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THE CODES OVER A FAMILY OF FINITE RINGS AND SOME APPLICATIONS

A. Dertli and Y. Cengellenmis

Abstract. Some types of codes have received much attention due to their applications in DNA computing. One of them is the skew cyclic codes. The other is the linear codes with special generator matrices. In this paper, by using two types of codes, the DNA codes are obtained. Firstly, the skew cyclic codes over a family of the finite rings $M_e = Z_4[u_1, ..., u_e]/ < u_i^3 - 1, u_iu_j - u_ju_i >$ are introduced, where $i, j = 1, 2, ..., e, i \neq j$. We define a non trivial automorphism θ_e over M_e and a generalized Gray map over M_e which preserves DNA reversibility. The DNA 3^i-mers are matched with the elements of the finite rings M_i, where $i = 1, ..., e$. The reversibility problem for DNA codes over a family of the finite rings M_e is solved, by using the skew cyclic codes over M_e. Secondly, in [4], a novel design strategy has been given to obtain DNA codes. We generalized it to the codes over a family of the finite rings M_e.

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1. Introduction

One type of biological computing is DNA computing. It computes faster with a lower energy consumption. It uses DNA as a data storage device to solve complex problems. As DNA (Deoxyribonucleic Acid) is a good platform to store more data effectively, designing DNA codes that satisfy some constraints has been a topic of popular research, recently. There are lots of methods to obtain them. The two methods are used in this paper to create them.

A DNA code C of length n is a subset of $S_{D_4}^n$, where $S_{D_4} = \{A, T, C, G\}$ is DNA alphabet.

The reversibility problem is very important in DNA computing. Let $(\alpha_1, \alpha_2) \in M_1^2$ be a codeword corresponds to $ATTGCC$. The reverse of the (α_1, α_2) is (α_2, α_1).

The vector (α_2, α_1) corresponding to $GCCATT$. It is not the reverse of $ATTGCC$. The reverse of $ATTGCC$ is $CCGTTA$.

Some authors used different approaches to solve this problem [1, 2, 6, 7, 8].

In [3], by defining a nontrivial automorphism, the skew cyclic codes over the finite ring $R_2 = F_4 + uF_4 + vF_4 + uvF_4, u^2 = u, v^2 = v, uv = vu$ were introduced. DNA 4-bases were matched with the elements 256 of the finite ring R_2. The reversible DNA codes were obtained from them.

In [5], the reversibility problem for DNA 2^a-bases was studied by using the skew cyclic codes over the finite ring R_a.

In the first part of this paper, motivated by these works, we study the reversibility problem for DNA 3^e-bases. Thanks to them, reversible DNA codes are obtained.

In [4], a new method was given to obtain DNA codes. In the second part of this paper, we generalize it to codes over a family of the finite rings M_e.

The rest of the paper is organized as follows. In Section II, preliminaries are presented. In Section III, a non-trivial automorphism on M_e is given to define the skew cyclic codes over M_e. In Section IV, a distance conserving map from M_i to $S_{D_4}^{3^i}$ is defined. By using a method in [4], we derive some conditions on the generator matrix of a linear code over M_i, for $i = 1, 2, 3, ...e$. We get the DNA codes that satisfy some constraints. In Section V, by using Reed-Muller types codes over M_i, the constructions of DNA codes are presented, where $i = 2, ..., e$. The parameters of the DNA codes obtained by this method are given. Some examples are obtained.

2. Preliminaries

A family of the finite rings $M_e = Z_4[u_1, ..., u_e]/ < u_i^3 - 1, u_iu_j - u_ju_i >$, where $i, j = 1, 2, ..., e, i \neq j$ contains the commutative the finite rings with characteristic 4 and cardinality 4^{3^e}. The finite rings of the family are written as recursively

$$M_j = M_{j-1} + u_jM_{j-1} + u_j^2M_{j-1}$$

where $j = 1, 2, ..., e$ and $u_j^3 = 1$.

Moreover $M_1 = Z_4 + u_1Z_4 + u_1^2Z_4, u_1^3 = 1$, where $M_0 = Z_4 = \{0, 1, 2, 3\}$.

We defined the Gray map as follows,

$$\begin{array}{rcl} \phi_i & : & M_i \longrightarrow M_{i-1}^3 \\ x_{i-1} + u_iy_{i-1} + u_i^2z_{i-1} & \longmapsto & (y_{i-1}, x_{i-1}, z_{i-1}) \end{array}$$

where $i = 2, .., e$ and

$$\begin{array}{rcl} \phi_1 & : & M_1 \longrightarrow M_0^3 \\ x_0 + u_1y_0 + u_1^2z_0 & \longmapsto & (y_0, x_0, z_0) \end{array}$$

Moreover

$$\begin{array}{rcl} \phi & : & M_i \longrightarrow M_0^{3^i} \\ \alpha_i = x_{i-1} + u_i y_{i-1} + u_i^2 z_{i-1} & \longmapsto & (\phi_1(\phi_2(...(\phi_i(\alpha_i))))) \end{array}$$

where $i = 1, 2, ..., e$.

