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HERMITE-HADAMARD TYPE INEQUALITIES FOR F_h-CONVEX INTERVAL-VALUED FUNCTIONS ON TIME SCALES

B.O. Fagbemigun, A.A. Mogbademu

Abstract. In this paper, a new concept of F_h-convex interval-valued functions is introduced and employed to establish new Hermite-Hadamard type integral inequalities on time scales. Furthermore, an economic application is included and discussed to support our results.

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1. Introduction

Hermite-Hadamard(H-H) inequality for convex functions was first discovered by Hermite and Hadamard in 1881 [4, 15]. Since its announcement, Hermite-Hadamard inequality has been regarded as one of the most useful inequalities in mathematical analysis and optimization theory. Several authors have extended, generalized and improved the Hermite-Hadamard integral inequality for uni- and multi-variate convex functions, as well as other classes of convex functions on classical intervals, see [9, 17, 18, 19, 20] and the references therein.

Interval analysis was initiated by Moore for providing reliable computations [16]. Significant work on the theory of interval analysis did not appear until the 1950's, though it dates back to Archimedes' computation of the circumference of a circle. Since then, interval analysis and interval-valued functions(IVF) have been extensively studied both in mathematics and its applications, [16, 23]. Several authors have, in recent times, extended, generalized, improved, varied and applied the Hermite-Hadamard inequality for different classes of interval-valued convex functions on classical intervals [2, 23, 25], but only a few authors have obtained results for H-H inequalities on time scales especially the ones involving F_h-convex functions, see [12, 13].

Recently, new developments of the theory and applications of the notions of convexity on time scales to Hermite-Hadamard inequalities, Calculus of Variations and Economics were made (see [1, 4, 8, 9, 12, 13, 14, 22]).

Definition 1. *[11] A mapping $f : I_{\mathbb{T}} \to \mathbb{R}$ is said to be F_h-convex, $f \in SX(F_h)$ on time scales if*

$$f(\lambda x + (1-\lambda)y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y), \tag{1.1}$$

for all $s \in [0,1], 0 \leq \lambda \leq 1$ and $x, y \in I_{\mathbb{T}}$.
If inequality (1.1) is reversed, then f is F_h-concave, that is, $f \in SV(F_h)$.

Definition 2. *[11] The diamond-F_h integral of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;*

$$\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \tag{1.2}$$

where $s \in [0,1]$, $0 \leq \lambda \leq 1$, provided that f has a delta and nabla integral on $[a,b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Definitions 1.1 and 1.2 have been employed to establish H-H type integral inequalities on time scales, see [12, 13] and the references therein. The results obtained are as follows.

Theorem 1. *[12] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is F_h-convex subset of the real $\mathbb{T}$ and $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$ with $a < b$ and $s \in [0,1]$. Then*

$$\begin{aligned} 2^s \left[h(\frac{1}{2})\right]^s f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x \\ &\leq f(a)\left[\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda - \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda\right] \\ &+ [f(a)+f(b)]\int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned}$$

Theorem 2. *[13] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is an F_h-convex subset of the real $\mathbb{T}$ and $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$ with $a < b$ and*

$s \in [0,1]$. *Then for real numbers* $l_f(\lambda)$ *and* $L_f(\lambda)$*; the lower and upper bounds of* f *respectively, we have*

$$\begin{aligned}
& 2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \\
\leq\ & l_f(\lambda) \leq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t \\
+\ & \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right) b - \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t) \diamond_{\frac{1}{2}} t \\
\leq\ & L_f(\lambda) \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s \left[\int_a^b f(t)\,\Delta t + \int_a^b f(t)\,\nabla t\right],
\end{aligned}$$

where

$$\begin{aligned}
l_f(\lambda) &= 2^s\left(h(\frac{1}{2})\right)^s \frac{1}{2}\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s + 1\right) a\right)\right] \\
&+ 2^s\left(h(\frac{1}{2})\right)^s \frac{1}{2}\left[\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left(\left(\left(\frac{\lambda}{h(\lambda)}\right)^s + 1\right) b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
L_f(\lambda) = & \left(\frac{\lambda}{h(\lambda)}\right)^s \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t + \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right] \\
& + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\Delta t + \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\nabla t\right].
\end{aligned}$$

On the other hand, several authors have, in recent times, extended their research by obtaining Hermite-Hadamard type inequalities for convex and classes of convex IVF on classical intervals, see [2, 6, 23, 25].

