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EXISTENCE AND LOCATION OF SOLUTIONS TO SOME EIGENVALUE DIRICHLET PROBLEMS

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To Professor Dan Pascali, at his 70's anniversary

Abstract

The aim of this paper is to obtain existence and additional qualitative information, including location properties, for the solutions of nonlinear Dirichlet problems, on a bounded domain $\Omega \subset \mathbb{R}^N$, that are obtained by perturbing the equation giving the Fučik spectrum with a term $f \in H^{-1}(\Omega)$.

1. Introduction and statements of results

In the present paper we develop a variational approach for studying an eigenvalue problem with Dirichlet boundary condition obtained as a perturbation of the equation describing the Fučik spectrum. Specifically, inspired from the definition of Fučik spectrum (see, e.g., [6]), we deal with the following boundary value problem: find $u : \Omega \to \mathbb{R}$ and $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta u(x) = \lambda u^+(x) - \mu u^-(x) + f(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Here and later on we use the standard notation $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

We are interested not only in the existence of solutions, but in obtaining additional qualitative properties too. For instance, such a basic property is the location of solutions. To this end we set up a variational approach suitable for

Key Words: Eigenvalue Dirichlet problems; Nonlinear Dirichlet problems; Fučik spectrum.

the eigenvalue problems of the type presented above whose idea originates in [3]. As general references for variational methods applied to nonlinear boundary value problems we indicate [1], [2], [5], [6].

In fact, our results establish alternatives in solving different eigenvalues problems with Dirichlet boundary conditions. We emphasize that in addition to the existence they supply significant information on the location of eigensolutions. Moreover, under the formulated assumptions, they provide explicit representations of the eigenvalues (regarded as pairs of real numbers following the pattern of Fučik spectrum). Results of this type have been obtained in [4] for superlinear elliptic boundary value problems. Notice that here we deal with sublinear problems, so the results in [4] are not applicable.

It is worth to point out that actually the location of eigensolutions in our results is obtained by means of the graph of an auxiliary function whose technical role is to create an artificial coercivity. This will be transparent from the relevant arguments in the proofs of these results.

Let us now precise the functional setting. Throughout the paper Ω stands for a bounded domain in $\mathbb{R}^N$. The space $H_0^1(\Omega)$ is endowed with the Hilbertian norm given by

$$\|v\|^2 = \int_\Omega |\nabla v(x)|^2 dx, \quad \forall v \in H_0^1(\Omega).$$

The dual of $H_0^1(\Omega)$ is denoted as usual by $H^{-1}(\Omega)$. In the sequel the notation $\langle \cdot, \cdot \rangle$ means the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. We denote by λ_1 the first eigenvalue of the negative Laplacian $-\Delta$ on $H_0^1(\Omega)$.

We can now state our results.

Theorem 1. *Let $f \in H^{-1}(\Omega)$ and a real number $a > \frac{1}{\lambda_1}$ such that there is a constant $\alpha > 0$ for which the Dirichlet problem*

$$(P_0) \qquad \begin{cases} -\Delta u = \frac{1}{a}(u^+ + f) \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega \\ \langle f, u \rangle \le -\alpha \end{cases}$$

is not solvable. Then for all $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ there exists a solution $(u, s) \in H_0^1(\Omega) \times \mathbb{R}$ of the Dirichlet problem with constraints

$$(P) \qquad \begin{cases} -\Delta u = \frac{1}{a+s^2}(u^+ + f) + \frac{s^2}{a+s^2}\, u^- \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega \\ \rho \le s \le r \\ \langle f, u \rangle \le \frac{\lambda_1}{a\lambda_1 - 1} \|f\|^2_{H^{-1}(\Omega)}. \end{cases}$$

An equivalent formulation of Theorem 1 is the following statement.

Theorem 1′. *Let $f \in H^{-1}(\Omega)$ and a real number $a > \frac{1}{\lambda_1}$ such that there is a constant $\alpha > 0$ for which the Dirichlet problem*

$$(P_0') \qquad \begin{cases} -\Delta u = \frac{1}{a}(-u^- + f) \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \langle f, u \rangle \le -\alpha \end{cases}$$

is not solvable. Then for all $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ there exists a solution $(u, s) \in H_0^1(\Omega) \times \mathbb{R}$ of the Dirichlet problem with constraints

$$(P') \qquad \begin{cases} -\Delta u = \frac{1}{a+s^2}(-u^- + f) - \frac{s^2}{a+s^2}\, u^+ \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \rho \le s \le r \\ \langle f, u \rangle \le \frac{\lambda_1}{a\lambda_1 - 1} \|f\|^2_{H^{-1}(\Omega)}. \end{cases}$$

