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CERTAIN RESULTS ON G-METRIC SPACES

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Abstract. The principal object of this paper is to present new contractive type condition for mappings defined on G-metric spaces. Further, we prove some new fixed point theorems concerning these mappings in G-metric space. Particular cases and examples to illustrate and support our results are also considered. Our findings extend and unify a known results.

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1. Introduction

The Banach contraction mapping principle is widely considered the source of metric fixed point theory, and its significance is in its application in several branches of mathematics. Hence, there are many numerous generalizations of the Banach contraction principle. In 2006 Zead Mustafa introduced the notion of G-metric spaces [24] as the generalization of ordinary metrics and analyzed the topological structure of the G-metric spaces see, e.g., [1-4, 5-8, 9-13, 16-28]. In 2012, Jleli and Samet [16] and Samet et.al [27] showed that some fixed point theorems in G-metric spaces can be deduced from standard metric spaces or quasi-metric spaces. Karapinar and Agarwal [17] proved that the approach of Jleli and Samet [16] and Samet et.al [27] cannot be applied if the contraction condition in the statement of the theorem can't be reducible to two variables and the introduced and proved diverse results in G-metric spaces. after that, several authors published many fixed-point results on the setting of G-metric space and G_b-metric space (see e.g. [14-15]). In this paper, we prove fixed point theorems for self-mappings satisfying some kind of contractive type conditions on complete G-metric spaces are generalizations of previous researchers.

Definition 1 (24)**.** *Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$*;*

(G2) $0 < G(x,x,y)$ *for all* $x, y \in X$ *with* $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x, y, x \in X$ *with* $y \neq z$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...,$*(symmetry in all three variables);*

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality).*

Then the function G *is called a* $G-metric$ *on* X*, and the pair* (X,G) *is called a* $G-metric$ *space.*

Another useful definitions are follow:

Definition 2 (24). *Let* (X,G) *be a* $G-metric$ *space. We say that* $\{x_n\}$ *is*

(1) *a G-Cauchy sequence if, for any* $\epsilon > 0$*, there is* $N \in \mathbb{N}$ *(the set of all positive integers) such that for all* $n, m, l \geq N$, $G(x_n, x_m, x_l) < \epsilon$;

(2) *a G-convergent sequence to* $x \in X$ *if, for any* $\epsilon > 0$*, there is* $N \in \mathbb{N}$ *such that for all* $n, m \geq N$, $G(x, x_n, x_m) < \epsilon$.

(3) *a G- metric space* (X,G) *is said to be complete if every G-Cauchy sequence in* X *is G-convergent in* X.

Definition 3 (24). *Let* (X,G) *and* $(\acute{X},\acute{G})$ *be two* $G-metric$ *spaces, and let* $f : (X,G) \to (\acute{X},\acute{G})$ *be a fuction, then* f *is said to be G-continuous at a point* $a \in X$ *if and only if, given* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $x, y \in X$*; and* $G(a,x,y) < \delta$ *implies* $\acute{G}(f(a), f(x), f(y)) < \epsilon$*. A function* f *is G-continuous at* X *if and only if it is G-continuous at all* $a \in X$.

Definition 4 (24). *A* $G-metric$ *space* (X,G) *is called symmetric* $G-metric$ *space if*
$G(x,y,y) = G(y,x,x)$ *for all* $x, y \in X$.

2. Certain Theorems for $G-metric$ space

In this section, we establish fixed point theorems for self mappings satisfying various contractive conditions on complete $G-metric$ spaces.

Theorem 1. *Let* (X,d) *be a complete* $G-metric$ *space and define the sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *by the iteration* $x_n = Tx_{n-1} = T^n x_0$ *and let* $T : X \to X$ *be a mapping such that*

$$\begin{aligned} G(Tx,Ty,Tz) \leq{} & \lambda_1 G(x,y,z) + \lambda_2 G(x,Tx,Tx) + \lambda_3 G(y,Ty,Ty) + \lambda_4 G(z,Tz,Tz) \\ & + \lambda_5 \{G(x,Ty,Ty) + G(y,Tx,Tx) + G(z,Tx,Tx) + G(x,Tz,Tz) \\ & + G(z,Ty,Ty) + G(y,Tz,Tz)\} \qquad (1) \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 < 1$ *for all* $x, y, z \in X$ *then there exists* $x^* \in X$ *such that* $x_n \to x^*$ *and* x^* *is a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = Tx_{n-1} = T^n x_0$. Assume $x_n \neq x_{n+1}$, $n = 1, 2, 3....$Then by (3.1), we get

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq \lambda_1 G(x_{n-1}, x_n, x_n) + \lambda_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \lambda_3 G(x_n, Tx_n, Tx_n) \\
&+ \lambda_4 G(x_n, Tx_n, Tx_n) + \lambda_5 \{G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1}) \\
&+ G(x_n, Tx_{n-1}, Tx_{n-1}) + G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_n) \\
&+ G(x_n, Tx_n, Tx_n)\} \\
&\leq \lambda_1 G(x_{n-1}, x_n, x_n) + \lambda_2 G(x_{n-1}, x_n, x_n) + \lambda_3 G(x_n, x_{n+1}, x_{n+1}) \\
&+ \lambda_4 G(x_n, x_{n+1}, x_{n+1}) + \lambda_5 \{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) \\
&+ G(x_n, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \\
&+ G(x_n, x_{n+1}, x_{n+1})\} \\
&\leq \lambda_1 G(x_{n-1}, x_n, x_n) + \lambda_2 G(x_{n-1}, x_n, x_n) + \lambda_3 G(x_n, x_{n+1}, x_{n+1}) \\
&+ \lambda_4 G(x_n, x_{n+1}, x_{n+1}) + \lambda_5 \{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\
&+ G(x_n, x_n, x_n) + G(x_n, x_n, x_n) + G(x_{n-1}, x_n, x_n) \\
&+ G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}
\end{aligned}$$