Example 1. *Let $e = 2$. Then the Gray map is $\phi(x_1 + u_2 y_1 + u_2^2 z_1) = \phi_1(\phi_2((x_1 + u_2 y_1 + u_2^2 z_1) = \phi_1(y_1, x_1, z_1) = (\phi_1(y_1), \phi_1(x_1), \phi_1(z_1)) \in M_0^9$.*

By defining the matching the elements of M_0 and $S_{D_4} = \{A, T, C, G\}$ which is given as $\xi_0(0) = A, \xi_0(3) = T, \xi_0(1) = C, \xi_0(2) = G$ and by using the Gray map from $M_1 = Z_4 + u_1 Z_4 + u_1^2 Z_4$ to Z_4^3, we define a ξ_1 correspondence between the elements of the finite ring $M_1 = Z_4 + u_1 Z_4 + u_1^2 Z_4$ and DNA 3-mers as follows

$$\begin{array}{rcl} \xi_1 & : & M_1 \longrightarrow S_{D_4}^3 \\ \alpha_1 = x_0 + u_1 y_0 + u_1^2 z_0 & \longmapsto & (\xi_0(y_0), \xi_0(x_0), \xi_0(z_0)) = \xi_1(\alpha_1) \end{array}$$

and give the following table,

elements α_1	DNA 3-mers $\xi_1(\alpha_1)$
0	AAA
1	ACA
2	AGA
3	ATA
u_1	CAA
$1 + u_1$	CCA
$2 + u_1$	CGA
$3 + u_1$	CTA
$2u_1$	GAA
$1 + 2u_1$	GCA
$2 + 2u_1$	GGA
$3 + 2u_1$	GTA
$3u_1$	TAA
$1 + 3u_1$	TCA
$2 + 3u_1$	TGA
$3 + 3u_1$	TTA
u_1^2	AAC
$1 + u_1^2$	ACC
$2 + u_1^2$	AGC
$3 + u_1^2$	ATC
$u_1 + u_1^2$	CAC
$1 + u_1 + u_1^2$	CCC

elements α_1	DNA 3-mers $\xi_1(\alpha_1)$
$2+u_1+u_1^2$	CGC
$3+u_1+u_1^2$	CTC
$2u_1+u_1^2$	GAC
$1+2u_1+u_1^2$	GCC
$2+2u_1+u_1^2$	GGC
$3+2u_1+u_1^2$	GTC
$3u_1+u_1^2$	TAC
$1+3u_1+u_1^2$	TCC
$2+3u_1+u_1^2$	TGC
$3+3u_1+u_1^2$	TTC
$2u_1^2$	AAG
$1+2u_1^2$	ACG
$2+2u_1^2$	AGG
$3+2u_1^2$	ATG
$u_1+2u_1^2$	CAG
$1+u_1+2u_1^2$	CCG
$2+u_1+2u_1^2$	CGG
$3+u_1+2u_1^2$	CTG
$2u_1+2u_1^2$	GAG
$1+2u_1+2u_1^2$	GCG
$2+2u_1+2u_1^2$	GGG
$3+2u_1+2u_1^2$	GTG
$3u_1+2u_1^2$	TAG
$1+3u_1+2u_1^2$	TCG
$2+3u_1+2u_1^2$	TGG
$3+3u_1+2u_1^2$	TTG
$3u_1^2$	AAT
$1+3u_1^2$	ACT
$2+3u_1^2$	AGT
$3+3u_1^2$	ATT
$u_1+3u_1^2$	CAT
$1+u_1+3u_1^2$	CCT
$2+u_1+3u_1^2$	CGT
$3+u_1+3u_1^2$	CTT
$2u_1+3u_1^2$	GAT
$1+2u_1+3u_1^2$	GCT
$2+2u_1+3u_1^2$	GGT
$3+2u_1+3u_1^2$	GTT
$3u_1+3u_1^2$	TAT
$1+3u_1+3u_1^2$	TCT
$2+3u_1+3u_1^2$	TGT
$3+3u_1+3u_1^2$	TTT

For $\alpha_1 \in M_1$, we have that the complement of $\xi_1(\alpha_1)$ is equal to $[\xi_1(\alpha_1^c)]$ where $\alpha_1^c = 3 + 3u_1 + 3u_1^2 - \alpha_1$.

Example 2. *Let $\alpha_1 = 3+u_1+2u_1^2 \in M_1$. Then $\xi_1(\alpha_1) = CTG$. On the other hand, we get $\xi_1(\alpha_1^c) = GAC$, since $\alpha_1^c = 2u_1 + u_1^2$.*

Similarly, we define a ξ_2 correspondence between elements of the ring $M_2 = (Z_4 + u_1Z_4 + u_1^2Z_4) + u_2(Z_4 + u_1Z_4 + u_1^2Z_4) + u_2^2(Z_4 + u_1Z_4 + u_1^2Z_4)$ and DNA 9-mers as follows:

$$\begin{array}{rcl} \xi_2 & : & M_2 \longrightarrow S_{D_4}^9 \\ \alpha_2 = x_1 + u_2y_1 + u_2^2z_1 & \longmapsto & (\xi_1(y_1), \xi_1(x_1), \xi_1(z_1)) \end{array}$$

and give the following table,

elements α_2	DNA 9-mers $\xi_2(\alpha_2)$
0	$AAAAAAAAA$
1	$AAAACAAAA$
2	$AAAAGAAAA$
3	$AAAATAAAA$
...	
...	

For $\alpha_2 \in M_2$, we have that the complement of $\xi_2(\alpha_2)$ is equal to $\xi_2(\alpha_2^c)$ where $\alpha_2^c = \mathbf{3 + 3u + 3u^2} - \alpha_2$, where $\mathbf{3 + 3u + 3u^2} \in M_2$ where its all coefficients are $3 + 3u_1 + 3u_1^2$.

