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ON MAWHIN'S CONTINUATION PRINCIPLES

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To Professor Dan Pascali, at his 70's anniversary

Abstract

The coincidence degree, introduced by J. Mawhin in 1972, is directed as a topological tool for the investigation of the semilinear equation Lu + Nu = f, where L is a linear Fredholm operator with zero index (not necessarily invertible) and N is a nonlinear perturbation. Continuation theorems involving these kind of pairs of mappings (L,N) became an effective procedure in proving the existence of solutions of a large variety of boundary value problems. We extend this method to the case when L is a quasi-linear operator or a duality map, in view of its application to problems involving a p-Laplacian.

Let X and Y be to real Banach spaces and let $M : X \cap \ dom\ M \ \to Y$ be a map. Assume that $X_1 = \ker M$ is a linear subspace of X and denote by X_2 its complement subspace, i. e., $X = X_1 \oplus X_2$. Likewise, let Y_1 and Y_2 be two complementary linear subspaces of Y so that $Y = Y_1 \oplus Y_2$.Assume that $\dim X_1 = \dim Y_1$. Let $P : X \to X_1$ and $Q : Y \to Y_1$be the coressponding orthogonal projectors. Denote by $J : Y_1 \to X_1$ a homeomorphism with $J(0) = 0$. The operator M is said to be *quasi-linear* if

(i) $\dim \ker M = \dim M^{-1}(0) = n < \infty$;

(ii)$R(M) = ImM = M(X \ \cap domM)$ is a closed subset in Y_2.

Let Ω be a bounded open subset of X, with $0 \in \Omega$, and consider a parameter family of perturbation (generally nonlinear) $N_\lambda : [0,1] \times \ \bar{\Omega} \to Y$ with $N_1 = N$. Denote by $\Sigma_\lambda \subset \Omega \times (0,1]$ the set of solutions of the operator equation

$$Mu = N_\lambda u, \ u \in \bar{\Omega}, \ \lambda \in (0,1]. \tag{1}$$

Key Words: Continuation methods; p-Laplacian operator.

The continuous operator $N_\lambda : [0,1] \times \bar{\Omega} \to Y$ is said to be $M-$*compact* if $(I-Q)\, N_\lambda \left(\bar{\Omega}\right) \subseteq ImM$ and there is a compact operator $R : [0,1] \times \bar{\Omega} \to X_2$ such that $R(0,x) = 0$, $R/_{\Sigma_\lambda} = (I-P)\,/_{\Sigma_\lambda}$ and

$$M(P+R) = (I-Q)\, N_\lambda$$

Finally, we introduce the intermediate map

$$S(\lambda,\cdot) = P + R(\lambda,\cdot) + JQN, \tag{2}$$

which is cleary compact, under the above assumptions, and we are interested in the solvability of the equation

$$Mu = Nu. \tag{3}$$

It easy to prove the following equivalence

Proposition 1. *Let $\Omega \subset X$ be a bounded nonempty domain, M be a quasi-linear operator and N_λ be a family of M - compact perturbations. Then $u \in \bar{\Omega}$ is a solution of the equation (1) if and only if it is a fixed point of the map S defined by (2).*

Our basic continuation result states:

Theorem 2. *If the assumptions of the above proposition are satisfied and in addition,we suppose that:*
(i) $Mu \neq N_\lambda u,\ \forall\ (\lambda, u) \in (0,1) \times \partial\Omega$;
(ii) $\deg(JQN,\ \Omega \cap domM,\ 0) \neq 0$,

then the equation (3) has at least one solution in $\bar{\Omega} \cap domM$.

From the previous proposition and the hypothesis (i), it follows that $u \neq S_\lambda u$ for all $(\lambda,u) \in (0,1) \times \partial\Omega$. Also, condition (ii) implies $u \neq S_0 u$ for $u \in \partial\Omega$. The existence of a fix point for S_1 is a consequence of the homotopy invariance property of the Leray-Schauder degree.