In [24], Zhao *et al.* introduced the concept of IVF to the theory of time scales for convex interval-valued functions on time scales thus:

Definition 3. *We say that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ is a convex interval-valued function if for all $x, y \in [a,b]_{\mathbb{T}}$ and $\alpha \in (0,1)$, we have*

$$\alpha f(x) + (1-\alpha)f(y) \subseteq f(\alpha x + (1-\alpha)y), \tag{1.3}$$

for which $\alpha x + (1-\alpha)y \in [a,b]_{\mathbb{T}}$.

Remark 1. *If the set inclusion (1.3) is reversed, then f is said to be a concave interval-valued function on time scales. If f is both convex and concave, then f is said to be affine. The set of all convex, concave and affine interval-valued functions are denoted by*
$SX([a,b]_{\mathbb{T}}, \mathbb{R}_I), SV([a,b]_{\mathbb{T}}, \mathbb{R}_I)$ and $SA([a,b]_{\mathbb{T}}, \mathbb{R}_I)$, respectively.

It is the purpose of this paper to establish New Hermite-Hadamard-type integral inequalities for F_h-convex interval-valued functions on time scales. Some economic applications are provided to illustrate our results.

2. Preliminaries

First, we give some basic concepts used in this paper and also refer interested researcher to the books [5, 16] for a detailed theory of time scales and interval analysis.

A real interval $[z]$ is the bounded, closed subset of $\mathbb{R}$ defined by

$$[z] = [\underline{z},\ \overline{z}] = \{x \in \mathbb{R} |\ \underline{z} \leq x \leq \overline{z}\},$$

where $\underline{z},\ \overline{z} \in \mathbb{R}$ and $\underline{z} \leq \overline{z}$. The left and right *endpoints* of $[\underline{z},\ \overline{z}]$ are $\underline{z}$ and $\overline{z}$ respectively.

A *time scale*, denoted $\mathbb{T}$, is any arbitrary nonempty closed subset of the real numbers. The interval $[a,b]_{\mathbb{T}}$ of $\mathbb{T}$ is defined by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Time scale $\mathbb{T}$ allows for classification of points in the following ways:

For all $t \in \mathbb{T}$, σ denotes the *forward jump operator* $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau \geq t\}$ while ρ denotes the *backward jump operator* $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau \leq t\}\, \forall\, t \in \mathbb{T}$.

The point t is said to be *right-scattered* if $\sigma(t) > t$, respectively *left-scattered* if $\rho(t) < t$. Points that are right-scattered and left-scattered at the same time are called *isolated*. The point t is called *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, respectively *left-dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are called *dense*.

The *forward and backward graininess functions* μ and ν are defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) = t - \rho(t)$ respectively.

A mapping $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at all right-dense point or maximal element of $\mathbb{T}$ and the left-sided limit exists (finite) at each left-dense point $t \in \mathbb{T}$. $C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ denotes the set of rd-continuous functions; f is said to be *ld-continuous* if it is continuous at all left-dense point or minimal element of $\mathbb{T}$ and the right-sided limit exists (finite) at each right-dense point $t \in \mathbb{T}$. $C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$ denotes the set of ld-continuous functions. The *set of continuous functions* $C([a,b]_{\mathbb{T}}, \mathbb{R})$ on $[a,b]_{\mathbb{T}}$ contains both C_{rd} and C_{ld}.

