([U]). *Suppose $G$ is an abelian group and $R$ is an $ND$-ring or an indecomposable commutative ring with 1. If $G$ is $R$-favorable and $RH \cong RG$*

*as $R$-algebras for any group $H$, then $H$ is $R$-favorable as well. Even more, $H \cong G$ provided $char(R) = 0$.*

We shall use and extend in the sequel this affirmation by showing that for rings of positive characteristic with non-empty set of invertible primes the $ND$-rings are the only ones that preserve the property of the torsion group basis to be ring-favorable. Thus we discover that the complete inheritance by $RG$ of this property for another group basis $H$ consists entirely of the specific decomposable ring structure of $R$.

The next proposition will be helpful for proving once again that the group $G$ modulo its torsion part $G_t$, that is $G/G_t$, can be invariantly retrieved by $RG$ over any commutative ring $R$ with identity (see [M], [Ma] and [D] for example).

**Proposition.** (May, 1976). *Suppose that $R$ is an indecomposable ring and suppose that $supp(G) \cap inv(R) = \emptyset$. Then $V(RG) = GW_{RG}$ and $G \cap W_{RG} = G_t$, where $W_{RG}$ is the multiplicative group (= the group of units) of the maximal integral subalgebra of $RG$ with augmentation 1 and $G_t$ is the maximal torsion subgroup of $G$.*

This enables us to give our first statement.

**Theorem.** *Suppose $R$ is a commutative ring with identity of $char(R) \neq 0$ and $G$ is a torsion $R$-favorable abelian group. Then $RG \cong RH$ as $R$-algebras over $R$ for any group $H$ will imply that $H$ is a torsion $R$-favorable abelian group if and only if either 1) $inv(R) = \emptyset$ or 2) $inv(R) \neq \emptyset$ and $R$ is an $ND$-ring.*

**Proof.** We foremost note that $G$ being torsion and $RG \cong RH$ being $R-$isomorphic force that $H$ is torsion as well (see [M] or [D] for instance).

After this, we consider two cases about $inv(R)$.

1) $inv(R) = \emptyset$.

Hence every torsion group is $R$-favorable, so $G$ and $H$ being torsion groups are both $R$-favorable.

2) $inv(R) \neq \emptyset$.

In this aspect, two subcases are valid:

2.1) If $R$ is indecomposable, we mention that it can be formally interpreted as an $ND$-ring and everything is done by the foregoing listed result of Ullery from [U].

2.2) Suppose for a moment that $R$ is decomposable, say $R = R_1 \times \cdots \times R_n$ for some rings $R_i$, $1 \leq i \leq n$, $n$ is natural. Notice that $RG \cong RH \iff R_iG \cong R_iH$, $\forall\, i : 1 \leq i \leq n$.

First of all, if $R$ is an $ND$-ring, we are done (e.g. [U]).

That is why, we shall presume that $R$ is not an $ND$-ring, whence $inv(R) \subset inv(R_i)$ $\forall\, i : 1 \leq i \leq n$. Thus, we distinguish two possibilities for $char(R)$.

a) Firstly, we note that $char(R) \neq p^k$, for all primes $p$ and all positive integers $k$. Otherwise, $char(R) = p^k$ for some prime number $p$ and natural number $k$ assures that $p \notin inv(R)$ and thus there is $l : 1 \leq l \leq n$ with $p \notin inv(R_l)$; $p \in inv(R_i) \forall i$ if and only if $p \in inv(R)$. But $(q, p) = 1$ $\forall$ prime numbers $q \neq p$, so $q \in inv(R) \subset inv(R_i)$ $\forall i$. Therefore $inv(R) = \pi \setminus \{p\} = inv(R_l)$, where $\pi$ is the set of all primes. Henceforth, $R$ is an $ND$-ring, a contradiction.

b) Secondly, $char(R) = m \neq p^k$, for all nonegative integers $k$ and over any prime number $p$. We show below that there exist three objects $R, G$ and $H$, such that $R$ is not an $ND$-ring, $char(R) = 6$ and $G$ is $R$-favorable while $H$ is not $R$-favorable but $RG \cong RH$ are $R$-isomorphic. This will substantiate our claim that for the group $H$ to be deduced as $R$-favorable, $R$ must be an $ND$-ring.

For a counterexample, let $R = F_2 \times F_3$, where $F_p$ is an algebraically closed field of characteristic $p$, let $T_p$ be an abelian $p$-group of cardinality $\aleph_0$, and put $G = T_2 \times T_3$ and $H = T_2 \times T_3 \times T_5$. Furthermore, it follows from well-known results due to May [M] on group algebras over algebraically closed fields that $F_2T_3 \cong F_2(T_3 \times T_5)$ and $F_3T_2 \cong F_3(T_2 \times T_5)$. Thus $RG \cong F_2G \times F_3G \cong (F_2T_3)T_2 \times (F_3T_2)T_3 \cong (F_2(T_3 \times T_5))T_2 \times (F_3(T_2 \times T_5))T_3 \cong F_2(T_2 \times T_3 \times T_5) \times F_3(T_2 \times T_3 \times T_5) \cong (F_2 \times F_3)(T_2 \times T_3 \times T_5) = RH$. So, the example is shown and the theorem is proved in full generality.

We now examine the extreme case when $char(R) = 0$ with $inv(R) \neq \emptyset$. Consider the ring $R = P \times L$ where $char(P) = char(L) = 0$. Since $inv(R) \neq \emptyset$ it follows that $inv(P) \cap inv(L) \neq \emptyset$, and even more that $inv(R) \subseteq inv(P) \cap inv(L)$. Note that we can take $inv(P) \neq inv(L)$ when $R$ is not an $ND$-ring.

Next, the following problem is actual.