Remark 3. When $L = M$ is a Fredholm linear operator with index zero, we define the (right) inverse K of $L/_{doml \cap X_2}$. We have

$$Y_2 = ImL,\ Y_1 = Y/ImL \text{ and } \dim X_1 = \dim Y_1 < \infty.$$

The operator $N_\lambda = \lambda N$ is L - compact. Define $R(\lambda,\cdot) = K(I-Q)\,N$. We can justify that

$$(I-Q)\,N_\lambda\left(\bar{\Omega}\right)=\lambda\,(I-Q)\,N\left(\bar{\Omega}\right)\subset ImL=Y_2,$$

$$R\,(\lambda,\cdot)\,/_{\Sigma_\lambda}=\lambda\,(I-Q)\,N/_{\Sigma_\lambda}=(I-P)\,/_{\Sigma_\lambda},$$

$$R\,(\lambda,\cdot)=\lambda\,(I-Q)\,N:\bar{\Omega}\to X\text{ is compact,}$$

$$L\,(P+R)=L\,[P+\lambda K\,(I-Q)\,N]=(I-Q)\,N_\lambda.$$

Thus, Theorem 2 can be regarded as an extension of Mawhin's continuation theorem [7].

As an application, we prove the existence of solutions of a boundary-value problem involving the one-dimensional p-Laplacian operator

$$\left(\Phi_p\,(u')\right)'+f\,(t,u)=0,\ t\in(0,1)\,, \tag{4}$$

with the boundary-value conditions

$$u\,(0)=0=\alpha u\,(\tau)-u\,(1)\,, \tag{5}$$

where $'=\frac{d}{dt}$, $\Phi_p\,(s)=|s|^{p-2}\,s$, while $p>1$ and $\alpha,\ \tau\in(0,1)$ are constants.
Assume that $f:[0,1]\times R\to R$ verifies the Carathéodory conditions.
Let $V=\left\{v\in C^1\,[0,1]\,\middle|\,\Phi_p\,(v')\in C^1\,[0,1]\text{ satisfying the conditions (5)}\right\}$ and look for the positive solutions $u\in V$, that is $u\,(t)>0$, for $t\in(0,1)$.

To apply the above continuation theorem, we take the spaces
$X=\{x\in C\,[0,1]\,|x\,(0)=0\}\,,Z=C\,[0,1]\,,Y=Z\times R$ and define the operator $M:X\cap domM\to Z\times\{0\}\subset Y$ by

$$M=\left(\frac{d}{dt}\left(\Phi_p\left(\frac{d}{dt}\right)\right),0\right).$$

It is easy to see that
$domM=V$, $\ker M=\{x=\alpha t\,|\alpha\in R\}$ and $ImM=Z\times\{0\}$.
If we label
$X_1=\ker M,\ X_2=\{x\in X\,|x\,(1)=0\}\,,\ Y_1=\{0\}\times R,\ Y_2=Z\times\{0\}\,,$
we can determine the projectors $P:X\to X_1,\ Q:Y\to Y_1$ by
$Px=x\,(1)\,t$ and $Qy=Q\,(z,a)=\begin{pmatrix}0\\a\end{pmatrix}$, with $z\in Z,\ a\in R.$

Clearly, we have $\dim X_1=\dim Y_1=1$.

For any $\Omega\subset V$ and $\lambda\in[0,1]$, define the family $N_\lambda:\bar{\Omega}\to Y$ by

$$(N_\lambda x)(t) = (-\lambda f(t, x(t)),\ \alpha x(\eta) - x(1)).$$

It is easy to show that
$(I - Q) N_\lambda \left(\bar{\Omega}\right) \subset Z \times \{0\} = ImM$ and $QN_\lambda \left(\bar{\Omega}\right) = 0.$
The homeomorphism $J : Y_1 \to X_1$ is given by $J(0, \alpha) = \alpha t$.
Now, we define $R : [0,1] \times \bar{\Omega} \to X_2$ in the form

$$(R(\lambda, x))(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) + c - \int_0^s \lambda f(\tau, x(\tau))\, d\tau\right] ds - x(1)\, t,$$

and, applying the Arzela-Ascoli theorem, we can prove that R is a continuous and compact operator. Moreover, for $x \in \Omega$ and $\lambda \in [0,1]$ given, the constant c is uniquely determined by the condition $(R(\lambda, x))(1) = 0$.