Example 3. *Let $\alpha_2 = 1+2u_1+u_2+2u_2^2 \in M_2$. Then $\xi_2(\alpha_2)) = \xi_2((1+2u_1)+u_2+2u_2^2) = (\xi_1(1), \xi_1(1+2u_1), \xi_1(2)) = ACAGCAAGA$. On the other hand, $\xi_2(\alpha_2^c) = \xi_2((3+3u_1+3u_1^2)-(1+2u_1)) + u_2((3+3u_1+3u_1^2)-1) + u_2^2((3+3u_1+3u_1^2)-2) = (\xi_1(2+3u_1+3u_1^2), \xi_1(2+3u_1+3u_1^2), \xi_1(1+3u_1+3u_1^2)) = TGTCGTTCT$, where $\alpha_2^c = (2+u_1+3u_1^2) + u_2(2+3u_1+3u_1^2) + u_2^2(1+3u_1+3u_1^2)$.*

Similarly, we can define ξ_i for $i = 3, ..., e$. Moreover for every $\alpha_i \in M_i$, we have that the complement of $\xi_i(\alpha_i)$ is equal to $[\xi_i(\alpha_i^c)]$ where $\alpha_i^c = \mathbf{3 + 3u + 3u^2} - \alpha_i$, where $\mathbf{3 + 3u + 3u^2} \in M_i$ where its all coefficients are $3+3u_1+3u_1^2$, where $i = 3, ...e$.

3. Skew Cyclic Codes over M_e

Definition 1. *Let B be a finite ring and θ be a non-trivial automorphism on B. A subset C of B^n is called a skew cyclic code of length n if C satisfies the following conditions,*

1. *C is a submodule of B^n*
2. *If $c = (c_0, c_1, ..., c_{n-1}) \in C$, then $\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), ..., \theta(c_{n-2})) \in C$, where σ_θ is the skew cyclic shift operator.*

By defining a non-trivial automorphism on M_e as follows, we can define the skew cyclic codes over M_e.

$$\begin{array}{rcl} \theta_i & : & M_i \longrightarrow M_i \\ x_{i-1} + u_i y_{i-1} + u_i^2 z_{i-1} & \longmapsto & \theta_{i-1}(x_{i-1}) + u_i \theta_{i-1}(z_{i-1}) + u_i^2 \theta_{i-1}(y_{i-1}) \end{array}$$

and

$$\begin{array}{rcl} \theta_1 & : & M_1 \longrightarrow M_1 \\ x_0 + u_1 y_0 + u_1^2 z_0 & \longmapsto & x_0 + u_1 z_0 + u_1^2 y_0 \end{array}$$

where $i = 2, 3, ..., e$. The order of θ_i is 2, where $i = 1, 2, ..., e$.

The rings

$$M_i[x, \theta_i] = \{b_0^i + b_1^i x + ... + b_{n-1}^i x^{n-1} : b_j^i \in M_i, n \in N, i = 1, ..., a, j = 0, ..., n-1\}$$

are called skew polynomial rings with the usual polynomial addition and multiplication as follows

$$(\varrho x^s)(\eta x^t) = \varrho \theta_i^s(\eta) x^{s+t}$$

where $i = 1, ..., e$. They are non-commutative rings.

In polynomial representation, a skew cyclic code of length n over M_i is defined as a left ideal of the quotient ring $M_{\theta_i,n} = M_i[x, \theta_i] / \langle x^n - 1 \rangle$, if the order of θ_i divides n, that is n is even. If the order of θ_i does not divide n, a skew cyclic code of length n over M_i is defined as a left $M_i[x, \theta_i]$-submodule of $M_{\theta_i,n}$, since the set $M_{\theta_i,n} = M_i[x, \theta_i] / \langle x^n - 1 \rangle = \{f_i(x) + \langle x^n - 1 \rangle : f_i(x) \in M_i[x, \theta_i]\}$ is a left $M_i[x, \theta_i]$-module with the multiplication from left defined by

$$r_i(x)(f_i(x) + \langle x^n - 1 \rangle) = r_i(x) f_i(x) + \langle x^n - 1 \rangle$$

where for any $r_i(x) \in M_i[x, \theta_i]$, for $i = 1, ..., e$.

In both cases, the following is held.

Theorem 1. *Let C_i be a skew cyclic code over M_i and let $f_i(x)$ be a polynomial in C_i of minimal degree, $i = 1, ..., e$. If the leading coefficient of $f_i(x)$ is a unit in M_i, then $C_i = \langle f_i(x) \rangle$, where $f_i(x)$ is a right divisor of $x^n - 1$.*

Definition 2. *For* $\mathbf{x} = \left(x_0^i, x_1^i, ..., x_{n-1}^i\right) \in M_i^n$, *the vector* $\left(x_{n-1}^i, x_{n-2}^i, ..., x_1^i, x_0^i\right)$ *is called the reverse of* $\mathbf{x}$ *and is denoted by* $\mathbf{x}^r$. *A linear code* C_i *of length* n *over* M_i *is called reversible if* $\mathbf{x}^r \in C_i$ *for every* $\mathbf{x} \in C_i$, *where* $i = 1, ..., e$.

We can express the matching the elements M_1 and $S_{D_4}^3 = S_{D_{64}} = \{AAA, TTT, ..., GGG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1 = x_0 + u_1 y_0 + u_1^2 z_0 \in M_1$ and $\theta_1(\alpha_1)$ are mapped to DNA 3-mers which are reverse of each other. Let ξ_1 be a correspondence between the elements of the finite ring M_1 and DNA 3-mers. For example

$$\xi_1(u_1) = CAA, \text{ while } \xi_1\left(\theta_1(u_1)\right) = AAC$$

This can be extended to a map γ_i from M_{i-1}^3 to 3^i-mers as follows,

$$\gamma_i(s_{i-1}, t_{i-1}, r_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}), \xi_{i-1}(r_{i-1}))$$

where $s_{i-1}, t_{i-1}, r_{i-1} \in M_{i-1}$, for $i = 1, ..., e$.