**Problem.** Given $P = Z[1/p, 1/q]$ and $L = Z[1/p, 1/s]$ as well as $G \cong Z(q^\infty) \times Z(s^\infty)$ and $H \cong Z(p^\infty) \times Z(q^\infty) \times Z(s^\infty)$. (We indicate that, because only $p \in inv(R)$, so $R$ is not an $ND$-ring, $G$ is $R$-favorable while $H$ is not.) Does it follow that $PG \cong PH$ as $P$-algebras and $LG \cong LH$ as $L$-algebras, respectively?

Notice that, because of the symmetry, only the first isomorphism of algebras have to be verified. If this question has a positive answer, we conclude that $RG \cong RH$ as $R$-algebras. Thereby, when $char(R) = 0$, the condition on $R$ to be an $ND$-ring cannot perhaps be omitted in general as well.

However, that possibility is probably fulfilled in all generality for the mixed case as the following example shows.

**Example.** ([U], Theorem 2). When $G$ is mixed, $R$ need not be however an $ND$-ring. Specifically, there exist three objects, namely $R, G$ and $H$, such that $R$ is not an $ND$-ring, $char(R) = 0$ and $inv(R) \neq \emptyset$ whereas both $G$ and $H$ are $R$-favorable mixed groups with $RG \cong RH$. Nevertheless, $G \ncong H$.

It may be given two another independent simple verifications of the periodicity of $H$.

In fact, if $G$ is torsion and $G_p = 1$ for every prime $p$, then $G = 1$ hence $H = 1$ and there is nothing to prove. If now $G_p \neq 1$ for some prime number $p$, then $p \notin inv(R)$ since $G$ is $R$-favorable. Furthermore, there exists a maximal ideal $J$ of $R$ with $p \in J$. So, $F = R/J$ is a field of $char(F) = p > 0$ and by the tensor multiplication over $F$, we infer that $RG \cong RH$ as $R$-algebras guarantees that $FG \cong FH$ as $F$-algebras. Thus [M], [Ma] or [D] can be employed to derive that $H$ is torsion, as wanted.

For the second confirmation in a special case for $G$, given that $P$ is a minimal prime ideal of $R$, whence $R/P$ is an integral domain and so indecomposable. Besides, we obviously observe that $RG \cong RH$ as $R$-algebras yields $(R/P)G \cong (R/P)H$ as $R/P$-algebras. We shall assume extraordinary that $G$ is chosen a priory as $R/P$-favorable, hence it is $R$-favorable since $p \in inv(R)$ therefore $p \in inv(R/P)$ whence $inv(R) \subseteq inv(R/P)$. Without harm of generality, we shall assume also that $(R/P)G = (R/P)H$. Since, by what we have argued above, both $G$ and $H$ are $R/P$-favorable groups, whence $supp(H) \cap inv(R/P) = \emptyset$, consulting with the second proposition of May we can write $V((R/P)G) = GW_{(R/P)G} = HW_{(R/P)H} = V((R/P)H)$. Because of the fact that the maximal integral subalgebra is an invariant for the group algebra, we establish that $GW_{(R/P)G}/W_{(R/P)G} \cong G/G_t \cong H/H_t \cong HW_{(R/P)H}/W_{(R/P)H}$. Thus $G/G_t \cong H/H_t$ and $G$ being torsion trivially leads us to $H$ is torsion, as desired.

The proofs are finished.

**Remarks.** W. May ([Ma], p. 489, line 13(-)) had claimed that $R_1 \leq R$, where $R$ is indecomposable, implies that $inv(R_1) \subseteq inv(R)$, but this is not immediate if it is true or otherwise it holds valid when $1_R \in R_1 \iff 1_{R_1} = 1_R$.

We emphasize that $char(R) = p$ gives $p.1_R = 0$ whence $p \notin inv(R)$ and $p \in zd(R)$, but $char(R) = 0$ when $zd(R) = \emptyset$.

Moreover, an appeal to the foregoing Theorem riches us with the implication that if $char(R) = p \neq 0$ and $G$ is $p$-mixed, then $RH \cong RG$ insures that $H$ is $p$-mixed, too. This is so since in that situation we have $G_R = \coprod_{q \neq p} G_q = 1$. In the special case when $G$ is also torsion, each $R$-favorable torsion group is a $p$-group.

In the variant when $inv(R)$ coincides with the set of all primes, $G$ being $R$-favorable torsion gives that $G = G_R = 1$, and so by combining $G = 1$ and $RH = RG$ we elementarily extract that $H = 1$.

Finally, we comment some aspects from the papers [Ulle] and [Ullery].

Inspired by the above argued Theorem, we detect that it is not necessarily in all of the hypotheses in [Ulle] the group $H$ to be a priory given as $R$-favorable torsion. In this direction, in [Ullery], $H$ need not be a $p$-group a priory, because as we have just seen $RG \cong RH$ along with $p \notin inv(R)$ imply $FG \cong FH$ for some field $F$ of $char(F) = p$, whence $G$ being a $p$-group yields that the same holds true for $H$, and besides $G_p = 1$ ensures $H_p = 1$. Thereby, it is rather natural to consider the another possibility for $p$. So, we state the following.

**Problem.** If $G_p = 1$ such that $p \in inv(R)$ and $RH \cong RG$, does it follow that $H_p = 1$?

The Theorem answers when $supp(G) \cap inv(R) = \emptyset$. This query considers the reverse cases when for all primes $q \neq p$ we have $G_q \neq 1$ but $q \in inv(R)$ that is $q \in supp(G) \cap inv(R)$, or when for almost all primes $q \neq p$ we have $G_q = 1$ but $q \notin inv(R)$ such that $supp(G) \cap inv(R) \neq \emptyset$.

However, such an investigation will be a work of some other appropriate research study.

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