$$\begin{aligned}
(1 - \lambda_3 - \lambda_4 - 4\lambda_5) G(x_n, x_{n+1}, x_{n+1}) &\leq (\lambda_1 + \lambda_2 + 2\lambda_5) G(x_{n-1}, x_n, x_n) \\
G(x_n, x_{n+1}, x_{n+1}) &\leq \frac{\lambda_1 + \lambda_2 + 2\lambda_5}{1 - \lambda_3 - \lambda_4 - 4\lambda_5} G(x_{n-1}, x_n, x_n) \qquad (2) \\
G(x_n, x_{n+1}, x_{n+1}) &\leq \eta G(x_{n-1}, x_n, x_n),
\end{aligned}$$

where $\eta = \dfrac{\lambda_1 + \lambda_2 + 2\lambda_5}{1 - \lambda_3 - \lambda_4 - 4\lambda_5}$

As $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 < 1$
and
$\lambda_1 + \lambda_2 + 2\lambda_5 < 1 - \lambda_3 - \lambda_4 - 4\lambda_5$

we get $\eta = \dfrac{\lambda_1 + \lambda_2 + 2\lambda_5}{1 - \lambda_3 - \lambda_4 - 4\lambda_5} < 1$ which yields the result

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &\leq \eta G(x_{n-1}, x_n, x_n) \\
&\leq \eta^2 G(x_{n-2}, x_{n-1}, x_{n-1})
\end{aligned}$$

Repeating this process we lead to

$$G(x_n, x_{n+1}, x_{n+1}) \leq \eta^n G(x_0, x_1, x_1) \tag{3}$$

Next, we show that $\{x_n\}_{n=1}^{\infty}$ is a G-Cauchy sequence in X. Let $m, n > 0$ with $m > n$

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &+ ... + G(x_{m-1}, x_m, x_m) \\ &\leq (\eta^n + \eta^{n+1} + ... + \eta^m) G(x_0, x_1, x_1) \\ &\leq \frac{\eta^n}{1-\eta} G(x_0, x_1, x_1). \end{aligned}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $\lim \frac{\eta^n}{1-\eta} G(x_0, x_1, x_1) = 0$, as $n, m \to \infty$. For $n, m, l \in N$, (**G5**) implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_l, x_m, x_m)$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u.
Let $T(u) \neq u$.
Now, we show that x^* is a fixed point of T. We have

$$\begin{aligned} G(x^*, Tx^*, Tx^*) &\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx^*, Tx^*) \\ &\leq G(x^*, x_{n+1}, x_{n+1}) + G(Tx_n, Tx^*, Tx^*) \\ &\leq G(x^*, x_{n+1}, x_{n+1}) + \lambda_1 G(x_n, x^*, x^*) \\ &+ \lambda_2 G(x_n, Tx_n, Tx_n) + \lambda_3 G(x^*, Tx^*, Tx^*) \\ &+ \lambda_4 G(x^*, Tx^*, Tx^*) + \lambda_5 \{G(x_n, Tx^*, Tx^*) \\ &+ G(x^*, Tx_n, Tx_n) + G(x^*, Tx_n, Tx_n) \\ &+ G(x_n, Tx^*, Tx^*) + G(x^*, Tx^*, Tx^*) + G(x^*, Tx^*, Tx^*)\} \\ &\leq G(x^*, x_{n+1}, x_{n+1}) + \lambda_1 G(x_n, x^*, x^*) + \lambda_2 G(x_n, x^*, x^*) \\ &+ \lambda_2 G(x^*, Tx_n, Tx_n) + \lambda_3 G(x^*, Tx^*, Tx^*) + \lambda_4 G(x^*, Tx^*, Tx^*) \\ &+ \lambda_5 \{G(x_n, x^*, x^*) + G(x^*, Tx^*, Tx^*) + G(x^*, Tx_n, Tx_n) \\ &+ G(x^*, Tx_n, Tx_n) + G(x_n, x^*, x^*) + G(x^*, Tx^*, Tx^*) \\ &+ G(x^*, Tx^*, Tx^*) + G(x^*, Tx^*, Tx^*)\} \end{aligned}$$

$$(1-\lambda_3-\lambda_4-4\lambda_5)G(x^*,Tx^*,Tx^*) \leq (1+\lambda_2+2\lambda_5)G(x^*,x_{n+1},x_{n+1})+(\lambda_1+\lambda_2+2\lambda_5)G(x_n,x^*,x^*)$$

$$G(x^*,Tx^*,Tx^*) \leq \frac{1+\lambda_2+2\lambda_5}{1-\lambda_3-\lambda_4-4\lambda_5}G(x^*,x_{n+1},x_{n+1})+\frac{\lambda_1+\lambda_2+2\lambda_5}{1-\lambda_3-\lambda_4-4\lambda_5}G(x_n,x^*,x^*).$$

Hence
$G(x^*,Tx^*,Tx^*) \leq 0$ as $n \to \infty$

That is $Tx^* = x^*$, then x^* is a fixed point of T.