A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ is said to be an *interval-valued function* of t on $[a,b]_{\mathbb{T}}$ if it assigns a nonempty interval

$$f(t) = [\underline{f}(t), \overline{f}(t)]$$

to each $t \in [a,b]_{\mathbb{T}}$. Thus, $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ is said to be *continuous* at $t_0 \in [a,b]_{\mathbb{T}}$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d(f(t), f(t_0)) < \epsilon,$$

whenever $|t - t_0| < \delta$. $C([a,b]_{\mathbb{T}}, \mathbb{R}_I)$ denotes the *set of continuous functions* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$. Thus, f is continuous at t_0 if and only if $\underline{f}$ and $\overline{f}$ are continuous at t_0.

3. Main Results

We introduce new concept of F_h-convex interval-valued functions on time scales, which is useful in the sequel.

Throughout this paper: Let $\mathbb{T}$ be a time scale and $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is F_h-convex interval-valued subset of the real $\mathbb{T}$.

Definition 4. *A mapping $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ is said to be F_h-convex interval-valued function on time scales or $f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ if*

$$f(\lambda x + (1-\lambda)y) \supseteq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y), \tag{3.1}$$

$\forall\ \lambda \in [0,1],\ s \in [0,1]$ *and for all* $x, y \in [a,b]_{\mathbb{T}}$.

Remark 2. *If the set inclusion (3.1) is reversed, then f is said to be F_h-concave interval-valued function on time scales. If f is both F_h-convex and F_h-concave, then f is F_h-affine interval-valued function on time scales.*
We denote by $SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, $SA(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ the set of all F_h-concave, F_h-affine interval-valued functions on time scales.

Remark 3. *Special cases of Definition 3.1 are discussed below.*

(i) $h(\lambda) = \lambda^{\frac{s}{s+1}}$, *then f is h-convex interval-valued function on time scales.*

(ii) *Choosing $s = 1$ and $h(\lambda) = 1$, then $f \in SX([a,b]_{\mathbb{T}}, \mathbb{R}_I)$, i.e, f satisfies definition 1.2 above. (see [24]).*

(iii) *If f is real-valued, then definition 1.1 is valid for F_h-convex function on time scales. (see [11]).*

(iv) *If $\underline{f}(x) = \overline{f}(x)$ and $h(\lambda) = \lambda^{\frac{s}{s+1}}$, then f is h-convex on time scales, see [11].*

(v) *For $s = 1$, $h(\lambda) = 1$, where f is real-valued, we say f is convex on time scales, see [8].*

(vi) *When $h(\lambda) = \lambda^{\frac{s}{s+1}}$ and $\mathbb{T} = \mathbb{R}$, f is h-convex interval-valued function on classical intervals, see [25].*

(vii) *The concept of h-convexity on classical intervals is obtained if $h(\lambda) = \lambda^{\frac{s}{s+1}}$, $\mathbb{T} = \mathbb{R}$ and $\underline{f}(x) = \overline{f}(x)$ (see [21]).*

(viii) *For $s = 1$, $h(\lambda) = 1$, and $\mathbb{T} = \mathbb{R}$, we recapture the concept of convex interval-valued function on classical intervals.*

(ix) *If $s = 1$, $h(\lambda) = 1$, $\mathbb{T} = \mathbb{R}$, where f is real-valued, then f is convex on classical intervals (see [4]).*

The following Proposition follows.

Proposition 1. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ be such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ for all $t \in [a,b]_{\mathbb{T}}$. Then,*

(i) *$f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ if and only if $\underline{f} \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ and $\overline{f} \in SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$,*

(ii) $f \in SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ *if and only if* $\underline{f} \in SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ *and* $\overline{f} \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$,

(iii) $f \in SA(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$ *if and only if* $\underline{f}, \overline{f} \in SA(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$.

Proof. According to definition 3.1, the proof is obvious.

