Acta Universitatis Apulensis No. 71/2022
ISSN: 1582-5329 pp. 137-145
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2022.71.10

ON FEKETE-SZEGÖ PROBLEM FOR A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY BERNOULLI POLYNOMIALS

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Abstract. In this paper, we introduce and investigate a new subclass of bi-univalent functions associated with the Bernoulli polynomials, which satisfy subordination conditions defined in the open unit disc. For this new subclass, we obtain estimates for the Taylor-Maclaurin coeffcients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $\left|a_3 - \mu a_2^2\right|$.

2010 *Mathematics Subject Classification*: 30C45, 30C50.

Keywords: Bernoulli polynomial, subordination, bi-univalent functions, analytic functions, Fekete-Szegö problem.

1. Introduction

Let A represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \Delta), \tag{1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A subclass of A is denoted by S whose members are univalent in Δ. The Koebe one quarter theorem [7] ensures that the image of Δ under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in \Delta) \text{ and } f\left(f^{-1}(\omega)\right) = \omega, (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

We say that $f \in A$ is bi-univalent in Δ if f and f^{-1} are univalent in Δ, and we denote by Σ the class of bi-univalent functions defined in the unit disk Δ. Since

$f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that $g = f^{-1}$ has the expansion

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2{a_2}^2 - a_3\right)\omega^3 + \ldots. \tag{2}$$

We know that the class Σ is not empty. For instance, the functions

$$f_1(z) = \frac{z}{z-1},\ f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z},\ f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega},\ f_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1},\ f_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

are elements of Σ.

However, the Koebe function is not a member of Σ.

In the paper [23] is revived the study of analytic and bi-univalent functions; this was followed by such works as those of [4, 5, 10, 14, 15, 18]. Several authors have introduced new subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [4, 5, 6, 14, 20, 22, 23]).

Let f and g be the analytic functions in Δ. We say that f is subordinate to g and denoted by

$$f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$ such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

If g is a univalent function in Δ, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

In [14], by means of Loewner's method, the Fekete-Szegö inequality for the coefficients of $f \in S$ is that

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2exp\left(\frac{-2\mu}{1-\mu}\right) \text{ for } 0 \leq \mu < 1.$$

As $\mu \to 1^-$, the elementary inequality $\left|a_3 - a_2^2\right| \leq 1$ is obtained. Moreover, the coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions, f in the open unit disk Δ plays an important role in geometric function theory. The problem of maximizing the absolute value of the functional $F_\mu(f)$ is called the Fekete-Szegö problem.

The Fekete–Szegö inequalities introduced in 1933, see [9], preoccupied researchers regarding different classes of univalent functions [8, 12, 16, 24]; hence, it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [1, 2, 25].

In 1980, Gradshteyn and Ryzhik [11] give an expression of the Bernoulli polynomials which have important applications in number theory and classical analysis. They appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-Maclaurin quadrature rule.

The Bernoulli polynomials $B_n(x)$ are usually defined (see, e.g, [17]) by means of the generating function:

$$F(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \ |t| < 2\pi, \tag{3}$$

where $B_n(x)$ are polynomials in x, for each nonnegative integer n.

The Bernoulli polynomials are easily computed by recursion since

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j(x) = nx^{n-1}, n = 2, 3, ... \tag{4}$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \ B_1(x) = x - \frac{1}{2}, \ B_2(x) = x^2 - x + \frac{1}{6}, \ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots. \tag{5}$$

First, we define new subclasses of bi-univalent functions, associated with Bernoulli polynomials.

Definition 1. *We say that f the form* (1) *is in the class $M_\Sigma(\tau, \vartheta; F)$, for $\tau \in \mathbb{C}\backslash\{0\}$ and $0 \le \vartheta \le 1$, if the following subordinations hold:*

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) \prec F(x, z), \tag{6}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega g''(\omega) - 1) \prec F(x, \omega), \tag{7}$$

$z, \omega \in \Delta$, F is given by (3), *and $g = f^{-1}$ is given by* (2).

2. Initial Taylor coefficients estimates for the class $M_{\Sigma}(\tau,\vartheta;F)$

Lemma 1. *([19], p.172) Assume that $w(z)=\sum_{n=1}^{\infty}w_n z^n,\ z\in\Delta$, be an analytic function in Δ such that $|w(z)|<1,\ z\in\Delta$. Then,*

$$|w_1|\le 1, |w_n|\le 1-|w_1|^2,\ n=2,3,\ldots.$$

For the functions belonging to a class $M_{\Sigma}(\tau,\vartheta;F)$, we will obtain upper bounds for the modulus of coefficients a_2 and a_3.

