AN EFFECTIVE METHOD FOR SOLVING FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a modification of variational iteration method is applied to solve fractional integro-differential equations. The fractional derivative is considered in the Caputo sense. Through examples, we will see the modified method performs extremely effective in terms of efficiency and simplicity to solve fractional integro-differential equations.

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1. Introduction

In recent years, it has turned out that many phenomena in physics, engineering, chemistry, and other sciences can be described very successfully by models using mathematical tool from fractional calculus, such as, frequency dependent damping behavior of materials, diffusion processes, motion of a large thin plate in a Newtonian fluid creeping and relaxation functions for viscoelastic materials. etc. However, most fractional differential equations do not have exact analytic solutions. There are only a few techniques for the solution of fractional integro-differential equations. Three of them are the Adomian decomposition method [1], the collocation method [2],and the fractional differential transform method [3].

The variational iteration method was first proposed by he [4-11] and has found a wide application for the solution of linear and nonlinear differential equations, for example, nonlinear wave equations [5], Fokker–Planck equation [6], Helmholtz equation [7], klein-Gordon equations [8], integro-differential equations [9], and space- and time-fractional KdV equation [10]. Meanwhile, the variational iteration method has been modified by many authors [11].

In this letter, we will set a new modified variational iteration method to solve fractional-integro-differential equations. It will show the modification of the method is a useful and simplify tool to solve fractional integro-differential equations as used in other fields.

2. Basic definitions

In this section, we give some basic definitions and properties of the fractional calculus theories which are used further in this paper.

Definition 1. *A real function* $f(x)(x>0)$*, is said to be in the space* $C_\mu(\mu \in R)$*, if there exists a real number* $p(>\mu)$ *such that* $f(x)=x^p f_1(x)$*, where* $f_1(x)\in C[0,\infty]$*, and it is said to be in the space* C_μ^n *iff* $f^{(n)}\in C_\mu, n\in N$*.*

Definition 2. *The Riemamann-Liouville's fractional integral operator of order* $\alpha \geq 0$*, of a function* $f\in C_\mu, \mu\geq -1$*, is defined as [1,3]*
$I^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \alpha>0, x>0,$
$I^0 f(x)=f(x).$

Properties of the operators I^α can be found in [1,3], we mention only the following: For $f\in C_\mu, \mu\geq -1, \alpha,\beta\geq 0$ and $\lambda>-1$:

1. $I^\alpha I^\beta f(x)=I^{\alpha+\beta}f(x).$
2. $I^\alpha I^\beta f(x)=I^\beta I^\alpha f(x).$
3. $I^\alpha x^\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}x^{\alpha+\lambda}.$

The Riemann-Liouville derivative has certain disadvantages when trying to model real word phenomena with fractional equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo.

Definition 3. *The fractional derivative of* $f(x)$ *in the Caputo sense is defined as[1,3]*

$$D^\alpha f(x)=I^{n-\alpha}\frac{d^n f(x)}{dx^n}=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, \tag{2.1}$$

for $n-1<\alpha\leq n, n\in N, x>0, f\in C_{-1}^n$*.*

Lemma 1. *If* $n-1<\alpha\leq n, n\in N$ *and* $f\in C_\mu^n, \mu\geq -1$*, then*

$$D^\alpha I^\alpha f(x)=f(x),$$

and,

$$I^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{x^k}{k!}, x>0.$$

3. Concept of the variational iteration method

To illustrate the basic idea of the variational iteration method, we consider the following general nonlinear equation

$$Lu(t) + Nu(t) = f(t) \tag{3.1}$$

where L is a linear operator, N is a nonlinear operator, and $f(t)$ is a known analytical function.

According to the variational iteration method, the terms of a sequence $u_n(t)(n \geq 0)$are constructed such that this sequence converges to the exact solution $u(t)$, $u_n(t)(n \geq 0)$ are calculated by a correction function as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ Lu_n(s) + \widetilde{N}u_n(s) - f(s) \right\} ds \tag{3.2}$$

whereλ is the general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation and $\widetilde{u_u}(t)$ is considered as a restricted variation, i.e. $\delta \widetilde{u}_n = 0$ [4].