Definition 5. *The diamond-F_h integral of an interval-valued function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ for all $t \in [a,b]_{\mathbb{T}}$ from a to b, where $a, b \in \mathbb{T}$ is given by*

$$\left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b [\underline{f}(t), \overline{f}(t)]\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b [\underline{f}(t), \overline{f}(t)]\nabla t = \int_a^b [\underline{f}(t), \overline{f}(t)]\diamond_{(F_{h(\lambda)})^s} t, \tag{3.2}$$

where $s \in [0,1]$ and $0 \le \lambda \le 1$.

The following result holds.

Theorem 3. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ be such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ for all $t \in [a,b]_{\mathbb{T}}$ from a to b, where $a, b \in \mathbb{T}$. If $f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I) \cup SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I) \cup SA(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, then $f \in I\mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s},\ [a,b]_{\mathbb{T}})}$.*

Proof. Let $f(t) = [\underline{f}(t), \overline{f}(t)] \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I) \cup SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I) \cup SA(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, for all $t \in [a,b]_{\mathbb{T}}$. According to proposition 3.1 and Theorem 3.3 of [**?**], it implies that $\underline{f}(t)$ and $\overline{f}(t)$ are continuous. Thus, by definition 3.2, $f \in I\mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s},\ [a,b]_{\mathbb{T}})}$.

Theorem 4. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ be an interval-valued function on a time scale $\mathbb{T}$ with $f \in \mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s}, [a,b]_{\mathbb{T}})}$; $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \ge 0$, where $\mathbb{J}_{\mathbb{T}}$ is an interval F_h-convex subset of the real $\mathbb{T}$. If $f \in C(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, then $f \in I\mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s},\ [a,b]_{\mathbb{T}})}$ and*

$$(I\mathbb{R}) \int_a^b f(t)\diamond_{(F_{h(\lambda)})^s} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \left[\int_a^b \underline{f}(t)\Delta t,\ \overline{f}(t)\Delta t\right] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[\int_a^b \underline{f}(t)\nabla t,\ \overline{f}(t)\nabla t\right],$$

$\forall\ \lambda \in [0,1],\ s \in [0,1]$ *and* $t \in [a,b]_{\mathbb{T}}$.

Proof. From Definition 3.2 and Theorem 3.1, the proof is obvious.

Now, we show representative applications of F_h-convex interval-valued functions, by establishing some new inequalities of H-H type via F_h-convex interval-valued functions on time scales.

Theorem 5. *Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}_I$ be an interval-valued function on a time scale $\mathbb{T}$ with $f \in \mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s}, [a,b]_{\mathbb{T}})}$; $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative continuous function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is F_h-convex interval-valued subset of the real $\mathbb{T}$. If $f \in SX(F_h, [a, b]_{\mathbb{T}}, \mathbb{R}_I)$, then*

$$\begin{aligned} 2^s \left[h(\frac{1}{2})\right]^s f\left(\frac{a+b}{2}\right) &\supseteq \frac{1}{b-a}\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x \\ &\supseteq f(a)\left[\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda - \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda\right] \\ &+ [f(a) + f(b)] \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned} \tag{3.3}$$

If $f \in SV(F_h, [a, b]_{\mathbb{T}}, \mathbb{R}_I)$, then

$$\begin{aligned} 2^s \left[h(\frac{1}{2})\right]^s f\left(\frac{a+b}{2}\right) &\subseteq \frac{1}{b-a}\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x \\ &\subseteq f(a)\left[\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda - \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda\right] \\ &+ [f(a) + f(b)] \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned}$$

Proof. By definition 3.1, f is F_h-convex interval-valued on $\mathbb{T}$.
By making the change of variables $x = \lambda a + (1-\lambda)b$, $y = (1-\lambda)a + \lambda b$ and $\lambda = \frac{1}{2}$, inequality (3.1) can be rewritten as

$$f\left(\frac{a+b}{2}\right) \supseteq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s [f(\lambda a + (1-\lambda)b) + f((1-\lambda)a + \lambda b)].$$

