Acta Universitatis Apulensis No. 76/2023
ISSN: 1582-5329 pp. 13-24
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.76.02

ON LIE-DERIVATIVE OF $M-$PROJECTIVE CURVATURE TENSOR AND $K-$CURVATURE INHERITANCE IN $G\mathfrak{B}K-5RF_n$

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Abstract. This work introduces a new class of Finsler spaces F_n , characterized by a specific relationship between the Cartan fourth curvature tensor K^i_{jkh} and given fifth-order covariant vector fields. We call such spaces generalized $\mathfrak{B}K-$fifth recurrent spaces. We prove that under certain conditions, the K^i_{jkh} exhibits both inheritance property and K-curvature inheritance with respect to the Berwald covariant derivative of fifth order. Additionally, we explore properties of the Lie derivative in F_n and its connection to tensor-curvature inheritance. Notably, we define the Lie derivative for the M-projective curvature tensor $\bar{W}^i_{jkh}$ and derive various identities for it within the $G\mathfrak{B}K-5RF_n$ framework.

2010 *Mathematics Subject Classification*: 53B40, 53C60.

Keywords: Finsler space, Lie-derivative, generalized $\mathfrak{B}K-$fifth recurrent space, M-projective curvature tensor $\bar{W}^i_{jkh}$ and $K-$curvature inheritance.

1. Introduction

Pandy's 1982 paper [14] introduced the concept of the Lie derivative for the curvature tensor. This derivative is defined as a specific infinitesimal transformation. Pandy also showed that the Lie derivative for Weyl projective curvature tensor vanishes. Duggal (1992) [7] then studied curvature inheritance symmetry in Riemannian spaces and found a number of applications, leading to increased research interest in the following years. Finally, Singh (2003) [22]studied infinitesimal transformations for which the Lie derivative of the curvature tensor is proportional to itself, naming them "curvature inheritance transformations."

Curvature inheritance is a physical phenomenon that can be explained by the Lie recurrence equation, a specific type of infinitesimal transformation. Building upon this, Singh (2004) [20] introduced a special kind of curvature inheritance called "projective curvature inheritance," characterized by its connection to projective motions. Further research explored specific instances of this phenomenon. Mishra and

Yadav (2007) [12] focused on projective curvature in an $NP - F_n$ space, while Gatoto and Singh (2008) [10] delved into both $\widetilde{k}-$curvature inheritance and projective $\widetilde{k}-$curvature inheritance. Saxena and Pandy (2011) [18] generalized several theorems related to Lie derivatives, further enriching the theoretical framework. In 2012, Singh [21] defined curvature inheritance in bi-recurrent Finsler spaces. Mishra and Lodhi (2012) [11] delved into the nuances of curvature inheritance symmetry and Ricci-inheriting symmetry in Finsler spaces, obtaining insightful results. Notably, Singh (2013) [19] introduced the study of curvature inheritance for the Weyl curvature tensor field. Finally, Gatoto (2014) [9] defined and analyzed $N-$curvature inheritance in recurrent Finsler spaces, uncovering valuable findings and discussing special cases.

In recent year Calvaruso and Zaemi (2016) [5] studied some application of a Lie-derivative for curvature tensors. Duggal (2017) [8] used mathematical models to establish a connection between curvature inheritance symmetry and Ricci Solitons. Opondo (2021) [13] defined and studied $W-$curvature inheritance in recurrent and bi-recurrent Finsler space. Ali, Salman, Rahaman, and Pundeer (2023) [1]studied a Lie-derivative of M-projective curvature and established some properties of this curvature tensor.