Theorem 2. *If the class $M_{\Sigma}(\tau,\vartheta;F)$ contains all the functions f given by (1), then*

$$|a_2|\le\frac{|\tau|\,|B_1(x)|\sqrt{|B_1(x)|}}{\sqrt{\left|3\tau(1+2\vartheta)(B_1(x))^2-2(1+\vartheta)^2B_2(x)\right|}},\tag{8}$$

and

$$|a_3|\le\frac{|\tau|\,B_1(x)}{3\,|1+2\vartheta|}+\frac{|\tau|^2[B_1(x)]^2}{4\,|1+\vartheta|^2}.\tag{9}$$

Proof. Let $f\in M_{\Sigma}(\tau,\vartheta;F)$ and $g=f^{-1}$. From the definition in formulas (6) and (7), we have

$$1+\frac{1}{\tau}(f'(z)+\vartheta zf''(z)-1)=F(x,Y(z))\tag{10}$$

and

$$1+\frac{1}{\tau}(g'(\omega)+\vartheta\omega g''(\omega)-1)=F(x,X(\omega))\tag{11}$$

where the functions Y and X are of the form

$$Y(z)=r_1z+r_2z^2+\ldots,\tag{12}$$

$$X(\omega)=s_1\omega+s_2\omega^2+\ldots,\tag{13}$$

are analytic in Δ with $Y(0)=0=X(0)$, and $|Y(z)|<1,\ |X(\omega)|<1$, for all $z,\omega\in\Delta$.

It follows that, from Lemma 1, that

$$|r_j|\le 1 \text{ and } |s_j|\le 1, \text{where } j\in\mathbb{N}.\tag{14}$$

If we replace (12) and (13) in (10) and (11), respectively, we obtain

$$1+\frac{1}{\tau}(f'(z)+\vartheta zf''(z)-1)+\cdots=B_0(x)+B_1(x)Y(z)+\frac{B_2(x)}{2!}Y^2(z)+\ldots\tag{15}$$

and

$$1+\frac{1}{\tau}(g'(\omega)+\vartheta g''(\omega)-1)+\cdots=B_0(x)+B_1(x)X(\omega)+\frac{B_2(x)}{2!}X^2(\omega)+\ldots. \quad (16)$$

In view of (1) and (2), from (15) and (16), we obtain

$$\begin{aligned} & 1+\frac{1}{\tau}\left(2a_2(1+\vartheta)z+3a_3(1+2\vartheta)z^2\right)+\ldots \\ = \ & 1+B_1(x)r_1z+\left[B_1(x)r_2+\frac{B_2(x)}{2!}r_1^2\right]z^2+\ldots \end{aligned}$$

and

$$\begin{aligned} & 1+\frac{1}{\tau}\left(-2a_2(1+\vartheta)\omega+3\left(2a_2^2-a_3\right)(1+2\vartheta)\omega^2\right)+\ldots \\ = \ & 1+B_1(x)s_1\omega+\left[B_1(x)s_2+\frac{B_2(x)}{2!}s_1^2\right]\omega^2+\ldots \end{aligned}$$

which yields the following relations:

$$2a_2(1+\vartheta)=\tau B_1(x)r_1, \quad (17)$$

$$3a_3(1+2\vartheta)=\tau\left[B_1(x)r_2+\frac{B_2(x)}{2!}r_1^2\right], \quad (18)$$

and

$$-2a_2(1+\vartheta)=\tau B_1(x)s_1, \quad (19)$$

$$3\left(2a_2{}^2-a_3\right)(1+2\vartheta)=\tau\left[B_1(x)s_2+\frac{B_2(x)}{2!}s_1^2\right]. \quad (20)$$

From (17) and (19), it follows that

$$r_1=-s_1, \quad (21)$$

and

$$\begin{aligned} 8a_2^2(1+\vartheta)^2 &= \tau^2[B_1(x)]^2\left(r_1^2+s_1^2\right) \\ a_2^2 &= \frac{\tau^2[B_1(x)]^2\left(r_1^2+s_1^2\right)}{8(1+\vartheta)^2}. \end{aligned} \quad (22)$$

Adding (18) and (20), using (22), we obtain

$$a_2^2=\frac{\tau^2[B_1(x)]^3(r_2+s_2)}{6\tau(1+2\vartheta)[B_1(x)]^2-4(1+\vartheta)^2B_2(x)}. \quad (23)$$

Using the relation (5), from (14) for r_2 and s_2 we get (8).