By Proposition 3.1,

$$f\left(\frac{a+b}{2}\right) \supseteq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s [\underline{f}(\lambda a+(1-\lambda)b)+\underline{f}((1-\lambda)a+\lambda b), \overline{f}(\lambda x+(1-\lambda)b)+\overline{f}((1-\lambda)a+\lambda b)].$$

$$= \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s [f(\lambda a + (1-\lambda)b)\Delta\lambda + f((1-\lambda)a + \lambda b)\nabla\lambda]. \tag{3.4}$$

Thus, integrating (3.4) with respect to λ on [0, 1], we get

$$f\left(\frac{a+b}{2}\right) \supseteq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_0^1 f(\lambda a + (1-\lambda)b)\Delta\lambda + \int_0^1 f((1-\lambda)a + \lambda b)\nabla\lambda\right]. \tag{3.5}$$

Substituting $x = \lambda a + (1-\lambda)b, \Delta x = (a-b)\Delta\lambda;\;\; y = (1-\lambda)a + \lambda b, \nabla y = (b-a)\nabla\lambda$ into (3.5), we obtain

$$2^s \left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x, \tag{3.6}$$

and the first inclusion is established.
Similarly, in light of Proposition 3.1, we have

$$\begin{aligned} & [\underline{f}(\lambda x + (1-\lambda)y), \overline{f}(\lambda x + (1-\lambda)y)] \\ \supseteq & \left[\left(\frac{\lambda}{h(\lambda)}\right)^s \underline{f}(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \underline{f}(y), \left(\frac{\lambda}{h(\lambda)}\right)^s \overline{f}(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \overline{f}(y)\right] \\ = & \left(\frac{\lambda}{h(\lambda)}\right)^s [\underline{f}(x), \overline{f}(x)] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [\underline{f}(y), \overline{f}(y)]. \end{aligned}$$

i.e.,

$$f(\lambda x + (1-\lambda)y) \supseteq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y). \tag{3.7}$$

Integrating (3.7) with $x = a$ and $y = b$, we get

$$\begin{aligned} \int_0^1 f(\lambda x + (1-\lambda)y) \diamond_{(F_{h(\lambda)})^s} \lambda \;\; \supseteq \;\; & f(a)\left[\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda - \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda\right] \\ + \;\; & [f(a) + f(b)]\int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned} \tag{3.8}$$

The second inclusion is established by performing the change of variable $x = \lambda a + (1-\lambda)b$ in (3.8) to get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x \;\; \supseteq \;\; & f(a)\left[\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda - \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda\right] \\ + \;\; & [f(a) + f(b)]\int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned} \tag{3.9}$$

Combining (3.6) and (3.9), the result follows easily.

Remark 4. *(i) If $h(\lambda) = \lambda^{\frac{s}{s+1}}$, then Theorem 3.3 can be obtained for an h-convex interval-valued function on time scales.*

(ii) A result for h-convex function on time scales is obtained if $h(\lambda) = \lambda^{\frac{s}{s+1}}$ and f is real-valued.

(iii) *If* $\underline{f}(x) = \overline{f}(x)$*, then Theorem 3.3 reduces to the result of Fagbemigun et al.([12], Theorem 2.1).*

(iv) *By choosing* $x = \lambda a + (1-\lambda)b,\ s = 1$ *and* $h(\cdot) = 1$ *in (3.3), where* f *is real-valued, we recover the second inequality in Theorem 3.9 of Dinu [9].*

(v) *When* $\mathbb{T} = \mathbb{R},\ h(\frac{1}{2}) = \frac{1}{2}\ s = 1$ *and* f *real-valued in (3.3), we recover the first part of the classical Hermite-Hadamard inequality. (see [4, 18]).*