In (1973) Sinha and Singh[23] studied the properties of recurrent tensor in recurrent Finsler space. The several works on recurrent Finsler space done in (1973,1987). Finsler geometry reveals intriguing recurrence properties in its curvature tensors: Verma (1991)[24] identified recurrence in Cartan's third, Dikshit (1992)[6] in Berwald's, and Qasem (2000)[15] explored conditions for recurrence in the general Berwald sense. Notably, Qasem and Abdallah (2016)[16] defined generalized -recurrent Finsler spaces, where both Berwald and Cartan's fourth tensors exhibit recurrence. Further, Qasem and Baleedi (2016)[17] introduced spaces where Cartan's fourth tensor exhibits a specific recurrence relation, proving the K-Ricci tensor, curvature vector, and scalar to be non-vanishing. Al-Qashbari and Qasem (2017)[2] extended this by studying generalized -trirecurrent Finsler spaces. Al-Qashbari (2020)[3] introduced some relations of generalized curvature tensors for B-recurrent Finsler Field. In (2023) Al-Qashbari and Al-Maisary [4] studied for generalized $\mathfrak{B}K-$fourth recurrent in Finsler space.

Let us consider the infinitesimal transformation point given by [14]:

$$x^{-i} = x^i + v^i(x)\varepsilon. \tag{1}$$

Within this framework, ε represent an infinitesimal point constant ,while $v^i(x)$ denotes a contravariant vector field solely dependent on positional coordinates x^i ,devoid of directional influence.infinitesimal method servves as a pivotal tool for constructing Lie-derivatives,invaluable for assessing the rate of change of a vector field

as they traverse the flow of another vector or tensor field,denote by L_v .The symbol signifies the Lie-differentiation operator ,specifically tailored to operate within the context of the transformation(1).
The vector y^i is Lie-invariant i.e.,

$$L_v y^i = 0. \tag{2}$$

The M-projective curvature collineation along a vector field $v^i(x)$ satisfies the relation [1] :

$$\bar{W}^i_{jkh} = 0. \tag{3}$$

The commutation formula for Lie-derivatives and other derivatives for tensors is a mathematical identity that relates the Lie derivative of a tensor field to other derivatives of that field. It is a powerful tool for studying the behavior of tensors T^i_j under transformations given by:

$$L_v(\dot{\partial}_l T^i_j) - \dot{\partial}_l(L_v T^i_j) = 0. \tag{4}$$

The Berwald curvature tensor's H^i_{jkh} covariant Lie derivative is proportional to the curvature itself if and only if the following relation holds [13] :

$$L_v H^i_{jkh} = \alpha(x) H^i_{jkh}. \tag{5}$$

For a non-zero scalar function $\alpha(x)$,the infinitesimal transformation (5) is a type of an H-curvature inheritance in $2R - F_n$,that is characterized by the constant, with respect to the vector field $v^i(x)$. Also the Lie-derivative of the Berwald curvature tensor H^i_{jkh} satisfies the relations:

$$\begin{cases} a)\ L_v H^i_{kh} = \alpha(x) H^i_{kh} \\ b)\ L_v H^r_{rkh} = \alpha(x) H^r_{rkh} \\ c)\ L_v H_{jk} = \alpha(x) H_{jk} \\ d)\ L_v H = \alpha(x) H \\ e) L_v H^i_j = \alpha(x) H^i_j. \end{cases} \tag{6}$$

In this paper, we consider an n-dimensional Finsler space F_n with the metric function F satisfying the following conditions [17]:

(i) Positively homogeneous: $F(x, ky) = kF(x, y), \quad k > 0.$

(ii) positively: $F(x, y) > 0, \quad y \neq 0.$

(iii) $\left\{\dot{\partial}_i \dot{\partial}_j F^2(x,\ y)\right\} \xi^i\ \xi^j, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}$ is the positive definite for all the variables ξ^i.