This equivalence is a direct consequence of the following facts. An element $u \in H_0^1(\Omega)$ is a solution of problem (P_0) (corresponding to $f \in H^{-1}(\Omega)$) if and only if $-u$ is a solution of problem (P_0') with f substituted by $-f$. Similarly, $(u, s) \in H_0^1(\Omega) \times \mathbb{R}$ is a solution of problem (P) (corresponding to $f \in H^{-1}(\Omega)$) if and only if $(-u, s)$ is a solution of problem (P') with f replaced by $-f$.

Theorem 2. *Let $f \in H^{-1}(\Omega)$ and a real number $a > 0$ such that there is a constant $\alpha > 0$ for which the Dirichlet problem*

$$(P_0'') \qquad \begin{cases} -\Delta u = \frac{1}{a}(-u^+ + f) \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \langle f, u \rangle \le -\alpha \end{cases}$$

is not solvable. Then for all $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ there exists a solution $(u, s) \in H_0^1(\Omega) \times \mathbb{R}$ of the Dirichlet problem with constraints

$$(P'') \qquad \begin{cases} -\Delta u = \frac{1}{a+s^2}(-u^+ + f) + \frac{s^2}{a+s^2}\, u^- \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \rho \le s \le r \\ \langle f, u \rangle \le \frac{1}{a} \|f\|^2_{H^{-1}(\Omega)}. \end{cases}$$

Equivalently, Theorem 2 can be stated as follows.

Theorem 2′. *Let $f \in H^{-1}(\Omega)$ and a real number $a > 0$ such that there is a constant $\alpha > 0$ for which the Dirichlet problem*

$$(P_0''') \qquad \begin{cases} -\Delta u = \frac{1}{a}(u^- + f) \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \langle f, u \rangle \le -\alpha \end{cases}$$

is not solvable. Then for all $\rho, r \in \mathbb{R}$ *with* $0 < \rho < r$ *there exists a solution* $(u,s) \in H_0^1(\Omega) \times \mathbb{R}$ *of the Dirichlet problem with constraints*

$$(P''') \qquad \begin{cases} -\Delta u = \frac{1}{a+s^2}(u^- + f) - \frac{s^2}{a+s^2}\, u^+ \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \\ \rho \le s \le r \\ \langle f, u\rangle \le \frac{1}{a}\, \|f\|^2_{H^{-1}(\Omega)}. \end{cases}$$

The equivalence between Theorems 2 and 2′ can be justified analogously as for Theorems 1 and 1′. Consequently, we have only to prove Theorems 1 and 2.

The proofs of Theorems 1 and 2 rely on the following minimax result in [3].

Lemma 1 . *Let* X *be a Banach space,* $F : X \times \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function and let numbers* $\delta > 0$ *and* ρ*,* r *with* $0 < \rho < r$ *such that*

(*i*) $F(0,0) \le 0$, $F(0,r) \le 0$;

(*ii*) $F(v,\rho) \ge \delta > 0$, $\forall v \in X$;

(*iii*) $F \in C^1(X \times \mathbb{R})$ *satisfies the Palais-Smale condition.*

Then the number

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} F(\gamma(\tau)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X \times \mathbb{R}) \,:\, \gamma(0) = (0,0),\ \gamma(1) = (0,r)\},$$

is a critical value of F*, i.e., there exists* $(u,s) \in X \times \mathbb{R}$ *such that*

$$F'(u,s) = (F_u(u,s), F_s(u,s)) = (0,0) \quad \text{and} \quad F(u,s) = c.$$

Moreover, one has the estimate

$$F(u,s) \ge \delta\,.$$

We see that in Theorems 1 and 1′ the condition $a > 1/\lambda_1$ is imposed, whereas in Theorems 2 and 2′ one has only $a > 0$. The proofs of Theorems 1 and 2 are presented in the next sections 2 and 3.

2. Proof of Theorem 1

According to the statement Theorem 1 we assume there is an $\alpha > 0$ such that problem (P_0) has no solutions because otherwise the result is proved. Fix numbers $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ and denote $\varepsilon = a - \frac{1}{\lambda_1}$. By hypothesis we know that $\varepsilon > 0$. Let us consider any C^1 function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying the properties

(β_1) $\beta(0) = \beta(r) = 0$;

(β_2) $\beta(\rho) = \frac{1}{2\varepsilon} \|f\|^2_{H^{-1}(\Omega)} + \frac{\alpha}{2}$;

(β_3) $\lim_{|t|\to\infty} \beta(t) = +\infty$;

(β_4) $\beta'(t) < 0 \iff t < 0 \ \text{or} \ \rho < t < r$;
and
$\beta'(t) = 0 \Longrightarrow t \in \{0, \rho, r\}$.

