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A STUDY OF UNIFIED INTEGRALS INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION(GMLF)

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Abstract. Many authors have developed integrals, involving a variety of special functions. Recently Khan et.al. have developed many integral formulas involving Whittaker function, MLF, Bessel function and generalized Bessel function. This paper deals with the integrals involving GMLF which are explicitly written in terms of GWHF. Several special cases are obtained as the application of main results. In view of diverse applications of MLF in mathematical physics, the results here may be potentially applicable in some related areas.

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1. Introduction

Many integral formulas involving Bessel function, Generalized Bessel function, Whittaker function and Mittag-Leffler function have been developed by many authors (see[13]-[20]). The importance of MLF and its generalization is realized during last one and a half decades due to its direct involvement in the problems of physics, biology, engineering and applied sciences. MLF naturally occurs as the solution of fractional order differential equations and fractional order integral equations. The special function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad (\alpha, z \in \mathbb{C}, \Re(\alpha) > 0) \tag{1}$$

with $\mathbb{C}$ being the set of complex numbers is called MLF. The function $E_\alpha(z)$ was introduced by Mittag-Leffler [9] [10], in connection with his method of summation of some divergent series. By assigning the values $\alpha = 2$ and $\alpha = 4$ respectively, we get

$$E_2(z) = \cosh(\sqrt{z}), z \in \mathbb{C} \tag{2}$$

$$E_4(z) = \frac{1}{2}\left[\cos(z^{\frac{1}{4}}) + \cosh(z^{\frac{1}{4}})\right], z \in \mathbb{C} \tag{3}$$

The generalization of $E_\alpha(z)$ was studied by Wiman [3] and he defined the function as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \quad (\alpha, \beta, z \in \mathbb{C}, [\Re(\alpha), \Re(\beta)] > 0) \tag{4}$$

by assigning the values $\alpha = 1, \beta = 2$ and $\alpha = 2, \beta = 2$ respectively, we get

$$E_{1,2}(z) = \frac{e^z - 1}{z}, z \in \mathbb{C} \tag{5}$$

$$E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}, z \in \mathbb{C} \tag{6}$$

Prabhakar [19] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the following form.

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta + \alpha n) n!}, (\alpha, \beta, \gamma, z \in \mathbb{C}, [\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0) \tag{7}$$

where $(\gamma)_n$ is the Pochammer's symbol [5] defined as

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma\gamma} \tag{8}$$

Now the further generalization of MLF was studied by Shukla and Prajapati [2] in the following form

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\beta + \alpha n) n!} \tag{9}$$

where $\alpha, \beta, \gamma, z \in \mathbb{C}$, $[\Re(\alpha), \Re(\beta), \Re(\gamma)] > 0$ and $q \in (0, 1) \cup \mathbb{N}$

Tariq O. Salim [17] introduced the function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ and defined as.

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta + \alpha n)(\delta)_n} \tag{10}$$

where $\alpha, \beta, \gamma, \delta, z \in \mathbb{C}$ and $[\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)] > 0$.

In 2012, a new generalization of Mittag-Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,s}(z)$ was introduced by Salim and Faraz[18], and defined as

$$E^{\gamma,\delta,q}_{\alpha,\beta,s}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\beta + \alpha n)(\delta)_{sn}} \tag{11}$$

where $\alpha, \beta, \gamma, \delta, z \in \mathbb{C}$ and $[\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)] > 0$.

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [4] and Wright [6]-[8], who studied the asymptotic expansion of the GWHF defined in [1] [11].