By using a map $\xi_i = \gamma_i \circ \phi_i$, we can explain the relationship between skew cyclic codes and DNA codes. $\xi_i(r_i)$ and $\xi_i\left(\theta_i(r_i)\right)$ are DNA reverse of each other, where $m_i = a_{i-1} + u_i b_{i-1} + u_i^2 c_{i-1}$ and $a_{i-1}, b_{i-1}, c_{i-1} \in M_{i-1}$, where $i = 1, ..., e$.

For $m_i = a_{i-1} + u_i b_{i-1} + u_i^2 c_{i-1} \in M_i$, we have

$$\begin{aligned} \xi_i(m_i) &= \gamma_i\left(\phi_i(a_{i-1} + u_i b_{i-1} + u_i^2 c_{i-1})\right) = \gamma_i\left(b_{i-1}, a_{i-1}, c_{i-1}\right) \\ &= (\xi_{i-1}(b_{i-1}), \xi_{i-1}(a_{i-1}), \xi_{i-1}(c_{i-1})) \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi_i\left(\theta_i(m_i)\right) &= \xi_i\left(\theta_{i-1}(a_{i-1}) + u_i\theta_{i-1}(c_{i-1}) + u_i^2\theta_{i-1}(b_{i-1})\right) \\ &= \gamma_i\left(\phi_i\left(\theta_{i-1}(a_{i-1}) + u_i\theta_{i-1}(c_{i-1}) + u_i^2\theta_{i-1}(b_{i-1})\right)\right) \\ &= \gamma_i\left(\theta_{i-1}(c_{i-1}), \theta_{i-1}(a_{i-1}), \theta_{i-1}(b_{i-1})\right) \\ &= \left(\xi_{i-1}\left(\theta_{i-1}(c_{i-1})\right), \xi_{i-1}\left(\theta_{i-1}(a_{i-1})\right), \xi_{i-1}\left(\theta_{i-1}(b_{i-1})\right)\right) \end{aligned}$$

where $i = 1, ..., e$.

This map can be extended as follows. For any $m_i = (m_0^i, ..., m_{n-1}^i) \in M_i^n$, where $i = 1, 2, ..., e$.

$$\left(\xi_i\left(m_0^i\right), \xi_i\left(m_1^i\right), ..., \xi_i\left(m_{n-1}^i\right)\right)^r = \left(\xi_i\left(\theta_i\left(m_{n-1}^i\right)\right), ..., \xi_i\left(\theta_i\left(m_1^i\right)\right), \xi_i\left(\theta_i\left(m_0^i\right)\right)\right)$$

Example 4. *If* $m_2 = u_1 + u_2(1 + 2u_1) + u_2^2 \in M_2$, *then we have*

$$\begin{aligned} \xi_2(m_2) &= \gamma_2\left(\phi_2(m_2)\right) = \gamma_2\left(1 + 2u_1, u_1, 1\right) \\ &= \left(\xi_1\left(1 + 2u_1\right), \xi_1(u_1), \xi_1(1)\right) = GCACAAACA \end{aligned}$$

On the other hand,

$$\begin{aligned}\xi_2(\theta_2(m_2)) &= \xi_2\left(\theta_1(u_1)+u_2\theta_1(1)+u_2^2\theta_1(1+2u_1)\right)\\ &= \gamma_2\left(\theta_1(1),\theta_1(u_1),\theta_1(1+2u_1)\right)\\ &= \left(\xi_1\left(\theta_1(1)\right),\xi_1\left(\theta_1(u_1)\right),\xi_1(\theta_1(1+2u_1))\right)\\ &= ACAAACACG\end{aligned}$$

Definition 3. *Let C_i be a code of length n over M_i, for $i=1,...,e$. If $\xi_i(\mathbf{c})^r \in \xi_i(C_i)$ for all $\mathbf{c}\in C_i$, then C_i or equivalently $\xi_i(C_i)$ is called a reversible DNA code.*

Definition 4. *Let $g_i(x)=b_0^i+b_1^ix+b_2^ix^2+...+b_s^ix^s$ be a polynomial of degree s over M_i, for $i=1,...,e$. $g_i(x)$ is called a palindromic polynomial if $b_j^i=b_{s-j}^i$ for all $j\in\{0,1,...,s\}$. $g_i(x)$ is called a θ_i-palindromic polynomial if $b_j^i=\theta_i(b_{s-j}^i)$ for all $j\in\{0,1,...,s\}$, for $i=1,...,e$.*

As the order of θ_i is 2, a skew cyclic code of odd length n over M_i with respect to θ_i is an ordinary cyclic code. So we will take the length n to be even, where $i=1,2,...,e$.

Theorem 2. *Let $C_i=\langle f_i(x)\rangle$ be a skew cyclic code of length n over M_i, for $i=1,...,e$, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a θ_i-palindromic polynomial and $f_i(x)=a_0^i+a_1^ix+...+a_{2s-1}^ix^{2s-1}$. So $a_d^i=\theta_i(a_{2s-1-d}^i)$, for all $d=0,1,...,s-1$. Let $h_i(x)=h_0^i+h_1^ix+...h_{2k-1}^ix^{2k-1}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l=1,...,n-1$. For any $t<n/2$, the coefficient of x^t in $h_i(x)f_i(x)$ is

$$b_t^i=\sum_{j=0}^{t}h_j^i\theta_i^j(a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i=\sum_{j=0}^{t}h_{2k-1-j}^i\theta_i^{2k-1-j}(a_{2s-1-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x)=\sum_{p=0}^{2k-1}h_p^ix^pf_i(x)$ corresponds a vector $b=(b_0^i,b_1^i,...,b_{n-1}^i)\in C_i$, for $i=1,...,a$.

