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AN IMPROVEMENT OF THE CONVERGENCE SPEED OF THE SEQUENCE $(\gamma_n)_{n\geq 1}$ CONVERGING TO EULER'S CONSTANT

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To Professor Dan Pascali, at his 70's anniversary

Abstract

There are a huge number of estimations for the convergence speed of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n.$$

We refer here to the result of J. Franel (e.g. [3], p.523) and we give a stronger double inequality related to the sequence $(\gamma_n)_{n\geq 1}$.

The Euler's constant $\gamma = 0,577...$ is defined as the limit of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n. \tag{1}$$

From the estimations

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}$$

established in [10] it follows the convergence speed of the sequence $(\gamma_n)_{n\geq 1}$,

$$\lim_{n\to\infty} n\left(\gamma_n - \gamma\right) = \frac{1}{2}.$$

Key Words: Convergence speed of a sequence; Euler constant.

The next step in the study of the convergence speed is to find other sequences which converge faster to γ. One method is to change the logarithmic term in (1). To be more precise, with the well-known notation

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n},$$

the sequence

$$R_n = H_n - \ln\left(n + \frac{1}{2}\right)$$

is strictly decreasing and convergent to γ. The convergence order of the sequence $(R_n)_{n\geq 1}$ is $1/24n^2$. Indeed, De Temple gave in [2] the estimations

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

Negoi proved in [4] that the sequence

$$T_n = H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24}\right)$$

is strictly increasing and convergent to γ. Moreover,

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}.$$

Later, Vernescu have found in [11] a fast convergent sequence to γ, by having the idea to replace the last term of the harmonic sum. He proved that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{2n} - \ln n$$

is strictly increasing and convergent to γ. In this case,

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}, \tag{2}$$

so the convergence order of the sequence $(x_n)_{n\geq 1}$ is $1/12n^2$. This fact is close related with the asymptotic develop of the sum H_n,

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \varepsilon_n, \tag{3}$$

where

$$0 < \varepsilon_n < \frac{1}{252n^2}.$$

In the same sense we can see that every other replacement of the term $1/n$ by α/n, with $\alpha \neq 1/2$, leads to a weaker convergence, because the term $1/2n$ from (3) dissapears only in case $\alpha = 1/2$,

$$H_n - \ln n - \frac{1}{2n} = -\frac{1}{12n^2} + \frac{1}{120n^4} - \varepsilon_n.$$

The estimations (2) allow us to establish the better estimation

$$\frac{1}{2n} - \frac{1}{12n^2} < \gamma_n - \gamma < \frac{1}{2n} - \frac{1}{12(n+1)^2} \tag{4}$$

for the sequence $(\gamma_n)_{n\geq 1}$. In this sense, from the identity

$$x_n = \gamma_n - \frac{1}{2n},$$

we deduce

$$\frac{1}{12(n+1)^2} < \gamma - \left(\gamma_n - \frac{1}{2n}\right) < \frac{1}{12n^2}.$$

Hence

$$\frac{1}{12(n+1)^2} - \frac{1}{2n} < \gamma - \gamma_n < \frac{1}{12n^2} - \frac{1}{2n},$$

so (4) is proved. Now mention that the estimations (4) we obtained here are stronger than the estimations

$$\frac{1}{2n} - \frac{1}{8n^2} < \gamma_n - \gamma < \frac{1}{2n}$$

due to J. Franel (e.g. [3], p.523), because

$$\frac{1}{2n} - \frac{1}{12n^2} > \frac{1}{2n} - \frac{1}{8n^2}$$

and obviously

$$\frac{1}{2n} - \frac{1}{12(n+1)^2} < \frac{1}{2n}.$$

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