It is clear that such a function $\beta(t)$ exists. We apply Lemma 1 setting $X = H^1_0(\Omega)$ and the function $F : H^1_0(\Omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(v,t) = \frac{1}{2}(a+t^2)\|v\|^2 + \beta(t) - \frac{1}{2}\int_\Omega (v^+)^2\,dx + \frac{t^2}{2}\int_\Omega (v^-)^2\,dx - \langle f, v\rangle \quad (1)$$

for all $(v,t) \in H^1_0(\Omega) \times \mathbb{R}$. Since

$$\int_0^{v(x)} \tau^+\,d\tau = \frac{1}{2}\,(v^+)^2(x) \quad \text{and} \quad -\int_0^{v(x)} \tau^-\,d\tau = \frac{1}{2}\,(v^-)^2(x),$$

F in (1) can be expressed as follows

$$F(v,t) = \frac{1}{2}(a+t^2)\|v\|^2 + \beta(t) - \int_\Omega\int_0^{v(x)} \tau^+\,d\tau\,dx$$

$$-t^2\int_\Omega\int_0^{v(x)} \tau^-\,d\tau\,dx - \langle f, v\rangle, \quad \forall (v,t) \in H^1_0(\Omega)\times\mathbb{R}. \quad (2)$$

By (2) we have that $F \in C^1(H^1_0(\Omega)\times\mathbb{R})$ (see, e.g., [1], [5]). Let us check assumptions (i)-(iii) of Lemma 1.

In view of (β_1) we have $F(0,0) = F(0,r) = 0$, which shows that assumption (i) in Lemma 1 holds true. Using (1) as well as (β_2) and the variational characterization of λ_1, we find the estimate

$$\begin{aligned} F(v,\rho) &\geq \frac{a}{2}\,\|v\|^2 + \beta(\rho) - \frac{1}{2}\,\|v\|^2_{L^2(\Omega)} - \|f\|_{H^{-1}(\Omega)}\,\|v\| \\ &\geq \beta(\rho) - \frac{1}{2\varepsilon}\|f\|^2_{H^{-1}(\Omega)} = \frac{\alpha}{2} \end{aligned}$$

for all $v \in H_0^1(\Omega)$, so (ii) of Lemma 1 is satisfied with $\delta = \alpha/2$.

We claim that the functional $F : H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ introduced in (1) satisfies the Palais-Smale condition on the product space $H_0^1(\Omega) \times \mathbb{R}$. Towards this let $\{(v_n, t_n)\} \subset H_0^1(\Omega) \times \mathbb{R}$ be a sequence such that

$$|F(v_n, t_n)| \leq M \quad \text{for all } n, \tag{3}$$

with a constant $M > 0$, and

$$F'(v_n, t_n) = (F_v(v_n, t_n), F_t(v_n, t_n)) \to 0 \;\text{ in } H^{-1}(\Omega) \times \mathbb{R} \text{ as } n \to \infty. \tag{4}$$

Taking into account (1), from (3) we have

$$M \geq F(v_n, t_n) \geq \beta(t_n) - \frac{1}{2\varepsilon} \|f\|^2_{H^{-1}(\Omega)}. \tag{5}$$

On the basis of property (β_3) we derive from (5) that

$$\{t_n\} \text{ is bounded in } \mathbb{R}. \tag{6}$$

Furthermore, from (3) and the variational characterization of λ_1 we get

$$M \geq F(v_n, t_n) \geq \frac{1}{2}\Big(a - \frac{1}{\lambda_1}\Big)\|v_n\|^2 + \beta(t_n) - \|f\|_{H^{-1}(\Omega)} \|v_n\|.$$

Making use of (6) we see that

$$\{v_n\} \text{ is bounded in } H_0^1(\Omega). \tag{7}$$

In view of (6) and (7), we have that, along a relabelled subsequence, $\{t_n\}$ is convergent in $\mathbb{R}$ and $\{v_n\}$ is weakly convergent in $H_0^1(\Omega)$. This yields that $\{v_n\}$ is strongly convergent in $L^2(\Omega)$ as well as the sequences $\{v_n^-\}$, $\{v_n^+\}$. On the other hand (4) implies

$$F_v(v_n, t_n) = (a + t_n^2)(-\Delta v_n) - v_n^+ - t_n^2 v_n^- - f \to 0 \;\text{ in } H^{-1}(\Omega) \text{ as } n \to \infty.$$