The following equation gives the relation between the metric function F and the components of the metric tensor g_{ij} :

$$\begin{cases} a)\ g_{ij}y^iy^j = F^2 \\ b)\ g_{ij}y^j = y_i \\ c)\ \delta^i_l = \dot{\partial}_l y^i. \end{cases} \tag{7}$$

$$d)\ g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \neq k. \end{cases}$$

Within the Finsler space under consideration, the metric tensor, along with Cartan's and Berwald's connection parameters, exhibit symmetry in their lower indices and demonstrate positive homogeneity of degree zero with respect to directional arguments. Furthermore, Berwald's covariant derivative applied to the metric function , the vectors y^i , y_i and the unit vector l^i, results in an identically vanishing outcome, mathematically expressed as follows:

$$\begin{cases} a)\ \mathfrak{B}_k F = 0. \\ b)\ \mathfrak{B}_k y^i = 0 \\ c)\ \mathfrak{B}_k y_i = 0 \\ d)\ \mathfrak{B}_k l^i = 0. \end{cases} \tag{8}$$

The Cartan's fourth curvature tensor K^i_{jkh} satisfies the following relations:

$$\begin{cases} a)\ K^i_{jkh} = R^i_{jkh} - C^i_{js}\ H^s_{kh} \\ b)\ H^i_{jkh} = K^i_{jkh} + \dot{\partial}_j\ K^i_{rkh}y^r \\ c)\ R^i_{jkh}\ y^j = K^i_{jkh}\ y^j = H^i_{kh}. \end{cases} \tag{9}$$

The Ricci tensor R_{jk} is given by

$$R_{jk}y^j = H_k. \tag{10}$$

Let us explore $G\mathfrak{B}K - RF_n$ for which Cartan's fourth curvature tensor K^i_{jkh} is defined as [17]:

$$\mathfrak{B}_m K^i_{jkh} = a_m K^i_{jkh} + b_m\Big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\Big), \quad K^i_{jkh} \neq 0,$$

is called generalized $\mathfrak{B}K-$ recurrent space, where $\mathfrak{B}_m$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^m. Taking the covariant derivative of the fifth order for above equation in the sense of Berwald with respect to x^m , x^n, x^l , x^q and x^s respectively, we obtain

$$\begin{aligned}&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \\ &- c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\right) \\ &- e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\right) - 2b_{qlnm}y^r\mathfrak{B}_r\left(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\right),\end{aligned} \tag{11}$$

where $a_{sqlnm} = (\mathfrak{B}_s a_{qlnm}) + a_{qlnm}\lambda_s$, $b_{sqlnm} = a_{qlnm}\mu_s + (\mathfrak{B}_s b_{qlnm})$, $c_{sqlnm} = (\mathfrak{B}_s c_{qlnm} + c_{qlnm}\mathfrak{B}_s)$, $d_{sqlnm} = (\mathfrak{B}_s d_{qlnm} + d_{qlnm}\mathfrak{B}_s)$ and $e_{sqlnm} = 2\left(\mathfrak{B}_s b_{lnm}y^r\mathfrak{B}_r + b_{lnm}y^r\mathfrak{B}_s\mathfrak{B}_r\right)$ are non-zero covariant vectors field of fifth order.we identify such spaces as generalized $\mathfrak{B}K$-fifth recurrent space and denoted $G\mathfrak{B}K - 5RF_n$.

2. A $K-$Curvature Inheritance in $G\mathfrak{B}K - 5RF_n$

According to Singh's (2003) findings, within the context of a bi-recurrent Finsler space space $2R-F_n$, if the Lie-derivative for the Berwald curvature tensor H^i_{jkh} exhibits proportionality to the curvature itself, a condition mathematically expressed by relation(5) with respect to the vector field $v^i(x)$ and $\alpha(x)$ involving a non-zero scalar function, then the infinitesimal transformation characterized by (5) is designated as an H-curvature inheritance within the designated space.
Taking the Lie- derivative of both sides of [(9)b] and using (2),[(7)c] and[(7)d] when $i \neq j$ also the Lie- derivative and partial derivative are commutative, we get

$$L_v H^i_{jkh} = L_v K^i_{jkh}. \tag{12}$$

Using (5) in (12), we get

$$L_v K^i_{jkh} = \alpha(x) H^i_{jkh}. \tag{13}$$

Since $\alpha(x)$ is a scalar function , we obtain

$$L_v K^i_{jkh} = \alpha(x) K^i_{jkh}. \tag{14}$$

Thus,the Cartan's fourth curvature tensor K^i_{jkh} has the inheritance property, hence it satisfying the relation(14).
$K-$curvature inheritance may be also called $K-$Lie recurrence. Singh(2012) [21] has proved that in a Finsler space F_n if the space is an isotropic then the properties does not different in directional , so that every $K-$curvature inheritance in $2R-F_n$ is also $K-$curvature inheritance in $5R-F_n$.