The vector $\xi_i(b)^r=\left(\left(\xi_i\left(b_0^i\right),\xi_i\left(b_1^i\right),...,\xi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\xi_i(z)$, where the vector z corresponds to polynomial $\sum_{p=0}^{2k-1}\theta_i(h_p^i)x^{2k-1-p}f_i(x)$,for $i=1,...,a$.

Since $z=(z_1^i,...,z_n^i)\in C_i$, then $\xi_i(C_i)$ is a reversible DNA code, for $i=1,...,a$.

Theorem 3. *Let $C_i=\langle f_i(x)\rangle$ be a skew cyclic code of length n over M_i, for $i=1,...,e$, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a palindromic polynomial with an even degree. $f_i(x) = a_0^i + a_1^i x + ... + a_{2s}^i x^{2s}$ and $a_d^i = a_{2s-d}^i$, for all $d = 0, 1, ..., s$. Let $h_i(x) = h_0^i + h_1^i x + ... + h_{2k}^i x^{2k}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l = 1, ..., n-1$. For any $t < n/2$, the coefficient of x^t in $h_i(x)f_i(x)$ is

$$b_t^i = \sum_{j=0}^{t} h_j^i \theta_i^j (a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i = \sum_{j=0}^{t} h_{2k-j}^i \theta_i^{2k-j}(a_{2s-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k} h_p^i x^p f_i(x)$ corresponds a vector $b = (b_0^i, b_1^i, ..., b_{n-1}^i) \in C_i$, for $i = 1, ..., a$.

The vector $\xi_i(b)^r = \left(\left(\xi_i\left(b_0^i\right), \xi_i\left(b_1^i\right), ..., \xi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\xi_i(z)$, where the vector z corresponds the polynomial $\sum_{p=0}^{2k} \theta_i(h_p^i) x^{2k-p} f_i(x)$,for $i = 1, ..., a$.

Since $z = (z_1^i, ..., z_n^i) \in C_i$, then $\xi_i(C_i)$ is a reversible DNA code, for $i = 1, ..., a$.

4. The other method that is used to obtain DNA codes

In [4], S. Das et al. derived a special generator matrix of a linear code over M_1. By using it, the DNA codes with some constraints are obtained. Moreover, they proposed a new construction of DNA codes using Reed Muller-type generator matrices.

In this part, we generalize it to codes over a family of the finite rings M_e.

The following definitions are in [4].

Definition 5. *For any DNA sequence $m = m_1...m_n$, the reverse DNA sequence is $m^r = m_n...m_1$ and the reverse complement DNA sequence is $m^{rc} = m_n^c...m_1^c$ where $A^c = T, T^c = A, C^c = G, G^c = C$.*

Definition 6. *A set $C \subset S_{D_4}^n$ of size M is called a DNA code with parameter (n, M, d_H), where the minimum distance $\mathrm{d}_H = min\{d_H(\mathbf{x}, \mathbf{y}) | \mathbf{x} \neq \mathbf{y}, \mathbf{x}, \mathbf{y} \in C\}$ and $d_H(\mathbf{x}, \mathbf{y})$ is the Hamming distance between DNA sequences $\mathbf{x}$ and $\mathbf{y}$.*

Moreover, there are some constraints on (n, M, d_H) DNA code C.The reverse constraint: For any two codewords $\mathbf{c_1}, \mathbf{c_2} \in C$ such that $\mathbf{c_1}^r \neq \mathbf{c_2}$, the DNA code holds reverse constraints, if $d_H(\mathbf{c_1}^r, \mathbf{c_2}) \geq \mathrm{d}_H$. The reverse complement constraint: For any two codewords $\mathbf{c_1}, \mathbf{c_2} \in C$ such that $\mathbf{c_1}^{rc} \neq \mathbf{c_2}$, the DNA code holds reverse complement constraints, if $d_H(\mathbf{c_1}^{rc}, \mathbf{c_2}) \geq \mathrm{d}_H$.

In [4], a mapping was defined and a table was given. We define the following mapping. The mapping is different from it in [4];

$$\begin{array}{rcl} \psi_1 & : & M_1 \longrightarrow S^3_{D_4} \\ x_0 + u_1 y_0 + u_1^2 z_0 & \longmapsto & (\xi_0(x_0), \xi_0(y_0), \xi_0(z_0)) = KLN \end{array}$$

where $\xi_0(0) = A, \xi_0(1) = G, \xi_0(2) = T, \xi_0(3) = C$ and $K, L, N \in S_{D_4} = \{A, T, C, G\}$.

We give the following table according to the mapping;

elements α_1	DNA 3-mers
0	AAA
u_1	AGA
$2u_1$	ATA
$3u_1$	ACA
u_1^2	AAG
$u_1 + u_1^2$	AGG
$2u_1 + u_1^2$	ATG
$3u_1 + u_1^2$	ACG
$2u_1^2$	AAT
$u_1 + 2u_1^2$	AGT
$2u_1 + 2u_1^2$	ATT
$3u_1 + 2u_1^2$	ACT
$3u_1^2$	AAC
$u_1 + 3u_1^2$	AGC
$2u_1 + 3u_1^2$	ATC
$3u_1 + 3u_1^2$	ACC
1	GAA
$1 + u_1$	GGA
$1 + 2u_1$	GTA
$1 + 3u_1$	GCA
$1 + u_1^2$	GAG
$1 + u_1 + u_1^2$	GGG
$1 + 2u_1 + u_1^2$	GTG
$1 + 3u_1 + u_1^2$	GCG
$1 + 2u_1^2$	GAT
$1 + u_1 + 2u_1^2$	GGT
$1 + 2u_1 + 2u_1^2$	GTT
$1 + 3u_1 + 2u_1^2$	GCT
$1 + 3u_1^2$	GAC