To solve (3.2) by the variational iteration method, we first determine the Lagrange multiplier λ that will be identified optionally via integration by parts. Then the successive approximation $u_n(t)(n \geq 0)$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(t)$. The zeroth approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximations $u_n(t)(n \geq 0)$ follow immediately.

Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n \to \infty} u_n(t). \tag{3.3}$$

4. Modification of the variational iteration method

Concerning the general fractional integro-differential equation of the type

$$D^\alpha y(t) = f\left(t, y(t), \int_0^t k(s, y) ds\right), \tag{4.1}$$

where D^α is the derivative of $y(t)$ in the sense of Caputo,and $n - 1 < \alpha < n$ $(n \in N)$, subject to the initial condition

$$y(0) = c. \tag{4.2}$$

According to the variational iteration method, we can construct the following correction functional

$$y_{n+1}(t) = y_n(t) + I^{\alpha}F(t), \tag{4.3}$$

where $F(t) = \lambda \left[D^{\alpha}y_n(t) - f\left(t, y_n, \int_0^t k(s, y_n)ds\right) dt\right]$, $y_n(t)$ is the nth approximation, and I^{α} is Riemann-Liouville's fractional integrate.

The Lagrange multiplier can not easy identified through (4.3), so an approximation of the correction functional can be expressed as follows

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda \left\{ \frac{d^n y_n(t)}{dt^n} - f\left(t, y_n(t), \int_0^t k(s, y_n)ds\right)\right\} dt. \tag{4.4}$$

Then the Lagrange multiplier can be easily determined by the variational theory in (4.4). Substituting the identified Lagrange multiplier into (4.3) results in the following iteration procedures

$$y_{n+1}(t) = y_n(t) + I^{\alpha}\lambda \left\{ D^{\alpha}y_n(t) - f\left(t, y_n(t), \int_0^t k(s, y_n)ds\right)\right\}, (n = 0, 1, 2, ...). \tag{4.5}$$

5. Applications

In order to illustrate its general process, in this section, we solve two examples. All the results are calculated by using the symbolic calculus software Maple 9.

Example 5.1 Let us consider the following linear fractional integro-differential equation that was studied by many authors [2,3]

$$y^{(0.75)}(t) = (\frac{-t^2e^t}{5})y(t) + \frac{6t^{2.25}}{\Gamma(3.25)} + e^t \int_0^t sy(s)ds \tag{5.1}$$

with the initial condition

$$y(0) = 0. \tag{5.2}$$

Its correctional functional reads

$$y_{n+1}(t) = y_n(t) + I^{0.75}\left\{\lambda[y_n^{0.75}(t) - g[(y_n(t)]]\right\}, \tag{5.3}$$

where

$$g[y_n(t)] = \frac{-t^2e^t}{5}y_n(t) + \frac{6t^{2.25}}{\Gamma(3.25)} + e^t \int_0^t sy_n(s)ds,$$

$y_n(t)$ is the nth approximation.

For ceil(0.75) = 1, it's approximating correctional functional can be expressed as follows

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda \left\{ y_n^{'}(\tau) - \widetilde{g}\left[y_n(\tau)\right] \right\} d\tau, \tag{5.4}$$

where $\widetilde{g}$ is considered as restricted variations, i.e.$\delta\widetilde{g} = 0$.