Finally, we show x^* is the unique fixed point of T. Let x' is another fixed point of T, then we have $Tx' = x'$ and

$$\begin{aligned}
G(x^*,x',x') &= G(Tx^*,Tx',Tx') \\
&\leq \lambda_1 G(x^*,x',x') + \lambda_2 G(x^*,Tx^*,Tx^*) \\
&+ \lambda_3 G(x',Tx',Tx') + \lambda_4 G(x',Tx',Tx') \\
&+ \lambda_5\{G(x^*,Tx',Tx') + G(x',Tx^*,Tx^*) \\
&+ G(x',Tx^*,Tx^*) + G(x^*,Tx',Tx') \\
&+ G(x',Tx',Tx') + G(x',Tx',Tx')\} \\
&\leq \lambda_1 G(x^*,x',x') + \lambda_2 G(x^*,x^*,x^*) \\
&+ \lambda_3 G(x',x',x') + \lambda_4 G(x',x',x') + \lambda_5\{G(x^*,x',x') \\
&+ G(x',x^*,x^*) + G(x',x^*,x^*) + G(x^*,x',x') \\
&+ G(x',x',x') + G(x',x',x')\} \\
&\leq \lambda_1 G(x^*,x',x') + \lambda_5\{G(x^*,x',x') \\
&+ G(x',x^*,x^*) + G(x',x^*,x^*) + G(x^*,x',x')\} \\
&\leq \lambda_1 G(x^*,x',x') + \lambda_5\{2G(x^*,x',x') + 2G(x',x^*,x^*)\} \\
(1-\lambda_1-2\lambda_5)G(x^*,x',x') &\leq (2\lambda_5)G(x',x^*,x^*) \\
G(x^*,x',x') &\leq \frac{2\lambda_5}{1-\lambda_1-2\lambda_5}G(x',x^*,x^*) \\
&\leq \frac{4\lambda_5}{1-\lambda_1-2\lambda_5}G(x',x^*,x^*).
\end{aligned}$$

which yields $x^* = x'$ so T has a unique fixed point.

Theorem 2. *Let (X,d) be a complete $G-$metric space and define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by the iteration $x_n = Tx_{n-1} = T^n x_0$ and let $T: X \to X$ be a mapping* *such that*

$$\begin{aligned} G(Tx,Ty,Tz) \leq{} & \lambda_1 G(x,y,z) + \lambda_2 G(x,Tx,Tx) + \lambda_3 G(y,Ty,Ty) + \lambda_4 G(z,Tz,Tz) \\ & + \lambda_5 G(x,Ty,Ty) + \lambda_6 G(y,Tx,Tx) + \lambda_7 G(z,Tx,Tx) + \lambda_8 G(x,Tz,Tz) \\ & + \lambda_9 G(z,Ty,Ty) + \lambda_{10} G(y,Tz,Tz) + \lambda_{11}[G(x,Ty,Ty) + G(y,Tx,Tx) \\ & + G(z,Tx,Tx) + G(x,Tz,Tz) + G(z,Ty,Ty) + G(y,Tz,Tz)] \end{aligned} \quad (4)$$

where $\min\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11}\} \geq 0$ *with* $\lambda_1+\lambda_2+\lambda_3+\lambda_4+2\lambda_5+\lambda_6+\lambda_7+2\lambda_8+\lambda_9+\lambda_{10}+6\lambda_{11} < 1$ *for all* $x,y,z \in X$*, then there exists* $x^* \in X$ *such that* $x_n \to x^*$ *and* x^* *is a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbi and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as $x_n = Tx_{n-1} = T^n x_0$,$n = 1,2,3...$ we obtain that

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) ={} & G(Tx_{n-1},Tx_n,Tx_n) \\ \leq{} & \lambda_1 G(x_{n-1},x_n,x_n) + \lambda_2 G(x_{n-1},Tx_{n-1},Tx_{n-1}) + \lambda_3 G(x_n,Tx_n,Tx_n) \\ & + \lambda_4 G(x_n,Tx_n,Tx_n) + \lambda_5 G(x_{n-1},Tx_n,Tx_n) + \lambda_6 G(x_n,Tx_{n-1},Tx_{n-1}) \\ & + \lambda_7 G(x_n,Tx_{n-1},Tx_{n-1}) + \lambda_8 G(x_{n-1},Tx_n,Tx_n) + \lambda_9 G(x_n,Tx_n,Tx_n) \\ & + \lambda_{10} G(x_n,Tx_n,Tx_n) + \lambda_{11}\{G(x_{n-1},Tx_n,Tx_n) + G(x_n,Tx_{n-1},Tx_{n-1}) \\ & + G(x_n,Tx_{n-1},Tx_{n-1}) + G(x_{n-1},Tx_n,Tx_n) + G(x_n,Tx_n,Tx_n) \\ & + G(x_n,Tx_n,Tx_n)\} \\ \leq{} & \lambda_1 G(x_{n-1},x_n,x_n) + \lambda_2 G(x_{n-1},x_n,x_n) + \lambda_3 G(x_n,x_{n+1},x_{n+1}) \\ & + \lambda_4 G(x_n,x_{n+1},x_{n+1}) + \lambda_5 G(x_{n-1},x_{n+1},x_{n+1}) + \lambda_6 G(x_n,x_n,x_n) \\ & + \lambda_7 G(x_n,x_n,x_n) + \lambda_8 G(x_{n-1},x_{n+1},x_{n+1}) + \lambda_9 G(x_n,x_{n+1},x_{n+1}) \\ & + \lambda_{10} G(x_n,x_{n+1},x_{n+1}) + \lambda_{11}\{G(x_{n-1},x_{n+1},x_{n+1}) + G(x_n,x_n,x_n) \\ & + G(x_n,x_n,x_n) + G(x_{n-1},x_{n+1},x_{n+1}) + G(x_n,x_{n+1},x_{n+1}) \\ & + G(x_n,x_{n+1},x_{n+1})\} \\ \leq{} & \lambda_1 G(x_{n-1},x_n,x_n) + \lambda_2 G(x_{n-1},x_n,x_n) + \lambda_3 G(x_n,x_{n+1},x_{n+1}) \\ & + \lambda_4 G(x_n,x_{n+1},x_{n+1}) + \lambda_5\{G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})\} \\ & + \lambda_6 G(x_n,x_n,x_n) + \lambda_7 G(x_n,x_n,x_n) + \lambda_8\{G(x_{n-1},x_n,x_n) \\ & + G(x_n,x_{n+1},x_{n+1})\} + \lambda_9 G(x_n,x_{n+1},x_{n+1}) + \lambda_{10} G(x_n,x_{n+1},x_{n+1}) \\ & + \lambda_{11}\{G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) + G(x_n,x_n,x_n) \\ & + G(x_n,x_n,x_n) + G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) \\ & + G(x_n,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},x_{n+1})\} \end{aligned}$$