elements α_1	DNA 3-mers
$1+u_1+3u_1^2$	GGC
$1+2u_1+3u_1^2$	GTC
$1+3u_1+3u_1^2$	GCC
2	TAA
$2+u_1$	TGA
$2+2u_1$	TTA
$2+3u_1$	TCA
$2+u_1^2$	TAG
$2+u_1+u_1^2$	TGG
$2+2u_1+u_1^2$	TTG
$2+3u_1+u_1^2$	TCG
$2+2u_1^2$	TAT
$2+u_1+2u_1^2$	TGT
$2+2u_1+2u_1^2$	TTT
$2+3u_1+2u_1^2$	TCT
$2+3u_1^2$	TAC
$2+u_1+3u_1^2$	TGC
$2+2u_1+3u_1^2$	TTC
$2+3u_1+3u_1^2$	TCC
3	CAA
$3+u_1$	CGA
$3+2u_1$	CTA
$3+3u_1$	CCA
$3+u_1^2$	CAG
$3+u_1+u_1^2$	CGG
$3+2u_1+u_1^2$	CTG
$3+3u_1+u_1^2$	CCG
$3+2u_1^2$	CAT
$3+u_1+2u_1^2$	CGT
$3+2u_1+2u_1^2$	CTT
$3+3u_1+2u_1^2$	CCT
$3+3u_1^2$	CAC
$3+u_1+3u_1^2$	CGC
$3+2u_1+3u_1^2$	CTC
$3+3u_1+3u_1^2$	CCC

Similarly, we can define a one-to-one correspondence between the DNA 3^i- mers and

the elements of M_i, where $i = 2, ..., e$ as follows.

$$\begin{array}{rcl} \psi_i & : & M_i \longrightarrow S_{D_4}^{3^i} \\ a_{i-1} + u_i b_{i-1} + u_i^2 c_{i-1} & \longmapsto & (\psi_{i-1}(a_i - 1), \psi_{i-1}(b_{i-1}), \psi_{i-1}(c_{i-1})). \end{array}$$

The map ψ_i satisfies the following two conditions:

1) For any $a_{i-1}+u_ib_{i-1}+u_i^2c_{i-1} \in M_i$, where $i = 1, 2, ..., e$, then $[\psi_i(a_{i-1}+u_ib_{i-1}+u_i^2c_{i-1})]^c = (\psi_{i-1}(a_{i-1}), \psi_{i-1}(b_{i-1}), \psi_{i-1}(c_{i-1}))^c = ((\psi_{i-1}(a_{i-1}))^c, (\psi_{i-1}(b_{i-1}))^c, (\psi_{i-1}(c_{i-1}))^c) = \psi_i(a_{i-1}+u_ib_{i-1}+u_i^2c_{i-1}+\mathbf{2}+\mathbf{2u}+\mathbf{2u^2})$, where all coefficients of $\alpha = \mathbf{2}+\mathbf{2u}+\mathbf{2u^2} \in M_i$ are $2 + 2u_1 + 2u_1^2$.

2) For any $a_{i-1}+u_ib_{i-1}+u_i^2c_{i-1} \in M_i$ where $i = 1, 2, ..., e$, then $[\psi_i(a_{i-1}+u_ib_{i-1}+u_i^2c_{i-1})]^r = (\psi_{i-1}(a_{i-1}), \psi_{i-1}(b_{i-1}), \psi_{i-1}(c_{i-1}))^r = \psi_i((a_{i-1} + u_ib_{i-1} + u_i^2c_{i-1})^r) = \psi_i(c_{i-1}^r + b_{i-1}^r u_i + a_{i-1}^r u_i^2)$, where $(a_0 + u_1b_0 + u_1^2c_0)^r = c_0 + u_1b_0 + u_1^2a_0$ for any $a_0 + u_1b_0 + u_1^2c_0 \in M_1$.

Example 5. *For $i = 2$, the element $\mathbf{2}+\mathbf{2u}+\mathbf{2u^2}$ is equal to $(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)$. For $i = 3$, the element $\mathbf{2}+\mathbf{2u}+\mathbf{2u^2}$ is equal to $[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)]+u_3[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)]+u_3^2[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)]$.*

Example 6. *Let $(1+2u_1+0u_1^2)+3u_2+0u_2^2+2u_3+1u_3^2 \in M_3$. $[\psi_3((1+2u_1+0u_1^2)+3u_2+0u_2^2+2u_3+1u_3^2)]^c = ((\psi_2((1+2u_1)+3u_2))^c, (\psi_2(2))^c, (\psi_2(1))^c) = ((\psi_1(1+2u_1), \psi_1(3), \psi_1(0))^c, (\psi_1(2), \psi_1(0), \psi_1(0))^c, (\psi_1(1), \psi_1(0), \psi_1(0))^c) = ((\psi_1\ (1+2u_1))^c, (\psi_1(3))^c, (\psi_1(0))^c, (\psi_1(2))^c, (\psi_1(0))^c, (\psi_1(0))^c, (\psi_1(1))^c, (\psi_1(0))^c, (\psi_1(0))^c) = ((GTA)^c(CAA)^c(AAA)^c(TAA)^c(AAA)^c(AAA)^c(GAA)^c(AAA)^c(AAA)^c) = CATGTTTTTATTTTTTTTCTTTTTTTT$.*

On the other hand, $\psi_3((1+2u_1+0u_1^2)+u_23+0u_2^2+2u_3+1u_3^2)+[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)]+u_3[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)]+u_3^2[(2+2u_1+2u_1^2)+u_2(2+2u_1+2u_1^2)+u_2^2(2+2u_1+2u_1^2)] = (\psi_1(3+2u_1^2), \psi_1(1+2u_1+2u_1^2), \psi_1(2+2u_1+2u_1^2), \psi(2u_1+2u_1^2), \psi_1(2+2u_1+2u_1^2), \psi_1(2+2u_1+2u_1^2), \psi_1(3+2u_1+2u_1^2), \psi_1(2+2u_1+2u_1^2), \psi_1(2+2u_1+2u_1^2)) = CATGTTTTTATTTTTTTTCTTTTTTTT$.