Using (21) and (22), by subtracting (20) from the relation (18), we get

$$\begin{aligned} a_3 &= \frac{\tau\left[B_1(x)(r_2-s_2)+\frac{B_2(x)}{2!}(r_1^2-s_1^2)\right]}{6(2\vartheta+1)}+a_2^2 \\ &= \frac{\tau\left[B_1(x)(r_2-s_2)+\frac{B_2(x)}{2!}(r_1^2-s_1^2)\right]}{6(2\vartheta+1)}+\frac{\tau^2[B_1(x)]^2\left(r_1^2+s_1^2\right)}{8(1+\vartheta)^2}. \end{aligned} \tag{24}$$

Once again applying (14) and using (5), for the coefficients r_1, s_1, r_2, s_2, we deduce (9).

3. The Fekete-Szegö problem for the Function Class $M_\Sigma(\tau, \vartheta; F)$

We will obtain the Fekete-Szegö inequality for the class $M_\Sigma(\tau, \vartheta; F)$, due to the result of Zaprawa, see [25].

Theorem 3. *If f given by* (1) *is in the class $M_\Sigma(\tau, \vartheta; F)$ where $\mu \in \mathbb{R}$, then, we have*

$$\left|a_3-\mu {a_2}^2\right| \leq \begin{cases} \frac{|\tau| B_1(x)}{3|1+2\vartheta|}, & if \quad |h(\mu)| \leq \frac{1}{6|1+2\vartheta|}, \\ 2\,|\tau|\,|h(\mu)|\,B_1(x)\,, & if \quad |h(\mu)| \geq \frac{1}{6|1+2\vartheta|}, \end{cases}$$

where

$$h(\mu)=\frac{(1-\mu)\tau[B_1(x)]^2}{6\tau(1+2\vartheta)[B_1(x)]^2-4(1+\vartheta)^2B_2(x)}.$$

Proof. If $f \in M_\Sigma(\tau, \vartheta; F)$ is given by (1), from (23) and (24), we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\tau B_1(x)(r_2-s_2)}{6(1+2\vartheta)}+(1-\mu)a_2^2 \\ &= \frac{\tau B_1(x)(r_2-s_2)}{6(1+2\vartheta)}+\frac{(1-\mu)\tau^2[B_1(x)]^3(r_2+s_2)}{6\tau(1+2\vartheta)\left(B_1(x)^2-4(1+\vartheta)^2B_2(x)\right)} \\ &= \tau B_1(x)\left[\frac{r_2}{6(1+2\vartheta)}-\frac{s_2}{6(1+2\vartheta)}+\frac{(1-\mu)\tau[B_1(x)]^2r_2}{6\tau(1+2\vartheta)\left(B_1(x)^2-4(1+\vartheta)^2B_2(x)\right)}\right. \\ &\quad \left.+\frac{(1-\mu)\tau[B_1(x)]^2s_2}{6\tau(1+2\vartheta)\left(B_1(x)^2-4(1+\vartheta)^2B_2(x)\right)}\right] \\ &= \tau B_1(x)\left[\left(h(\mu)+\frac{1}{6(1+2\vartheta)}\right)r_2+\left(h(\mu)-\frac{1}{6(1+2\vartheta)}\right)s_2\right], \end{aligned}$$

where

$$h(\mu) = \frac{\tau(1-\mu)[B_1(x)]^2}{6\tau(1+2\vartheta)\left([B_1(x)]^2 - 4(1+\vartheta)^2 B_2(x)\right)}.$$

Now, by using (5)

$$a_3 - \mu a_2^2 = \tau\left(x - \frac{1}{2}\right)\left[\left(h(\mu) + \frac{1}{6(1+2\vartheta)}\right) r_2 + \left(h(\mu) - \frac{1}{6(1+2\vartheta)}\right) s_2\right],$$

where

$$h(\mu) = \frac{\tau(1-\mu)\left(x-\frac{1}{2}\right)^2}{6\tau(1+2\vartheta)\left(x-\frac{1}{2}\right)^2 - 4(1+\vartheta)^2\left(x^2 - x + \frac{1}{6}\right)}.$$

Therefore, given (5) and (14), we conclude that the required inequality holds.

4. Conclusions

In this paper, we introduced and investigated a new subclass of bi-univalent functions in the open unit disk defined by Bernoulli polynomials and satisfies subordination conditions. Furthermore, we obtain upper bounds for $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $\left|a_3 - \mu {a_2}^2\right|$ for functions in this subclass.

Also, the approach presented here has been extended to establish new subfamilies of bi-univalent functions with the other special functions. The related outcomes may be left to the the researchers for practice.

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