Definition 1. *In a fifth recurrent Finsler space if the space is an isotropic and the Lie-derivative of Cartan's fourth curvature tensor K^i_{jkh} is equal the curvature itself if and only if the relation* (14) *W.r.t the vector field $v^i(x)$ where $\alpha(x)$ is a non-zero scalar function and $K^i_{jkh} \neq 0$, then the infinitesimal transformation*(14) *is called an $K-$curvature inheritance in $5R - F_n$.*

Taking the covariant derivative of 5^{th} order for (14) with respect to x^m , x^n, x^l , x^q and x^s respectively in the sense of Berwald, we get

$$\begin{gathered}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vK^i_{jkh}) = \\ [\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x)]K^i_{jkh} + \alpha(x)[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mK^i_{jkh}].\end{gathered} \tag{15}$$

Usig (11)in (15), we get

$$\begin{gathered}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vK^i_{jkh}) = \\ [\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x)]K^i_{jkh} + \alpha(x)\Big[a_{sqlnm}K^i_{jkh} + b_{sqlnm}\left(\delta^i_hg_{jk} - \delta^i_kg_{jh}\right) \\ - c_{sqlnm}\left(\delta^i_hC_{jkn} - \delta^i_k\,C_{jhn}\right) - d_{sqlnm}\left(\delta^i_hC_{jkl} - \delta^i_k\,C_{jhl}\right) \\ - e_{sqlnm}\left(\delta^i_hC_{jkq} - \delta^i_k\,C_{jhq}\right) - 2b_{qlnm}y^r\mathfrak{B}_r\left(\delta^i_hC_{jks} - \delta^i_k\,C_{jhs}\right)\Big].\end{gathered} \tag{16}$$

Taking the Lie-derivative of bth sides of (11) and since the components of the Kronecker delta δ^i_h are constant functions (one or zero), so $L_v\delta^i_h = 0$, thus we get

$$\begin{aligned}&L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mK^i_{jkh}) = (L_va_{sqlnm})K^i_{jkh} + a_{sqlnm}L_v(K^i_{jkh}) \\ &+(L_vb_{sqlnm})\left(\delta^i_hg_{jk} - \delta^i_kg_{jh}\right) + b_{sqlnm}\left(\delta^i_hL_vg_{jk} - \delta^i_kL_vg_{jh}\right) \\ &-(L_vc_{sqlnm})\left(\delta^i_hc_{jkn} - \delta^i_kc_{jhn}\right) - c_{sqlnm}\left(\delta^i_hL_vc_{jkn} - \delta^i_kL_vc_{jhn}\right) \\ &-(L_vd_{sqlnm})\left(\delta^i_hc_{jkl} - \delta^i_kc_{jhl}\right) - d_{sqlnm}\left(\delta^i_hL_vc_{jkl} - \delta^i_kL_vc_{jhl}\right) \\ &-(L_ve_{sqlnm})\left(\delta^i_hc_{jkq} - \delta^i_kc_{jhq}\right) - e_{sqlnm}\left(\delta^i_hL_vc_{jkq} - \delta^i_kL_vc_{jhq}\right) \\ &-2(L_vb_{qlnm})y^r\mathfrak{B}_r\left(\delta^i_hC_{jks} - \delta^i_k\,C_{jhs}\right) - 2b_{qlnm}y^r\mathfrak{B}_r\left(\delta^i_hL_vC_{jks} - \delta^i_k\,L_vC_{jhs}\right).\end{aligned} \tag{17}$$