(vi) *If* $h(\lambda) = \lambda^{\frac{s}{s+1}}$*,* f *is real-valued and* $\mathbb{T} = \mathbb{R}$*, we obtain Theorem 6 due to Sarikaya et al. [19].*

(vii) *If* $s = 1,\ h(\frac{1}{2}) = \frac{1}{4},\ \mathbb{T} = \mathbb{R}$ *and* f *is real-valued, we recover the first inequality of Theorem 3.1 due to [10].*

(viii) *If* $h(\lambda) = \lambda^{\frac{s}{s+1}},\ \mathbb{T} = \mathbb{R}$*, where* $s = 1$*, we recover Theorem 4.1 in [23].*

Theorem 6. *Let* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ *be an interval-valued function on a time scale* $\mathbb{T}$ *with* $f \in \mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s}, [a,b]_{\mathbb{T}})}$*;* $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ *be a non zero non negative continuous function with the property that* $h(t) > 0$ *for all* $t \geq 0$*, where* $\mathbb{J}_{\mathbb{T}}$ *is* F_h*-convex interval-valued subset of the real* $\mathbb{T}$*. Then for real numbers* $l_f(\lambda)$ *and* $L_f(\lambda)$*; the lower and upper bounds of* f *respectively, and for* $f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$*, we have*

$$\begin{aligned}
&2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right)\\
\supseteq\ & l_f(\lambda) \supseteq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t\\
+\ & \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1 - \left(\frac{\lambda}{h(\lambda)}\right)^s\right) b - \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t) \diamond_{\frac{1}{2}} t\\
\supseteq\ & L_f(\lambda) \supseteq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s \left[\int_a^b f(t)\,\Delta t + \int_a^b f(t)\,\nabla t\right],
\end{aligned} \tag{3.10}$$

where $l_f(\lambda)$ *and* $L_f(\lambda)$ *are as stated in Theorem 1.2 above.*

If $f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, then

$$\begin{aligned}
&2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right)\\
\subseteq\ & l_f(\lambda) \subseteq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t\\
+\ & \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right) b - \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t) \diamond_{\frac{1}{2}} t\\
\subseteq\ & L_f(\lambda) \subseteq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s \left[\int_a^b f(t)\,\Delta t + \int_a^b f(t)\,\nabla t\right].
\end{aligned}$$

Proof. Using Proposition 3.1, Theorem 3.2 and the results of Fagbemigun and Mogbademu ([11], Theorem 2.1), the proof follows easily.

Remark 5. *Theorem 3.4 refines and generalizes previous results in literature as follows:*

(i) If $\underline{f}(x) = \overline{f}(x)$, then a result of Fagbemigun and Mogbademu ([11], Theorem 2.1) is obtained.

(ii) For $\underline{f}(x) = \overline{f}(x)$, the first three inequalities in ([9], (5.7)) are obtained by applying (3.3) for $\lambda = \frac{1}{2}$, $s = 1$, $h(\lambda) = 1$ and $h(\frac{1}{2}) = \frac{1}{2}$ in (3.10), which is a further refinement and improvement of ([9], Theorem 3.9), ([15], Theorem 1.1.) and ([22], Theorem 2.E.).

(iii) If $\mathbb{T} = \mathbb{R}$, and f is real-valued, then (3.10) is the same as inequality (5.1) of [9].

Corollary 7. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ be an interval-valued function on a time scale $\mathbb{T}$ such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ with $f \in \mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s}, [a,b]_{\mathbb{T}})}$; $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative continuous function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is F_h-convex interval-valued subset of the real $\mathbb{T}$. Then for $f \in SX(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$, we have the following inclusions for lower and upper bounds $l_f(\lambda)$ and $L_f(\lambda)$ of real numbers respectively.*

$$\begin{aligned}
&2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right)\\
\supseteq\ & \sup_{\lambda\in[0,1]} l_f(\lambda) \supseteq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t
\end{aligned}$$

$$+\frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s}}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^{s}\right)b-\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s}a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^{s}b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s}a}^{b} f(t)\diamond_{\frac{1}{2}}t$$

$$\supseteq \inf_{\lambda\in[0,1]} L_f(\lambda) \supseteq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^{s}\left[\int_a^b f(t)\,\Delta t+\int_a^b f(t)\,\nabla t\right], \tag{3.11}$$

where $l_f(\lambda)$ and $L_f(\lambda)$ are as stated in Theorem 1.2.