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1 \ , \ A_1), & \ldots.., & (\alpha_p \ , \ A_p); \\ \\ (\beta_1 \ , \ B_1), & \ldots.., & (\beta_q \ , \ B_q); \end{matrix} \ z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \ \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \ \Gamma(\beta_j + B_j k)} \ \frac{z^k}{k!} \tag{12} $$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$ (i) 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \quad and \quad 0 < |z| < \infty; z \neq 0 $$

$$ (ii) 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \quad and \quad 0 < |z| < A_1^{-A_1} \cdots A_p^{-A_p} B_1^{-B_1} \cdots B_q^{-B_q} $$

A special case of (12) is the Wright function [11, p-21]

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1 \ , \ 1), & \ldots.., & (\alpha_p \ , \ 1); \\ \\ (\beta_1 \ , \ 1), & \ldots.., & (\beta_q \ , \ 1); \end{matrix} \ z \right] = \frac{\prod_{j=1}^{p} \ \Gamma(\alpha_j)}{\prod_{j=1}^{q} \ \Gamma(\beta_j)} \ {}_pF_q \left[\begin{matrix} \alpha_1, & \ldots.., & \alpha_p \ ; \\ \\ \beta_1, & \ldots.., & \beta_q \ ; \end{matrix} \ z \right] \tag{13} $$

where ${}_pF_q$ is the generalized hypergeometric function defined by [5].

$$ \begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, & \ldots.., & \alpha_p \ ; \\ \\ \beta_1, & \ldots.., & \beta_q \ ; \end{matrix} \ z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \ \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \cdots, \alpha_p; \ \beta_1, \cdots \beta_q; \ z) \end{aligned} \tag{14} $$

Now for our present investigation, the following interesting and useful results due to Lavoie and Trottier [12] will be required.

$$ \int_0^1 x^{\alpha-1} \ (1-x)^{2\beta-1} \left(1 - \frac{x}{3}\right)^{2\alpha-1} \left(1 - \frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha} \ \frac{\Gamma(\alpha) \ \Gamma(\beta)}{\Gamma(\alpha + \beta)} \tag{15} $$

where $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. Integral formulas involving GMLF

In this section we establish two unified integral formulas involving GMLF, which are expressed in terms GWHF and then transformed into GHF.

Theorem 1. *The following integral formula holds:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma\Gamma\delta}{\Gamma\gamma}\,{}_3\Psi_3\left[\begin{array}{cccc}(\rho,1) & (\gamma,q) & (1,1); & \\ & & & \frac{4y}{9}\\ (\rho+\sigma,1) & (\delta,s) & (\beta,\alpha); & \end{array}\right] \tag{16}$$

where $\rho,\sigma,\alpha,\beta,\gamma,\delta,y\in\mathbb{C}$, $q,s\in\mathbb{N}$ *and* $[\Re(\rho),\Re(\sigma),\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)]>0$.

Proof: By using (11) in the left hand side of (16) and then interchanging the order of integral and summation which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}y^n}{\Gamma(\alpha n+\beta)(\delta)_{sn}}\int_0^1 x^{\rho+n-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho+2n-1}\left(1-\frac{x}{4}\right)^{\sigma-1}$$

Now using (15) in above equation, we get

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)(\gamma)_{qn}(4y/9)^n n!}{\Gamma(\alpha n+\beta)\Gamma(\rho+\sigma+n)(\delta)_{sn}n!}$$

which upon using (8) yields (16).

Theorem 2. *The following integral formula holds:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho\Gamma\delta}{\Gamma\gamma}\,{}_3\Psi_3\left[\begin{array}{cccc}(\sigma,1) & (\gamma,q) & (1,1); & \\ & & & y\\ (\rho+\sigma,1) & (\delta,s) & (\beta,\alpha); & \end{array}\right] \tag{17}$$

where $\rho,\sigma,\alpha,\beta,\gamma,\delta,y\in\mathbb{C}$, $q,s\in\mathbb{N}$ *and* $[\Re(\rho),\Re(\sigma),\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)]>0$

Proof: Theorem 2, can be obtained by similar steps as in proof of Theorem 1.

Next we consider the other variations of our main results 16 and 17 . In fact we establish integral formulas for the GMLF which are expressed in terms of GHF.