$$(1-\lambda_3-\lambda_4-\lambda_5-\lambda_8-\lambda_9-\lambda_{10}-4\lambda_{11})G(x_n,x_{n+1},x_{n+1}) \leq (\lambda_1+\lambda_2+\lambda_5+\lambda_8+2\lambda_{11})G(x_{n-1},x_n,x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\lambda_1 + \lambda_2 + \lambda_5 + \lambda_8 + 2\lambda_{11}}{1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_8 - \lambda_9 - \lambda_{10} - 4\lambda_{11}} G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \eta G(x_{n-1}, x_n, x_n)$$

where $\eta = \dfrac{\lambda_1 + \lambda_2 + \lambda_5 + \lambda_8 + 2\lambda_{11}}{1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_8 - \lambda_9 - \lambda_{10} - 4\lambda_{11}} < 1$

As $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 + 2\lambda_8 + \lambda_9 + \lambda_{10} + 6\lambda_{11} < 1$

$(\lambda_1 + \lambda_2 + \lambda_5 + \lambda_8 + 2\lambda_{11}) < (1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_8 - \lambda_9 - \lambda_{10} - 4\lambda_{11})$
$\dfrac{\lambda_1 + \lambda_2 + \lambda_5 + \lambda_8 + 2\lambda_{11}}{1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_8 - \lambda_9 - \lambda_{10} - 4\lambda_{11}} < 1$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq \eta G(x_{n-1}, x_n, x_n) \\ &\leq \eta^2 G(x_{n-2}, x_{n-1}, x_{n-1}) \end{aligned}$$

Continuing this process, we find that,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \eta^n G(x_0, x_1, x_1)$$

.
Now, we show that$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &+ ... + G(x_{m-1}, x_m, x_m) \\ &\leq (\eta^n + \eta^{n+1} + ... + \eta^m) G(x_0, x_1, x_1) \\ &\leq \frac{\eta^n}{1-\eta} G(x_0, x_1, x_1) \end{aligned}$$

Then $\lim G(x_n, x_m, x_m) = 0$ as $m, n \to \infty$.

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence,

$\{x_n\}$ converges to $x^* \in X$.
Now we show that x^* is a fixed point of T.

$$\begin{aligned}
G(x^*,Tx^*,Tx^*) &\leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n+1},Tx^*,Tx^*)\\
&\leq G(x^*,x_{n+1},x_{n+1}) + G(Tx_n,Tx^*,Tx^*)\\
&\leq G(x^*,x_{n+1},x_{n+1}) + \lambda_1 G(x_n,x^*,x^*) + \lambda_2 G(x_n,Tx_n,Tx_n)\\
&+ \lambda_3 G(x^*,Tx^*,Tx^*) + \lambda_4 G(x^*,Tx^*,Tx^*) + \lambda_5 G(x_n,Tx^*,Tx^*)\\
&+ \lambda_6 G(x^*,Tx_n,Tx_n) + \lambda_7 G(x^*,Tx_n,Tx_n) + \lambda_8 G(x_n,Tx^*,Tx^*)\\
&+ \lambda_9 G(x^*,Tx^*,Tx^*) + \lambda_{10} G(x^*,Tx^*,Tx^*) + \lambda_{11}\{G(x_n,Tx^*,Tx^*)\\
&+ G(x^*,Tx_n,Tx_n) + G(x^*,Tx_n,Tx_n) + G(x_n,Tx^*,Tx^*)\\
&+ G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\}\\
&\leq G(x^*,x_{n+1},x_{n+1}) + \lambda_1 G(x_n,x^*,x^*) + \lambda_2 G(x_n,x^*,x^*)\\
&+ \lambda_2 G(x^*,Tx_n,Tx_n) + \lambda_3 G(x^*,Tx^*,Tx^*) + \lambda_4 G(x^*,Tx^*,Tx^*)\\
&+ \lambda_5\{G(x_n,x^*,x^*) + G(x^*,Tx^*,Tx^*)\} + \lambda_6 G(x^*,Tx_n,Tx_n)\\
&+ \lambda_7 G(x^*,Tx_n,Tx_n) + \lambda_8\{G(x_n,x^*,x^*) + G(x^*,Tx^*,Tx^*)\}\\
&+ \lambda_9 G(x^*,Tx^*,Tx^*) + \lambda_{10} G(x^*,Tx^*,Tx^*) + \lambda_{11}\{G(x_n,x^*,x^*)\\
&+ G(x^*,Tx^*,Tx^*) + G(x^*,Tx_n,Tx_n)\\
&+ G(x^*,Tx_n,Tx_n) + G(x_n,x^*,x^*) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\}
\end{aligned}$$

which implies that,

$$G(x^*,Tx^*,Tx^*) \leq \frac{1+\lambda_2+\lambda_6+\lambda_7+2\lambda_{11}}{1-\lambda_3-\lambda_4-\lambda_5-\lambda_8-\lambda_9-\lambda_{10}-4\lambda_{11}} G(x^*,x_{n+1},x_{n+1})$$

$$+ \frac{\lambda_1+\lambda_2+\lambda_5+\lambda_8+2\lambda_{11}}{1-\lambda_3-\lambda_4-\lambda_5-\lambda_8-\lambda_9-\lambda_{10}-4\lambda_{11}} G(x_n,x^*,x^*)$$