Using [(7)d] when $i \neq j$ and (14) in (17) ,we get

$$L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mK^i_{jkh}) = (L_va_{sqlnm} + \alpha(x)a_{sqlnm})K^i_{jkh}\ . \tag{18}$$

Using [(7)d] when $i \neq j$ in (16) ,we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vK^i_{jkh}) = [\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x) + \alpha(x)a_{sqlnm}]K^i_{jkh}\ . \tag{19}$$

Subtracting (18) from (19) ,we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vK^i_{jkh}) - L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mK^i_{jkh}) = [\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x) - L_va_{sqlnm}]K^i_{jkh}\ .$$

Which can be written as

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vK^i_{jkh}) - L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mK^i_{jkh}) = 0. \tag{20}$$

If and only if

$$L_va_{sqlnm} = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\alpha(x). \tag{21}$$

Thus, we conclude

Theorem 1. *In $G\mathfrak{B}K-5RF_n$, the K-curvature inheritance and the Berwald covariant derivative of the fifth order are commutative if and only if the Berwald covariant derivative of the fifth order of the scalar function $\alpha(x)$ is equal L_va_{sqlnm} .*

Applying commutation formula (4) for H^i_j and using [(6)e], we get

$$L_v(\dot{\partial}_lH^i_j) = \alpha(x)(\dot{\partial}_lH^i_j). \tag{22}$$

Thus, we conclude

Theorem 2. *In $G\mathfrak{B}K-5RF_n$, wich admits the H-curvature inheritance the partial derivative with respect to y^l for the deviation tensor H^i_j satisfies the inheritance property .*

We have the equation [13]:

$$L_vW^i_j = \alpha(x)W^i_j. \tag{23}$$

Since $\alpha(x)$ is a scalar function , we obtain

$$L_vW^i_j = \alpha(x)H^i_j. \tag{24}$$

Using [(6)e] in (24), we get

$$L_vW^i_j = L_vH^i_j. \tag{25}$$

Similarly, using (14)in (12) ,we get

$$L_vH^i_{jkh} = \alpha(x)K^i_{jkh}. \tag{26}$$

Since $\alpha(x)$ is a scalar function , we obtain

$$L_vH^i_{jkh} = \alpha(x)W^i_{jkh}. \tag{27}$$

Also, we have [13]:

$$L_v W^i_{jkh} = \alpha(x) W^i_{jkh}. \tag{28}$$

Using (28) in (27) ,we get

$$L_v H^i_{jkh} = L_v W^i_{jkh}. \tag{29}$$

By applying the precedent paces on any two tensors-curvature inheritance of the same type we obtain that the Lie-derivative for any two tensors-curvature inheritance of the same type are equal .
Thus, we conclude

Theorem 3. *In $G\mathfrak{B}K-5RF_n$,wich admits tensor-curvature inheritance,the Lie-derivatives for the tensors of the same type are equql .*

Adding [(6)e] and (23) ,we get

$$L_v(W^i_j + H^i_j) = \alpha(x)(W^i_j + H^i_j). \tag{30}$$

Similarly,adding (5) and (28) ,we get

$$L_v(W^i_{jkh} + H^i_{jkh}) = \alpha(x)(W^i_{jkh} + H^i_{jkh}). \tag{31}$$

By applying the precedent paces on any two tensors-curvature inheritance of the same type we obtain that the addition for any two tensors-curvature inheritance of the same type is also tensor-curvature inheritance .
Thus, we conclude

Theorem 4. *In $G\mathfrak{B}K-5RF_n$,which admits tensor-curvature inheritance,the addition for any two tensors-curvature inheritance of the same type is also tensor-curvature inheritance .*

3. A Lie-Derivative of $M-$Projective Curvature Tensor in $G\mathfrak{B}K-5RF_n$

Definition 2. *The M- projective curvature tensor $\bar{W}^i_{jkh}$ is given by [1]:*

$$\bar{W}^i_{jkh} = R^i_{jkh} - \frac{1}{6}\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k\right). \tag{32}$$