3. Variation of 16 and 17

Corollary 3. *Let the conditions of Theorem 1 be satisfied, with the restriction $\alpha \in \mathbb{N}$, then the following integral holds:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\sigma)\Gamma(\rho)}{\Gamma(\rho+\sigma)\Gamma(\beta)}\ {}_{q+1}F_{\alpha+s+1}\left[\begin{array}{cccc}\Delta(q;\gamma), & -, & \rho; & \\ & & & \frac{4yq^q}{9\alpha^\alpha s^s}\\ \Delta(\alpha;\beta), & \Delta(s;\delta), & \rho+\sigma; & \end{array}\right] \tag{18}$$

Corollary 4. *Let the conditions of Theorem 2 be satisfied, with the restriction $\alpha \in \mathbb{N}$, then the following integral holds:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,q}_{\alpha,\beta,s}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\sigma)\Gamma(\rho)}{\Gamma(\rho+\sigma)\Gamma(\beta)}\ {}_{q+1}F_{\alpha+s+1}\left[\begin{array}{cccc}\Delta(q;\gamma), & -, & \sigma; & \\ & & & \frac{yq^q}{\alpha^\alpha s^s}\\ \Delta(\alpha;\beta), & \Delta(s;\delta), & \rho+\sigma; & \end{array}\right] \tag{19}$$

Proof: Corollary 3 and Corollary 4 can be obtained from Theorem 1 and Theorem 2 respectively by using the following result (see[5])

$$(\lambda)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n, \qquad n \in Z^+ \ \textit{and}\ m \in N$$

and summing up the given series with the help of (14), we easily arrive at the right hand side of (18) and (19), where $\Delta(k;\lambda)$ abbreviates the arrangement of k parameters as $\frac{\lambda}{k}, \frac{\lambda+1}{k}, ..., \frac{\lambda+k-1}{k}$ where $k \geq 1$.

4. Special cases

All the following special cases are true under the same conditions, as the condition of the main results in the form of Theorem 1 and Theorem 2.

Corollary 5. *If we put q=s=1 in (16) and use (10), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta,}_{\alpha,\beta,}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma\Gamma\delta}{\Gamma\gamma}{}_3\Psi_3\left[\begin{array}{cccc}(\rho,1) & (\gamma,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (\delta,1) & (\beta,\alpha); & \end{array}\right] \tag{20}$$

Corollary 6. *If we put $\delta = s = 1$ in (16) and use (9), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,q}_{\alpha,\beta,}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma}{\Gamma\gamma}{}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (\gamma,q); & \frac{4y}{9}\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{21}$$

Corollary 7. *If we put $\delta = s = q = 1$ in (16) and use (7), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma}_{\alpha,\beta,}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma}{\Gamma\gamma}{}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (\gamma,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{22}$$

Corollary 8. *If we put $\gamma = \delta = s = q = 1$ in (16) and use (4), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E_{\alpha,\beta}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{23}$$

Corollary 9. *If we put $\gamma = \delta = \alpha = s = q = 1$ and $\beta = 2$ in (16) and use (5), then we get the following integral formula:*

$$\int_0^1 x^{\rho-2}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-3}\left(1-\frac{x}{4}\right)^{\sigma-1}[e^{yx\left(1-\frac{x}{3}\right)^2}-1]dx$$

$$= y\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (2,1); & \end{array}\right] \tag{24}$$

Corollary 10. *If we put $\gamma=\delta=s=q=1$ and $\alpha=\beta=2$ in (16) and use (6), then we get the following integral formula:*

$$\int_0^1 x^{\rho-3/2}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-2}\left(1-\frac{x}{4}\right)^{\sigma-1}\sinh\left[\sqrt{xy}\left(1-\frac{x}{3}\right)\right]dx$$

$$= \sqrt{y}\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (2,2); & \end{array}\right] \tag{25}$$

Corollary 11. *If we put $\gamma=\delta=s=\beta=q=1$ in (16) and use (1), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_\alpha\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (1,\alpha); & \end{array}\right] \tag{26}$$

Corollary 12. *If we put $\gamma=\delta=s=\beta=q=1$ and $\alpha=2$ in (16) and use (2), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\cosh\left[\sqrt{xy}\left(1-\frac{x}{3}\right)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (1,2); & \end{array}\right] \tag{27}$$