$\Rightarrow G(x^*,Tx^*,Tx^*) \leq 0$ as $n \to \infty$
i.e $Tx^* = x^*$. i.e x^* is a fixed point of T.
Now we show x^* is the unique fixed point of T. Assume that x' is another fixed point

of T, then we have $Tx' = x'$ and

$$\begin{aligned}
G(x^*, x', x') &= G(Tx^*, Tx', Tx') \\
&\leq \lambda_1 G(x^*, x', x') + \lambda_2 G(x^*, Tx^*, Tx^*) \\
&+ \lambda_3 G(x', Tx', Tx') + \lambda_4 G(x', Tx', Tx') + \lambda_5 G(x^*, Tx', Tx') \\
&+ \lambda_6 G(x', Tx^*, Tx^*) + \lambda_7 G(x', Tx^*, Tx^*) + \lambda_8 G(x^*, Tx', Tx') \\
&+ \lambda_9 G(x', Tx', Tx') + \lambda_{10} G(x', Tx', Tx') + \lambda_{11}\{G(x^*, Tx', Tx') \\
&+ G(x', Tx^*, Tx^*) + G(x', Tx^*, Tx^*) + G(x^*, Tx', Tx') \\
&+ G(x', Tx', Tx') + G(x', Tx', Tx')\} \\
&\leq \lambda_1 G(x^*, x', x') + \lambda_2 G(x^*, x^*, x^*) \\
&+ \lambda_3 G(x', x', x') + \lambda_4 G(x', x', x') + \lambda_5 G(x^*, x', x') \\
&+ \lambda_6 G(x', x^*, x^*) + \lambda_7 G(x', x^*, x^*) + \lambda_8 G(x^*, x', x') \\
&+ \lambda_9 G(x', x', x') + \lambda_{10} G(x', x', x') + \lambda_{11}\{G(x^*, x', x') \\
&+ G(x', x^*, x^*) + G(x', x^*, x^*) + G(x^*, x', x') \\
&+ G(x', x', x') + G(x', x', x')\} \\
&\leq \lambda_1 G(x^*, x', x') + \lambda_5 G(x^*, x', x') + \lambda_6 G(x', x^*, x^*) \\
&+ \lambda_7 G(x', x^*, x^*) + \lambda_8 G(x^*, x', x') + \lambda_{11}\{G(x^*, x', x') \\
&+ G(x', x^*, x^*) + G(x', x^*, x^*) + G(x^*, x', x')\}
\end{aligned}$$

$$\begin{aligned}
(1 - \lambda_1 - \lambda_5 - \lambda_8 - 2\lambda_{11}) G(x^*, x', x') &\leq (\lambda_6 + \lambda_7 + 2\lambda_{11}) G(x', x^*, x^*) \\
G(x^*, x', x') &\leq \frac{\lambda_6 + \lambda_7 + 2\lambda_{11}}{1 - \lambda_1 - \lambda_5 - \lambda_8 - 2\lambda_{11}} G(x', x^*, x^*)
\end{aligned}$$

which implies that $x^* = x'$ so T has a unique fixed point.

Theorem 3. *Let (X, d) be a complete $G-$metric space. let T be a self mapping on X such that*

$$\begin{aligned}
G(Tx, Ty, Tz) \leq{}& \alpha\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\} + \beta\{G(x, Ty, Ty) + G(x, Tz, Tz) \\
&+ G(y, Tx, Tx) + G(y, Tz, Tz) + G(z, Ty, Ty) + G(z, Tx, Tx)\} + \gamma\{G(y, Ty, Ty) \\
&+ G(y, Tx, Tx) + G(y, Tz, Tz)\} + \delta\{G(x, Tx, Tx) + G(x, Ty, Ty) + G(x, Tz, Tz)\},
\end{aligned} \tag{5}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ such that
$4\beta + \gamma + 2\delta < 1$,for every $x, y, z \in X$,then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined by the recursion. $x_n = Tx_{n-1} = T^n x_0$ for every $n \in N$

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq \alpha\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x_n, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_n)\} \\
&+ \beta\{G(x_{n-1}, Tx_n, Tx_n) + G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1}) \\
&+ G(x_n, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1})\} \\
&+ \gamma\{G(x_n, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1}) + G(x_n, Tx_n, T_n)\} \\
&+ \delta\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x_{n-1}, Tx_n, Tx_n) + G(x_{n-1}, Tx_n, Tx_n)\} \\
&\leq \alpha\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\} \\
&+ \beta\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) \\
&+ G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} \\
&+ \gamma\{G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\
&+ \delta\{G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1})\} \\
&\leq \alpha\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\} \\
&+ \beta\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\
&+ G(x_n, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \\
&+ G(x_n, x_n, x_n)\} + \gamma\{G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\
&+ \delta\{G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x_n) \\
&+ G(x_n, x_{n+1}, x_{n+1})\} \\
&\leq \alpha\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\} + \beta\{2G(x_{n-1}, x_n, x_n) \\
&+ 4G(x_n, x_{n+1}, x_{n+1})\} + \gamma\{2G(x_n, x_{n+1}, x_{n+1})\} + \delta\{3G(x_{n-1}, x_n, x_n) \\
&+ 2G(x_n, x_{n+1}, x_{n+1})\}
\end{aligned}$$