Example 7. *$[\psi_3((1+2u_1+0u_1^2)+u_23+0u_2^2+2u_3+1u_3^2)]^r = [\psi_2(1+2u_1)+3u_2), \psi_2(2), \psi_2(1)]^r = [(\psi_1(1+2u_1), \psi_1(3), \psi_1(0)), (\psi_1(2), \psi_1(0), \psi_1(0)), (\psi_1(1), \psi_1(0), \psi_1(0))]^r = [GTACAAAAATAAAAAAAAGAAAAAAAA]^r = AAAAAAAAGAAAAAA AATAAAAACATC$.*

On the other hand, $\psi_3(1^r + 2^r u_3 + u_3^2(1+2u_1+3u_2)^r) = (\psi_2(1^r), \psi_2(2^r), \psi_2((1+2u_1)+3u_2)^r) = (\psi_1(0), \psi_1(0), \psi_1(u_1^2), \psi_1(0), \psi_1(0), \psi_1(2u_1^2), \psi_1(0), \psi_1(3u_1^2), \psi_1(2u_1+u_1^2)) = AAAAAAAAGAAAAAAAATAAAAACATC$.

In [4], S. Das et al. defined the Gau distance d_G. This mapping

$$d_G \quad : \quad M_1 \times M_1 \longrightarrow R$$

was defined by $d_G(x,y) = min\{1,(l+3l^{'})mod4\}+min\{1,(j+3j^{'})mod4\}+min\{1,(k+3k^{'})mod4\}$ where (l,j,k) and $(l^{'},j^{'},k^{'})$ are the positions of two elements x and y, the letter R represents real numbers. It is shown that this map is a metric. Moreover they defined the minimum Gau distance for any code C, as follows.

For any two elements $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ of length n over M_1, the Gau distance between x and y is defined by

$$d_G(x,y) = \sum_{v=1}^{n} d_G(x_v,y_v)$$

For any code C, the minimum Gau distance d_G is defined by $\mathrm{d}_G = min\{d_G(x,y)| x,y \in C, x \neq y\}$.

Similarly, we define the Gau distance on M_i as follows

$$\begin{array}{rcl} d_G & : & M_i \times M_i \longrightarrow R \\ (x,y) & \longmapsto & d_G(x,y) = \sum_{s=1}^{3^i} min\{1,(t_s+3j_s)mod4\} \end{array}$$

where $(t_1,...,t_{3^i})$ and $(j_1,...,j_{3^i})$ are the positions of two elements x and y, where $i=1,...,e$. The mapping is also a metric. The Gau distance of two elements and the Gau distance of any code C_i are defined similarly, where $i=1,...,e$.

Proposition 1. *The map ψ_i is a distance conserving map from (M_i^n, d_G) to $(S_{D_4}^{3^i n}, d_H)$ for $i=1,2,..,e$, where $\mathrm{d}_H = min\{d_H(\mathbf{a},\mathbf{b})|\mathbf{a}\neq\mathbf{b},\mathbf{a},\mathbf{b}\in C_{DNA}\}$ and $d_H(\mathbf{a},\mathbf{b})$ is the Hamming distance between the DNA sequences $\mathbf{a}$ and $\mathbf{b}$.*

Proof. If t_s and j_s are the same, then $min\{1,(t_s+3j_s)mod4\}$ is equal to 0, otherwise 1 , where $s=1,...,3^i$. It is easily seen that $d_G(\mathbf{a},\mathbf{b}) = d_H(\psi_i(\mathbf{a}),\psi_i(\mathbf{b}))$ for every $\mathbf{a},\mathbf{b}\in M_i$, where $i=2,...,e$.

Example 8. *Let $\mathbf{a}=(1+2u_1+u_1^2,3u_1)$, $\mathbf{b}=(2u_1,1+3u_1^2)$ be in M_1^2. Then $d_G(\mathbf{a},\mathbf{b}) = d_G(1+2u_1+u_1^2,2u_1)+d_G(3u_1,1+3u_1^2) = 2+3 = 5$. On the other hand, $d_H(GTCACA,ATAGAC) = 5$.*

Proposition 2. *For two elements* $\mathbf{a}$ *and* $\mathbf{b}$ *in* M_i^n*, the map* ψ_i *satisfies*

$$\psi_i^{-1}(\psi_i(s\mathbf{a}+p\mathbf{b})^r) = s\psi_i^{-1}(\psi_i(\mathbf{a})^r) + p\psi_i^{-1}(\psi_i(\mathbf{b})^r)$$

where $\psi_i^{-1}(\psi_i(\mathbf{a})^r) = (\psi_i^{-1}(\psi_i(a_{i,n})^r), ..., \psi_i^{-1}(\psi_i(a_{i,1})^r))$ *for* $\mathbf{a} = (a_{i,1}, ..., a_{i,n}) \in M_i$ *for* $i = 1, 2, ..., e$.