It follows that

$$\{(a + t_n^2)(-\Delta v_n)\} \text{ is convergent in } H^{-1}(\Omega).$$

Since then $\{-\Delta v_n\}$ is convergent in $H^{-1}(\Omega)$, we conclude that up to a subsequence $\{v_n\}$ is convergent in $H_0^1(\Omega)$. Hence F satisfies the Palais-Smale condition which means that (iii) in Lemma 1 is verified.

Applying Lemma 1 to the function $F \in C^1(H_0^1(\Omega) \times \mathbb{R})$ in (1) provides a point $(u, s) \in H_0^1(\Omega) \times \mathbb{R}$ such that

$$(a + s^2)(-\Delta u) - u^+ - s^2 u^- - f = 0 \;\text{ in } H^{-1}(\Omega), \tag{8}$$

$$s\|u\|^2 + \beta'(s) + s\int_\Omega (u^-)^2\,dx = 0. \tag{9}$$

$$\frac{1}{2}(a+s^2)\|u\|^2 + \beta(s) - \frac{1}{2}\int_\Omega (u^+)^2\,dx + \frac{s^2}{2}\int_\Omega (u^-)^2\,dx - \langle f,u\rangle \geq \frac{\alpha}{2}\,. \tag{10}$$

Notice that (9) gives

$$s^2\|u\|^2 + s\beta'(s) + s^2\int_\Omega (u^-)^2\,dx = 0.$$

This equality enables us to deduce

$$s\beta'(s) \leq 0. \tag{11}$$

In view of property (β_4), it turns out from (11) that either $s=0$ or $\rho \leq s \leq r$.

Consider first the situation $s=0$. Then (8) and (10) with $s=0$ become

$$-\Delta u = \frac{1}{a}(u^+ + f), \tag{12}$$

$$\frac{a}{2}\,\|u\|^2 - \frac{1}{2}\int_\Omega (u^+)^2\,dx - \langle f,u\rangle \geq \frac{\alpha}{2}\,, \tag{13}$$

respectively. By (12) we get

$$\|u\|^2 - \frac{1}{a}\int_\Omega (u^+)^2\,dx - \frac{1}{a}\langle f,u\rangle = 0. \tag{14}$$

Combining (13) and (14) yields $\langle f,u\rangle \leq -\alpha$, which together with (12) ensures that u is a solution of problem (P_0). This contradicts our assumption on problem (P_0) corresponding to the positive number α.

It remains to consider the situation

$$\rho \leq s \leq r\,. \tag{15}$$

From (8) we see that (u,s) is a solution of the equation in problem (P). Moreover, by (8) we find

$$(a+s^2)\|u\|^2 - \int_\Omega (u^+)^2\,dx + s^2\int_\Omega (u^-)^2\,dx - \langle f,u\rangle = 0. \tag{16}$$

Substituting (16) in (10) leads to

$$\beta(s) - \frac{1}{2}\langle f,u\rangle \geq \frac{\alpha}{2}\,. \tag{17}$$

We note that the properties (β_4), (β_2), (15) and the definition of ε imply

$$\beta(s) \le \beta(\rho) = \frac{1}{2\varepsilon}\,\|f\|^2_{H^{-1}(\Omega)} + \frac{\alpha}{2} = \frac{\lambda_1}{2(a\lambda_1 - 1)}\,\|f\|^2_{H^{-1}(\Omega)} + \frac{\alpha}{2}\,.$$

It now suffices to use (17) for obtaining the estimate

$$\langle f, u\rangle \le \frac{\lambda_1}{a\lambda_1 - 1}\|f\|^2_{H^{-1}(\Omega)}.$$

Recalling that (15) holds, it follows that the pair $(u, s) \in H^1_0(\Omega) \times \mathbb{R}$ solves problem (P). This completes the proof. $\square$

Remark 1. A direct proof of Theorem $1'$, without passing as in section 1 through Theorem 1, can be done by arguing like in the proof of Theorem 1 excepting that now in place of $F(v, t)$ in (1) we take the function $F : H^1_0(\Omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(v,t) = \frac{1}{2}(a + t^2)\|v\|^2 + \beta(t) + \frac{t^2}{2}\int_\Omega (v^+)^2\,dx - \frac{1}{2}\int_\Omega (v^-)^2\,dx - \langle f, v\rangle$$

for all $(v, t) \in H^1_0(\Omega) \times \mathbb{R}$, where $\beta \in C^1(\mathbb{R})$ satisfies the requirements (β_1)-(β_4).