Taking the covariant derivative of 5^{th} order for (32) with respect to x^m , x^n, x^l , x^q and x^s respectively in the sense of Berwald ,we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\bar{W}^i_{jkh} = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh} \\ -\frac{1}{6}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k\right). \tag{33}$$

Taking the Lie-derivative of both sides of (33), we get

$$L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\bar{W}^i_{jkh}\right) = L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh}\right) \\ -\frac{1}{6}L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k\right)\right]. \tag{34}$$

Thus, we conclude

Theorem 5. *In $G\mathfrak{B}K-5RF_n$, the Lie-derivative of Berwald covariant derivative of the fifth order for the M-projective curvature tensor $\bar{W}^i_{jkh}$ is giving by* (34).

From (34) we have

$$L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\bar{W}^i_{jkh}\right) = L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh}\right). \tag{35}$$

If and only if

$$-\frac{1}{6}L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k\right)\right] = 0. \tag{36}$$

Thus, we conclude

Theorem 6. *In $G\mathfrak{B}K-5RF_n$, the Lie-derivatives of Berwald covariant derivative of the fifth order for the M-projective curvature tensor $\bar{W}^i_{jkh}$ and the Cartan's third curvature tensor R^i_{jkh} both are equal if and only if* (36) *holds good .*

Using [(9)a] in (34), we get

$$L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\bar{W}^i_{jkh}\right) = L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh}\right). \tag{37}$$

If and only if

$$L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(C^i_{jt}H^t_{kh})\right] \\ -\frac{1}{6}L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k\right)\right] = 0. \tag{38}$$

Thus,we conclude

Theorem 7. *In $G\mathfrak{B}K-5RF_n$, the Lie-derivatives of Berwald covariant derivative of the fifth order for the M-projective curvature tensor $\bar{W}^i_{jkh}$ and the Cartan's fourth curvature tensor K^i_{jkh} both are equal if and only if* (38) *holds good .*

Transvecting (34) by y^j ,using (2), [(7)b], [(8)b], [(9)c] and (10), we get

$$L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\bar{W}^i_{jkh}y^j\right)=L_v\left(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mH^i_{kh}\right) \tag{39}$$
$$-\frac{1}{6}L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(H_k\delta^i_h-H_h\delta^i_k+y_kR^i_h-y_hR^i_k\right)\right].$$

Thus, we conclude

Theorem 8. *In $G\mathfrak{B}K-5RF_n$, the Lie-derivative of Berwald covariant derivative of the fifth order for the tensor ($\bar{W}^i_{jkh}y^j$) is giving by* (39).

Taking the Lie-derivative of both sides of (32), we get

$$L_v\bar{W}^i_{jkh}=L_vR^i_{jkh}-\frac{1}{6}L_v\left(R_{jk}\delta^i_h-R_{jh}\delta^i_k+g_{jk}R^i_h-g_{jh}R^i_k\right). \tag{40}$$

Taking the covariant derivative of 5^{th} order for (40) with respect to x^m , x^n, x^l , x^q and x^s respectivelyin the sense of Berwald ,we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_v\bar{W}^i_{jkh})=\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vR^i_{jkh}) \tag{41}$$
$$-\frac{1}{6}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left[L_v\left(R_{jk}\delta^i_h-R_{jh}\delta^i_k+g_{jk}R^i_h-g_{jh}R^i_k\right)\right].$$

Using (3) in (41) ,we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vR^i_{jkh})=\frac{1}{6}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left[L_v\left(R_{jk}\delta^i_h-R_{jh}\delta^i_k+g_{jk}R^i_h-g_{jh}R^i_k\right)\right]. \tag{42}$$

Thus, we conclude

Theorem 9. *In $G\mathfrak{B}K-5RF_n$, the Lie-derivative of Berwald covariant derivative of the fifth order for the Cartan's third curvature tensor R^i_{jkh} is giving by* (42) *if the M-projective curvature collineation along the vector field.*

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