Theorem 4. *For any given generator matrix* G_i *over* M_i*, where* $i = 1, 2, ..., e$*, the code* $\psi_i(< G_i >)$ *is closed under complement DNA sequences, if* $\mathbf{2}+\mathbf{2u}+\mathbf{2u^2}$ $\in < G_i >$ *where* $\mathbf{2}+\mathbf{2u}+\mathbf{2u^2}$ *stands for the vector of length* n *with all* $2+2u_1+2u_1^2$.

Proof. Since $a^c = a + 2$, for each $a \in M_0$, the proof is easily seen.

Theorem 5. *The DNA code* $\psi_i(< G_i >)$ *is closed under reverse DNA sequence if* $\mathbf{g}^r = \psi_i^{-1}(\psi_i(\mathbf{g})^r) = (\psi_i^{-1}(\psi_i(g_{i,n})^r))), ..., \psi_i^{-1}(\psi_i(g_{i,1})^r))) \in < G_i >$ *for each row* $\mathbf{g}$ *where* $i = 1, ..., e$.

Proof. For $i = 1$, it was proven in the Theorem 2 in [4]. Similarly, it is proven for $i = 2, ..., e$.

Corollary 6. *The code* $\psi_i(< G_i >)$ *satisfies the reversible complement constraints, for a given* G_i *over* M_i*, if* $\psi_i(< G_i >)$ *is closed under both reverse and complement DNA sequences, where* $i = 1, ..., e$.

For any code C_i of length n over M_i, $\psi_i(C_i) = \{\psi_i(\mathbf{x}) | each \mathbf{x} \in C_i\} \subset S_{D_4}^{3^i n}$ represents the DNA code of length $3^i n$, where $i = 1, 2, ..., e$.

Theorem 7. *If* C_i *is a linear* (n, M, d_G) *code over* M_i *where* $i = 1, ..., e$ *and the matrix* G_i *is associated generator matrix of* C *such that the rows of* G_i *hold the conditions described in Theorem 20 and Theorem 21, then* $\psi_i(C_i)$ *is a DNA code with the parameters* $(3^i n, M, \mathrm{d}_G = \mathrm{d}_H)$ *and the code* $\psi_i(C_i)$ *holds reversible and reversible complement constraints.*

For any ν from $T = \{2, 2u_1, 2u_1^2, 2u_2, 2u_2^2, ...2u_i, 2u_i^2, ..., 2u_1^2 2u_2^2 ... 2u_i^2\}$, where $|T| = 3^i$, we get that the number of elements of ideal generated by ν is equal to 2^{3^i}, where $i = 1, ..., e$.

Theorem 8. *Let* C_i *be a linear code over* M_i*, where* $i = 1, ..., e$*. For any* $\nu \in T$*, if the generator matrix* G_i *over* $< \nu >$ *satisfies the conditions in Theorem 20 and 21, the DNA code* $\psi_i(< G_i >)$ *satisfies reversible, reversible complement constraint.*

Proof. It follows from Theorem 4 and Theorem 5.

5. Reed Muller type codes over M_i

In [4], by using Reed-Muller types codes over M_1, new constructions of DNA codes were presented. The parameters of the DNA codes obtained by this method were given.

In this section, we construct DNA codes, by using Reed Muller types codes over M_i, where $i = 2, ..., e$.

The generator matrix $G^i_{1,m}$ of the Reed Muller type code over $R(1, m)$ of length 2^m is

$$G^i_{1,j+1} = \begin{pmatrix} G^i_{1,j} & G^i_{1,j} \\ \mathbf{0}_{2^j} & \nu_{2^j} \end{pmatrix}$$

$$G^i_{1,1} = \begin{pmatrix} \nu & \nu \\ 0 & \nu \end{pmatrix}$$

where $\mathbf{0}_{2^j} = [000...0], \nu_{2^j} = [\nu\nu...\nu]$ with $\nu \in M_i$, $i = 1, ..., e$ and $j = 1, 2, ..., m-1$. The order of this matrix is $(m+1) \times 2^m$.

Theorem 9. *Let $R(1, m)$ be the code over M_i, where $i = 1, ..., e$. Then there exists a DNA code $\psi_i(R(1, m))$ and the code is with the parameter $(3^i 2^m, (4^{3^i})^{m+1}$, $\mathrm{d}_H = 2^{m-1})$. Moreover, it satisfies both reversible and reversible complement constraints.*

Theorem 10. *Let $R(1, m)$ be the code over M_i and $\nu \in T$, where $i = 1, ..., e$. Then the code $R(1, m)$ over M_i has the length 2^m, size $(2^{3^i})^{m+1}$ and the minimum Gau distance $\mathrm{d}_G = 2^{m-1}$.*

Example 9. *For $m = 4$ and $i = 3$, the $\left(432, 2^{270}, 8\right)$-DNA code $\psi_3\left(R\left(1, 4\right)\right)$ holds the reversible and reversible complement constraints.*

Example 10. *For $m = 5, i = 5$ and $\nu = 2u_5^2$, the $R\left(1, 5\right)$ is a $\left(32, 2^{1458}, 16\right)$-DNA code. Also, the DNA code $\psi_3\left(R\left(1, 4\right)\right)$ satisfies reversible and reversible complement constraints.*

6. Conclusion

It is shown that the skew cyclic codes over the ring M_i can be used to construct the reversible DNA codes and Reed-Muller types codes over M_i can be used to construct the reversible and reversible complement DNA codes, where $i = 1, 2, ..., e$.

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Abdullah Dertli
Department of Mathematics, Faculty of Science,
University of Ondokuz Mayıs,
Samsun, Turkey
email: *abdullah.dertli@gmal.com*

Yasemin Cengellenmis
Department of Mathematics, Faculty of Science,
University of Trakya,
Samsun, Turkey
email: *ycengellenmis@gmail.com*