3. Proof of Theorem 2

We follow the same lines as in the proof of Theorem 1 in section 2, but now taking into account that we only have that $a > 0$. We point out only the differences in the treatment. As in the proof of Theorem 1 we assume there is an $\alpha > 0$ such that problem (P_0) has no solutions and fix numbers $\rho, r \in \mathbb{R}$ with $0 < \rho < r$. Consider now $\beta \in C^1(\mathbb{R})$ which fulfils the conditions (β_1)-(β_4) with $\varepsilon = a$. We introduce the C^1 function $F : H^1_0(\Omega) \times \mathbb{R} \to \mathbb{R}$ by

$$F(v,t) = \frac{1}{2}(a + t^2)\|v\|^2 + \beta(t) + \frac{1}{2}\int_\Omega (v^+)^2\,dx + \frac{t^2}{2}\int_\Omega (v^-)^2\,dx - \langle f, v\rangle \quad (18)$$

for all $(v, t) \in H^1_0(\Omega) \times \mathbb{R}$. Assumption (i) in Lemma 1 is justified as in the proof of Theorem 1. By (18) and (β_2) we derive

$$\begin{aligned} F(v,\rho) &\ge \frac{a}{2}\,\|v\|^2 + \beta(\rho) - \|f\|_{H^{-1}(\Omega)}\,\|v\| \\ &\ge \beta(\rho) - \frac{1}{2a}\|f\|^2_{H^{-1}(\Omega)} = \frac{\alpha}{2}, \quad \forall v \in H^1_0(\Omega). \end{aligned}$$

Thus (ii) of Lemma 1 is valid with $\delta = \alpha/2$.

Let us check that (iii) of Lemma 1 is verified, i.e. the Palais-Smale condition is true for the functional $F : H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ in (18). To this end let $\{(v_n, t_n)\} \subset H_0^1(\Omega) \times \mathbb{R}$ be a sequence such that (3) and (4) hold for F in (18). We infer from (3) and (18) that

$$M \geq \beta(t_n) - \frac{1}{2a}\|f\|^2_{H^{-1}(\Omega)}.$$

This in conjunction with (β_3) entails (6). Again by (3) and (18) we get

$$M \geq \frac{a}{2}\|v_n\|^2 + \beta(t_n) - \|f\|_{H^{-1}(\Omega)}\|v_n\|.$$

Due to (6) we conclude that (7) is also true. Proceeding now as in the proof of Theorem 1 we deduce that condition (iii) in Lemma 1 is satisfied.

We are in a position to apply Lemma 1 to the functional $F(v,t)$ in (18) which produces a point $(u,s) \in H_0^1(\Omega) \times \mathbb{R}$ such that we have (9) and

$$(a+s^2)(-\Delta u) + u^+ - s^2 u^- - f = 0 \ \text{ in } H^{-1}(\Omega), \tag{19}$$

$$\frac{1}{2}(a+s^2)\|u\|^2 + \beta(s) + \frac{1}{2}\int_\Omega (u^+)^2\,dx + \frac{s^2}{2}\int_\Omega (u^-)^2\,dx - \langle f, u\rangle \geq \frac{\alpha}{2}. \tag{20}$$

Because of (9), we may justify as in the proof of Theorem 1 that either $s = 0$ or $\rho \leq s \leq r$. Furthermore, we may eliminate like before the case $s = 0$. To handle the remaining situation $\rho \leq s \leq r$ we observe that by means of (19) and (20) we arrive at (17). From now on the proof goes on in a similar way to the one of Theorem 1. □

Remark 2. A proof of Theorem 2′, which is different from the one in section 1 based on Theorem 2, is to follow the same reasoning as in the proof of Theorem 2 considering in place of (18) the function $F : H_0^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ given by

$$F(v,t) = \frac{1}{2}(a+t^2)\|v\|^2 + \beta(t) + \frac{t^2}{2}\int_\Omega (v^+)^2\,dx + \frac{1}{2}\int_\Omega (v^-)^2\,dx - \langle f, v\rangle$$

for all $(v,t) \in H_0^1(\Omega) \times \mathbb{R}$, where $\beta \in C^1(\mathbb{R})$ satisfies conditions (β_1)-(β_4) taking $\varepsilon = a$. □

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