$$\begin{aligned}
(1 - 2\alpha - 4\beta - 2\gamma - 2\delta)G(x_n, x_{n+1}, x_{n+1}) &\leq (\alpha + 2\beta + 3\delta)G(x_{n-1}, x_n, x_n) \\
G(x_n, x_{n+1}, x_{n+1}) &\leq \frac{\alpha + 2\beta + 3\delta}{1 - 2\alpha - 4\beta - 2\gamma - 2\delta} G(x_{n-1}, x_n, x_n) \\
\Rightarrow G(x_n, x_{n+1}, x_{n+1}) &\leq kG(x_{n-1}, x_n, x_n)
\end{aligned}$$

where $k = \dfrac{\alpha + 2\beta + 3\delta}{1 - 2\alpha - 4\beta - 2\gamma - 2\delta} < 1$

$$\Rightarrow G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n)$$

$$\Rightarrow G(x_n, x_{n+1}, x_{n+1}) \leq k^2 G(x_{n-1}, x_n, x_n)$$

continuing this process, we have

$$\Rightarrow G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_{n-1}, x_n, x_n)$$

Thus T is a contractive mapping
Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N}, m > n$, then

$$\begin{aligned}G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &+ ... + G(x_{m-1}, x_m, x_m) \\ &\leq (k^n + k^{n+1} + ... + k^{m-1})G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k}G(x_0, x_1, x_1)\end{aligned}$$

Letting $n \to \infty$,we get

$$\lim_{n\to\infty} G(x_n, x_m, x_m) = 0,$$

as $n, m \to \infty$, since $\frac{k^n}{1-k} < 1$
hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in X. Since X is complete then $\exists u$ in X s.t.

$$\lim_{n\to\infty} x_n = x^*(\in X)$$

Now, we prove that x^* is the fixed point of T.

$$\begin{aligned}
G(x^*,Tx^*,Tx^*) &\leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n+1},Tx^*,Tx^*)\\
&\leq G(x^*,x_{n+1},x_{n+1}) + G(Tx_n,Tx^*,Tx^*)\\
&\leq G(x^*,x_{n+1},x_{n+1}) + \alpha\{G(x_n,Tx_n,Tx_n) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,Tx^*,Tx^*)\} + \beta\{G(x_n,Tx^*,Tx^*) + G(x_n,Tx^*,Tx^*)\\
&+ G(x_n,Tx_n,Tx_n) + G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,Tx_n,Tx_n)\} + \gamma\{G(x^*,Tx^*,Tx^*) + G(x^*,Tx_n,Tx_n)\\
&+ G(x^*,Tx^*,Tx^*)\} + \delta\{G(x_n,Tx_n,Tx_n) + G(x_n,Tx^*,Tx^*)\\
&+ G(x_n,Tx^*,Tx^*)\}\\
&\leq G(x^*,x_{n+1},x_{n+1}) + \alpha\{G(x_n,x_{n+1},x_{n+1}) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,Tx^*,Tx^*)\} + \beta\{G(x_n,Tx^*,Tx^*) + G(x_n,Tx^*,Tx^*)\\
&+ G(x_n,x_{n+1},x_{n+1}) + G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,x_{n+1},x_{n+1})\} + \gamma\{G(x^*,Tx^*,Tx^*) + G(x^*,x_{n+1},x_{n+1})\\
&+ G(x^*,Tx^*,Tx^*)\} + \delta\{G(x_n,x_{n+1},x_{n+1}) + G(x_n,Tx^*,Tx^*)\\
&+ G(x_n,Tx^*,Tx^*)\}\\
&\leq G(x^*,x_{n+1},x_{n+1}) + \alpha\{G(x_n,x^*,x^*) + G(x^*,x_{n+1},x_{n+1})\\
&+ G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\} + \beta\{G(x_n,x^*,x^*)\\
&+ G(x^*,Tx^*,Tx^*) + G(x_n,Tx^*,Tx^*) + G(x_n,x^*,x^*)\\
&+ G(x^*,x_{n+1},x_{n+1}) + G(x^*,Tx^*,Tx^*) + G(x^*,Tx^*,Tx^*)\\
&+ G(x^*,x_{n+1},x_{n+1})\} + \gamma\{G(x^*,Tx^*,Tx^*) + G(x^*,x_{n+1},x_{n+1})\\
&+ G(x^*,Tx^*,Tx^*)\} + \delta\{G(x_n,x^*,x^*) + G(x^*,x_{n+1},x_{n+1})\\
&+ G(x_n,x^*,x^*) + G(x^*,Tx^*,Tx^*)\\
&+ G(x_n,x^*,x^*) + G(x^*,Tx^*,Tx^*)\}
\end{aligned}$$

$$\begin{aligned}
(1-2\alpha-4\beta-\gamma-2\delta)G(x^*,Tx^*,Tx^*) &\leq (\alpha+2\beta+3\delta)G(x_n,x^*,x^*)\\
&+ (1+\alpha+2\beta+\gamma+\delta)G(x^*,x_{n+1},x_{n+1})
\end{aligned}$$

$$G(x^*,Tx^*,Tx^*) \leq \frac{\alpha+2\beta+3\delta}{1-2\alpha-4\beta-\gamma-2\delta}G(x_n,x^*,x^*) + \frac{1+\alpha+2\beta+\gamma+\delta}{1-2\alpha-4\beta-\gamma-2\delta}G(x^*,x_{n+1},x_{n+1})$$