If the set inclusions (3.11) are reversed, then $f \in SV(F_h, [a,b]_{\mathbb{T}}, \mathbb{R}_I)$.

4. Applications in Economics

The concept of the theory of time scales is applicable, not only to the variational calculus as seen in [4] but equally finds flexible and capable modeling application techniques in the field of Economics, whose dynamic processes can be typically described with discrete or continuous time systems, variables or models(see [1, 14]).

The Household Utility Problem

The household utility problem in Economics is a dynamic optimization problem which is set up in the following form: a representative consumer seeks to maximize his/her lifetime utility U subject to certain budgetary constraints A (see [7]). There is the (constant) discount factor δ, which satisfies $0 \leq \delta \leq 1$, C_s is consumption during period s, $u(C_s)$ is the utility the consumer derives from consuming C_s units of consumption in periods $s= 0, 1, 2, ...,T$. Utility is assumed to be concave: $u(C_s)$ has $u(C_s)'>0$ and $u(C_s)''<0$. The consumer receives some income Y in a time period s and decides how much to consume and save during that same period. If the consumer consumes more today, the utility or satisfaction he derives from consumption, is forgone tomorrow as the detterrence. Normally, the consumer is insatiable. However, each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period (Law of diminishing marginal utility, LDMU) (see [1]).

The individual is constrained by the fact that the value function of his consumption, $u(C)$ must be equal to the value function of his income Y_s, plus the assets/debts, A_s that he might accrue in a period s. Hence, A_{s+1} is the amount of assets held at the beginning of period $s + 1$. Also, A could be positive or negative; the consumer might save for the future or borrow against the future at interest rate r in any given period s but the value of A_T, which is the debt accrued with limit or the last period asset holding, has to be nonnegative (the optimal level is naturally zero).

In order to state the necessary and sufficient condition for optimization in the formulation of a dynamic optimization problem, it is important to present the simplest form of optimal control problem in terms of the diamond-F_h integral as;

$$\max J_{\diamond_{(F_{h(\lambda)})^s}}[x,u] = \int_a^b L(t,x,u)\diamond_{(F_{h(\lambda)})^s} t$$
$$= \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b L(t,x^\sigma,u^\sigma)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b L(t,x^\rho,u^\sigma)\nabla t, \tag{4.1}$$

for all $\lambda \in [0,1]$ and $s \in [0,1]$, among all pairs (x,u) such that $x^\Delta = f(t,x^\sigma,u^\sigma)$ and $x^\Delta = f(t,x^\rho,u^\sigma)$, together with appropriate endpoint conditions $u^{\diamond_{(F_{h(\lambda)})^s}{}'}(t) = L(t,u,p)$, $x(0) = u_0$, $u(T)$ free for all $t \in [0,T]$.

A simple utility maximization model of household consumption in Economics for a function of single variable can be refined and solved in time scales settings in order to obtain better estimates of the maximized utility function, using the same intuition as that of the dynamic optimization problem presented earlier, by employing our developed concepts as follows. The model assumes a perfect foresight.