Consider first $\lambda \neq 0$ and take the restriction of R on the solution set

$$\Sigma_\lambda = \left\{ u \in \bar{\Omega}\ | Mu = N_\lambda u \right\} \subseteq \left\{ u \in \bar{\Omega} \,\middle|\, (\Phi_p(u'))' = -\lambda f(t, u) \right\}.$$

We can write

$$(R(\lambda, x))(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(u(1)) + c + \int_0^s (\Phi_p(u'(\tau)))'\, d\tau\right] ds - u(1)\, t$$

$$= \int_0^t \Phi_p^{-1}\left[\Phi_p(u(1)) + c + \Phi_p(u'(s)) - \Phi_p(u'(0))\right] ds - u(1)\, t. \qquad (6)$$

Now, choose $c = \Phi_p(u'(0)) - \Phi_p(u(1))$ and obtain the above mentioned claim, namely

$$(R(\lambda, x))(1) = \int_0^1 \Phi_p^{-1}\left[\Phi_p(x(s))\right] ds - x(1) = x(1) - x(1) = 0.$$

Since the constant c is unique, the same choice in (6) yields
$(R(\lambda, x))(t) = \int_0^1 \Phi_p^{-1}\left[\Phi_p(u(1)) - \Phi_p(u(1)) + \Phi_p(u'(0)) + \right.$

$$\left.+\int_0^s \left(\Phi_p\left(u'\left(\tau\right)\right)\right)' d\tau\right] ds - u\left(1\right)t = u\left(t\right) - u\left(1\right)t = \left[\left(I-P\right)u\right]\left(t\right).$$

Finally, when $\lambda = 0$ we take $c = 0$ and then $\left(R\left(0,x\right)\right)\left(t\right) = 0$ holds for any $x \in \Omega$.

Therefore, $R : [0,1] \times \bar{\Omega} \rightarrow X_2$ fulfils all properties assumed by M - compact operators.

The condition (ii) in Theorem 2 represents in fact an a priori estimate. We point out a simple example related to the problem (4) - (5) considered above. For thes aim, consider the space X endowed with the norm

$$\|x\|_X = \max_{0\le t\le 1} \|x\left(t\right)\|.$$

Proposition 4. *Suppose* $0 < \alpha < 1$ *and there is a constant* $r > 0$ *such that*

$$f\left(t,r\right) < 0 < f\left(t,-r\right),\ t \in [0,1]. \tag{7}$$

Then the problem (4) - (5) has at least one solution $u \in V$ *with* $\|u\|_X \le r$.

Indeed, consider the problem

$$\begin{cases} \Phi_p\left(u'\right)' + \lambda f\left(t,u\right) = 0, \\ \alpha u\left(\eta\right) - u\left(1\right) = 0, \end{cases}$$

on X, which is equivalent to

$$Mu = N_\lambda u,\ \lambda \in [0,1],$$

where M and N_λ are defined above. Take $B_r = \{x \in X\, | \|x\|_X < r\}$ and prove, by contradiction, that

$$Mu \neq N_\lambda u \text{ for } \left(\lambda, u\right) \in \left(0,1\right) \times \partial B_r.$$

Therefore, the sign-change condition (7) is a sufficient condition for the continuation method in the case of boundary value problem (4)-(5).

Recently, a great deal of attention has been paid to problems involving p-Laplacian-like operators. It is worth mentioning the basic contribution of Chaitan P.Gupta and Raul Manasevich (cf.[1],[3],[5] and the references therein) with applications to m-point boundary value problems at resonance. We treated a simpler case to follow the continuation argument based on the coincidence degree [2],[8]. A general approach of periodic solutions was performed in [5]. In the case of null Dirichlet conditions, even in an n-dimensional domain Ω, the p-Laplacian leads to the duality map on the Sobolev space $W_0^p(\Omega)$. Monotonicity and compactness methods for Dirichlet problems with a p-Laplacian operator are surveyed in [4].

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