Taking the limit as $n \to \infty$, we get
$\lim_{n\to\infty} G(x^*,Tx^*,Tx^*) = 0$
$\Rightarrow Tx^* = x^*$
$\Rightarrow x^*$ is the fixed point of T
Uniqueness of fixed point.
Further, let us suppose that x' is another fixed point of T.Then $Tx' = x'$and

$$
\begin{aligned}
G(x^*, x', x') &= G(Tx^*, Tx', Tx') \\
&\leq \alpha\{G(x^*, Tx^*, Tx^*) + G(x', Tx', Tx') + G(x', Tx', Tx')\} \\
&+ \beta\{G(x^*, Tx', Tx') + G(x^*, Tx', Tx') + G(x', Tx^*, Tx^*) \\
&+ G(x', Tx', Tx') + G(x', Tx', Tx') + G(x', Tx^*, Tx^*)\} \\
&+ \gamma\{G(x', Tx', Tx') + G(x', Tx^*, Tx^*) + G(x', Tx', Tx')\} \\
&+ \delta\{G(x^*, Tx^*, Tx^*) + G(x^*, Tx', Tx') + G(x^*, Tx', Tx')\} \\
&\leq \alpha\{G(x^*, x^*, x^*) + G(x', x', x') + G(x', x', x')\} \\
&+ \beta\{G(x^*, x', x') + G(x^*, x', x') + G(x', x^*, x^*) \\
&+ G(x', x', x') + G(x', x', x') + G(x', x^*, x^*)\} \\
&+ \gamma\{G(x', x', x') + G(x', x^*, x^*) + G(x', x', x')\} \\
&+ \delta\{G(x^*, x^*, x^*) + G(x^*, x', x') + G(x^*, x', x')\} \\
&\leq \beta\{G(x^*, x', x') + G(x^*, x', x') + G(x', x^*, x^*) \\
&+ G(x', x^*, x^*)\} + \gamma\{G(x', x^*, x^*)\} + \delta\{G(x^*, x', x') \\
&+ G(x^*, x', x')\} \\
(1 - 2\beta - 2\delta)G(x^*, x', x') &\leq (2\beta + \gamma)G(x', x^*, x^*).
\end{aligned}
$$

which implies that $x^* = x'$ so T has a unique fixed point.

Finally, we give an application of the Theorem 2.1.

Example 1. *Let $X = \{0, 1/2, 1\}$ and let $G : X^3 \to [0, \infty)$ be defined by*

$$
\begin{aligned}
G(0, 1, 1) = 6 = G(1, 0, 0), \quad G(0, 1/2, 1/2) = 4 = G(1/2, 0, 0) \\
G(1/2, 1, 1) = 5 = G(1, 1/2, 1/2), \quad G(0, 1/2, 1) = 15/2 \\
G(x, x, x) = 0 \quad \forall x \in X
\end{aligned}
$$

(X, G) is a symmetric G-complete G-metric space.
Let $T : X \to X$ be defined by $T0 = 1, \quad T1/2 = 1/2, \quad T1 = 0.$

where $\lambda_1 = 1/2, \quad \lambda_2 = 1/20, \quad \lambda_3 = 1/25, \quad \lambda_4 = 1/30 \quad \lambda_5 = 1/6.$

$$
\begin{aligned}
&G(T0, T1/2, T1/2) = G(0, 1/2, 1/2) = 4; G(T0, T1, T1) = G(1, 0, 0) = 6 \\
&G(T1/2, T1, T1) = G(1/2, 0, 0) = 4; G(T0, T1/2, T1) = G(1, 1/2, 0) = 15/2
\end{aligned}
$$

we have

$$\begin{aligned}
5 = G(T0, T1/2, T1/2) = G(1, 1/2, 1/2) \\
&\leq \frac{1}{2}G(1, 1/2, 1/2) + \frac{1}{20}G(1, T0, T0) + \frac{1}{25}G(1/2, T1/2, T1/2) \\
&+ \frac{1}{30}G(1/2, T1/2, T1/2) + \frac{1}{6}\{G(1, T1/2, T1/2) + G(1/2, T0, T0) \\
&+ G(1/2, T0, T0) + G(1, T1/2, T1/2) + G(1/2, T1/2, T1/2) \\
&+ G(1/2, T1/2, T1/2)\} \\
&\leq \frac{5}{2} + \frac{20}{6} = \frac{35}{6}.
\end{aligned}$$

Again,

$$\begin{aligned}
6 = G(T0, T1, T1) = G(1, 0, 0) \\
&\leq \frac{1}{2}G(1, 0, 0) + \frac{1}{20}G(1, T0, T0) + \frac{1}{25}G(0, T1, T1) \\
&+ \frac{1}{30}G(0, T1, T1) + \frac{1}{6}\{G(1, T1, T1) + G(0, T0, T0) \\
&+ G(0, T0, T0) + G(1, T1, T1) + G(0, T1, T1) + G(0, T1, T1)\} \\
&\leq \frac{6}{2} + \frac{20}{6} = \frac{19}{3}.
\end{aligned}$$

Also,

$$\begin{aligned}
4 = G(T1/2, T1, T1) = G(1/2, 0, 0) \\
&\leq \frac{1}{2}G(1/2, 0, 0) + \frac{1}{20}G(1/2, T1/2, T1/2) + \frac{1}{25}G(0, T1, T1) \\
&+ \frac{1}{30}G(0, T1, T1) + \frac{1}{6}\{G(1/2, T1, T1) + G(0, T1/2, T1/2) \\
&+ G(0, T1/2, T1/2) + G(1/2, T1, T1) + G(0, T1, T1) + G(0, T1, T1)\} \\
&\leq \frac{4}{2} + \frac{16}{6} = \frac{14}{3}.
\end{aligned}$$

finally,

$$\begin{aligned}\frac{15}{2} = G(T0, T1/2, T1) = G(1, 1/2, 0) \\ &\leq \frac{1}{2}G(1, 1/2, 0) + \frac{1}{20}G(1, T0, T0) + \frac{1}{25}G(1/2, T1/2, T1/2) \\ &+ \frac{1}{30}G(0, T1, T1) + \frac{1}{6}\{G(1, T1/2, T1/2) + G(1/2, T0, T0) \\ &+ G(0, T0, T0) + G(1, T1, T1) + G(0, T1/2, T1/2) + G(1/2, T1, T1)\} \\ &\leq \frac{15}{4} + \frac{30}{6} = \frac{35}{4}.\end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied and T has a unique fixed point in X.