Theorem 8. *Let $u : [a,b]_{\mathbb{T}} \to \mathbb{R}_I$ be an interval-valued function on a time scale $\mathbb{T}$ with $u \in \mathbb{R}_{(\diamond_{(F_{h(\lambda)})^s},[a,b]_{\mathbb{T}})}$; $h: \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(\frac{1}{2}) \neq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. Then*

$$Maximize\ U_{\diamond_{(F_{h(\lambda)})^s}} = \sup_{\lambda\in[0,1]} l_F(\lambda) \supseteq \int_0^T u(C(t))\, e_{-\delta}(t,\ 0) \diamond_{(F_{h(\lambda)})^s} t \supseteq \inf_{\lambda\in[0,1]} L_F(\lambda), \tag{4.2}$$

subject to the budget constraints

$$a^\nabla(t) = (rA + Y - C)(\rho(t)), \quad \frac{r}{1\ +\ r\mu(t)} a^\sigma(t) + \frac{1}{1+r\mu(t)} y^\sigma(t) - \frac{1}{1+r\mu(t)} c^\sigma(t), \tag{4.3}$$
$$a^\Delta(t) = \qquad a(0) = a_0, \qquad\qquad a(T) = a_T,$$

where u is F_h-concave ($u'(C) > 0$ and $u''(C) < 0$), $s \in [0,1], \lambda \in [0,1]$, $l_F(\lambda)$, $L_F(\lambda)$, $A^\Delta, A^\nabla, r, \delta, A, Y$ and e are as defined above.

Proof. Let $f(t)$ be a function satisfied by the consumption function path that would maximize lifetime utility $U(C(t))e_{-\delta}(t,\ 0)$ in (4.2), then the condition for a functional of the form

$$\int_a^b L(t,x,u)\diamond_{(F_{h(\lambda)})^s} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b L(t,x^\sigma,u^\sigma)\,\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b L(t,x^\rho,u^\sigma)\,\nabla t,$$

for all $s \in [0,1]$ and $0 \le \lambda \le 1$, to have a local extremum for a function $u(t)$ and the sufficient condition for an absolute maximum(minimum) of the functional hold. Since both local and absolute extrema hold, then the functional satisfies the sufficient conditions for optimization, which in turn satisfies Theorem 3.3.

The model (4.2)-(4.3) can then be analyzed by writing (4.2) in terms of (3.2), the maximum principle and the Hamiltonian function for the model. The result follows easily.

Remark 6. *The $\diamond_{(F_{h(\lambda)})^s}$ interval-valued household utility model (4.2)-(4.3) unifies and extends both the discrete and continuous classical models, as well as Δ, ∇, $\diamond_\alpha$ and $\diamond_{F_h}$ time scale models as special cases:*

(i) The $\diamond_{(F_{h(\lambda)})^s}$ time scales model reduces to $\diamond_\alpha$ model if $F_h = \alpha,\ s = 1,\ h(\lambda) = 1$ and f is real-valued.

(ii) If $F_h = 1,\ s = 1,\ h(\lambda) = 1$ and $u(t) = [\underline{u}(t), \overline{u}(t)]$, then (4.2)-(4.3) is reduced to the Δ household utility model of [1].

(iii) The standard ∇ time scales model is obtained for $F_h = 0,\ s = 1,\ h(\lambda) = 1$ and $u(t) = [\underline{u}(t), \overline{u}(t)]$ (see [14]).

(iv) When $\mathbb{T} = \mathbb{Z}$ and f is real-valued, (4.2) yields the result of Chiang on classical intervals, see [7].

(v) Choosing $\mathbb{T} = \mathbb{R}$ gives the result of [7] for real-valued function f.

(vi) If f is real-valued, then Theorem 4.1 gives Theorem 3.1 of [[13], Theorem 3.1].

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Bosede Opeyemi Fagbemigun
Department of Mathematical Sciences, Faculty of Natural Sciences,
Ajayi Crowther University,
Oyo, Nigeria.
email: *bo.fagbemigun@acu.edu.ng*

Adesanmi Alao Mogbademu
Department of Mathematics,
Faculty of Science,
University of Lagos,
Lagos, Nigeria.
email: *amogbademu@unilag.edu.ng*