Corollary 13. *If we put $\gamma=\delta=s=\beta=q=1$ and $\alpha=4$ in (16) and use (3), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[\cos\left[xy\left(1-\frac{x}{3}\right)\right]^{\frac{1}{4}}\right.$$

$$\left.+\cosh\left[xy\left(1-\frac{x}{3}\right)\right]^{\frac{1}{4}}\right]dx = 2\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (1,1); & \frac{4y}{9}\\ (\rho+\sigma,1) & (1,4); & \end{array}\right] \tag{28}$$

Corollary 14. *If we put $s = q = 1$ in (17) and use (10), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,\delta}_{\alpha,\beta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho\Gamma\delta}{\Gamma\gamma}{}_3\Psi_3\left[\begin{array}{cccc}(\sigma,1) & (\gamma,1) & (1,1); & \\ & & & y\\ (\rho+\sigma,1) & (\delta,1) & (\beta,\alpha); & \end{array}\right] \tag{29}$$

Corollary 15. *If we put $\delta = s = 1$ in (17) and use (9), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma,q}_{\alpha,\beta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho}{\Gamma\gamma}{}_2\Psi_2\left[\begin{array}{ccc}(\rho,1) & (\gamma,q); & \\ & & y\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{30}$$

Corollary 16. *If we put $\delta = s = q = 1$ in (17) and use (7), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma}_{\alpha,\beta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho}{\Gamma\gamma}{}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (\gamma,1); & \\ & & y\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{31}$$

Corollary 17. *If we put $\gamma = \delta = s = q = 1$ in (17) and use (4), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} E_{\alpha,\beta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\ \ {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (\beta,\alpha); & \end{array}\right] \tag{32}$$

Corollary 18. *If we put $\gamma = \delta = s = q = 1$ and $\alpha = 1, \beta = 2$ in (17) and use (5), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-3}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-2}[e^{y(1-x)^2\left(1-\frac{x}{4}\right)}-1]dx$$

$$= y\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho \;\; {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (2,1); & \end{array}\right] \tag{33}$$

Corollary 19. *If we put $\gamma=\delta=s=q=1$ and $\alpha=2,\beta=2$ in (17) and use (6), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-2}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-3/2}\sinh\left[\sqrt{y\left(1-\frac{x}{4}\right)}(1-x)\right]dx$$

$$= \sqrt{y}\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho \;\; {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (2,2); & \end{array}\right] \tag{34}$$

Corollary 20. *If we put $\gamma=\delta=s=\beta=q=1$ in (17) and use (1), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_\alpha\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\rho \;\; {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (1,\alpha); & \end{array}\right] \tag{35}$$

Corollary 21. *If we put $\gamma=\delta=s=\beta=q=1$ and $\alpha=2$ in (17) and use (2), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\cosh\left[\sqrt{y\left(1-\frac{x}{4}\right)}(1-x)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\rho \;\; {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (1,2); & \end{array}\right] \tag{36}$$

Corollary 22. *If we put $\gamma=\delta=s=\beta=q=1$ and $\alpha=4$ in (17) and use (3), then we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[\cos\left[y(1-x)\left(1-\frac{x}{4}\right)\right]^{\frac{1}{4}}\right.$$

$$\left.+\cosh\left[y(1-x)\left(1-\frac{x}{4}\right)\right]^{\frac{1}{4}}\right]dx = 2\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho \;\; {}_2\Psi_2\left[\begin{array}{ccc}(\sigma,1) & (1,1); & \\ & & y\\ (\rho+\sigma,1) & (1,4); & \end{array}\right] \tag{37}$$

5. Concluding Remark

In the present paper, I have investigated two unified integrals involving GMLF ,which are expressed in terms of GWHF and GHF. The result presented in this paper are easily converted in various forms of MLF after assigning some particular values to the parameters. Since MLF are associated with wide range of problems in diverse fields of mathematical physics. The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions of mathematical physics.

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