References

[1] R. P. Agarwal, E. Karapınar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed Point Theory Appl. 1(2013),1-33.

[2] M. A. Alghamdi, E. Karapınar, *G-- contractive-type mappings and related fixed point theorems*, J. Inequal. Appl., 1(2013),1-16.

[3] A.H. Ansari, O. Ege, S. Radenovic, *Some fixed point results on complex-valued G_b-metric spaces*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Seria A. Matematicas, 112(2) (2018), 463-472.

[4] A.H. Ansari, M.K. Jain, N. Saleem, *Inverse-C-class function on weak semi-compatibility and fixed point theorems for expansive mappings in G-metric spaces*, Mathematica Moravica, 24(1) (2020), 93-108.

[5] H. Aydi, E. Karapınar, W. Shatanawi, *Tripled fixed point results in generalized metric spaces*, J. Appl. Math., 2012(2012), Article ID 314279.

[6] H. Aydi, M. Postolache, W. Shatanawi, *Coupled fixed point results for (,)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl., 63(1)(2012),298-309.

[7] N. Bilgili, E. Karapınar, *Cyclic contractions via auxiliary functions on G-metric spaces*, Fixed Point Theory Appl., 1(2013),1-16.

[8] D. Das, L.N. Mishra, V.N. Mishra, H.G. Rosales, A. Dhaka, F.E.L. Monteagudo, E.G. Fernández, T.A. Ramirez-delReal, *C*-Algebra Valued Modular G-metric Spaces with Applications in Fixed Point Theory*, Symmetry, 2021, 13, 2003. https://doi.org/10.3390/sym13112003.

[9] H. S. Ding, E. Karapınar, *A note on some coupled fixed-point theorems on G-metric spaces*, J. Inequal. Appl., 1(2012),1-9.

[10] O. Ege, *Complex valued Gb -metric spaces*, J. Comput. Anal. Appl., 21(2) (2016), 363-368.

[11] O. Ege, *Some fixed point theorems in complex valued Gb-metric spaces*, J. Nonlinear Convex Anal., 18(11) (2017), 1997-2005.

[12] O. Ege, I. Karaca, *Common fixed point results on complex valued G_b-metric spaces*, Thai J. Math., 16(3) (2018), 775-787.

[13] O. Ege, C. Park, A.H. Ansari, *A different approach to complex valued Gb-metric spaces*, Adv. Difference Equ., 152(2020), 1-13.

[14] V. Gupta, O. Ege, R. Saini, M. de la Sen, *Various fixed point results in complete G_b–metric spaces*, Dynam. Systems Appl., 30(2) (2021), 277-293.

[15] V. Gupta, O. Ege, R. Saini, *Extended G_b-metric spaces and some fixed point results with an application to Fredholm integral equation*, Electron. J. Mat. Anal. and Appl., 10(2) (2022), 219-227.

[16] M. Jleli, B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appli., 1(2012),1-7.

[17] E. Karapınar, R. P. Agarwal, *Further fixed point results on G-metric spaces*, Fixed Point Theory Appli., 1(2013),1-14.

[18] Z. Mustafa, *A New Structure for Generalized Metric Spaces: With Applications to Fixed Point Theory*, Ph.D. thesis, University of New castle, 2005.

[19] Z. Mustafa, H. Aydi, E. Karapinar, *Mixed G-monotone property and quadruple fixed point theorems in partially ordered metric spaces*, Fixed Point Theory and Appl., 1(2012),1-19.

[20] Z. Mustafa, H. Aydi, E. Karapınar, *Generalized meir–keeler type contractions on G-metric spaces*, Appl. Math. Comput., 219(2013):10441-10447.

[21] Z. Mustafa, M. Khandagji, W. Shatanawi, *Fixed point results on complete G-metric spaces*, Studia Sci. Math. Hungar., 48(3)(2011),304-319.

[22] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed point theory and Appli., 2008(2008), Article ID: 189870.

[23] Z. Mustafa, W. Shatanawi, M. Bataineh, A. Volodin, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Math. Sci., 2009(2009), Article ID: 283028.

[24] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., 7(2)(2006), 289-297.

[25] Z. Mustafa, B. Sims, *Fixed point theorems for contractive mappings in complete-metric spaces*, Fixed point theory and Appli., 2009(2009), Article ID: 917175.

[26] Roldán López de Hierro, A.F., Karapınar, E., D. O'Regan, *Coincidence point theorems on quasi-metric spaces via simulation functions and applications to G-metric spaces*, J. Fixed Point Theory Appl., 20, 112 (2018). https://doi.org/10.1007/s11784-018-0582-x.

[27] B. Samet, C. Vetro, F. Vetro, *Remarks on-metric spaces*, Int. J. Anal., 2013(2013), Article ID 917158.

[28] W. Shatanawi, Z. Mustafa, *On coupled random fixed point results in partially ordered metric spaces*, Matematički vesnik, 64(2)(2012),139-146.

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