Its stationary conditions are given by

$$\lambda'(\tau) = 0, 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{5.5}$$

The Lagrange multiplier, therefore, there can be easily identified as $\lambda = -1$. Substituting the identified multiplier into(5.3), we have the following iteration formula

$$y_{n+1}(t) = y_n(t) - I^{0.75}\left\{ y_n^{0.75}(t) - g[y_n(t)] \right\}. \tag{5.6}$$

We start with $y_0(t) = 0$, by the variational iteration formula (5.6), we have

$$y_1(t) = y_0(t) - I^{0.75}\left\{ y_0^{0.75}(t) - \tfrac{-t^t e^t}{5} y_0(t) - e^t \textstyle\int_0^t s y_0(s) ds \right\} = t^3, \tag{5.7}$$

$$y_2(t) = y_1(t) - I^{0.75}\left\{ y_1^{0.75}(t) - \tfrac{-t^t e^t}{5} y_1(t) - e^t \textstyle\int_0^t s y_1(s) ds \right\} = t^3, \tag{5.8}$$

$$\vdots$$

$$y(t) = \lim_{n\to\infty} y_n(t) = t^3, \tag{5.9}$$

which is the exact solution.

Example 5.2 Consider the following linear system of fractional integro-differential equations [1,3]

$$\begin{cases} D^\alpha y_1(t) = 1 + t + t^2 - y_2(t) - \int_0^t \left(y_1(x) + y_2(x)\right) dx, \\ \\ D^\alpha y_2(t) = -1 - t + y_1(t) - \int_0^t \left(y_1(x) - y_2(x)\right) dx, 0 < \alpha < 1, \end{cases} \tag{5.10}$$

subject to the initial conditions

$$\begin{cases} y_1(0) & = 1, \\ y_2(0) & = -1. \end{cases} \tag{5.11}$$

The correct functional formulas for the above system,obviously, can be expressed as

$$\begin{cases} y_{1,n+1}(t) = y_{1,n}(t) - I^{\alpha}\Big\{D^{\alpha}y_{1,n}(t) - \Big[1+t+t^2-y_{2,n}(t) \\ \qquad - \int_0^t \left(y_{1,n}(x)+y_{2,n}(x)\right)dx\Big]\Big\}, \\ \\ y_{2,n+1}(t) = y_{2,n}(t) - I^{\alpha}\Big\{D^{\alpha}y_{2,n}(t) - \Big[-1-t+y_{1,n}(t) \\ \qquad - \int_0^t \big(y_{1,n}(x)-y_{2,n}(x)\big)dx\Big]\Big\}. \end{cases} \tag{5.12}$$

We start $y_{1,0}(t) = 1$, and $y_{2,0}(t) = -1$, by the variational iteration formulas (5.12), we have

$$\begin{cases} y_{1,1}(t) = 1 + \frac{2}{\Gamma(1+\alpha)}t^{\alpha} + \frac{1}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{2}{\Gamma(3+\alpha)}t^{2+\alpha}, \\ \\ y_{2,1}(t) = -1 - \frac{2}{\Gamma(1+\alpha)}t^{\alpha} - \frac{3}{\Gamma(2+\alpha)}t^{1+\alpha}. \end{cases} \tag{5.13}$$

$$\begin{cases} y_{1,2}(t) = 1 + \frac{2}{\Gamma(1+\alpha)}t^{\alpha} + \frac{1}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{2}{(\alpha^2+3\alpha+2)\Gamma(1+\alpha)}t^{2+\alpha} \\ \qquad + \frac{1}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{1}{\Gamma(2+2\alpha)}t^{1+2\alpha} - \frac{1}{\Gamma(3+2\alpha)}t^{2+2\alpha}, \\ \\ y_{2,2}(t) = -1 - \frac{3}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{1}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{1}{\Gamma(2+2\alpha)}t^{1+2\alpha} \\ \qquad - \frac{1}{\Gamma(3+2\alpha)}t^{2+2\alpha}. \end{cases} \tag{5.14}$$

$$\vdots$$

and so on. Then choosing fixed α and n, the numerical solutions (fixed x)of the system of the fractional integro-differential equations can be obtained.

6. Conclusions

In this paper, we applied the modified variational iteration method for solving the fractional integro-differential equations. Comparison with other traditional methods, the simplicity of the method and the obtained exact results show that the modified variational iteration method is a powerful mathematical tool for solving fractional integro-differential equations. Although the examples are given in this paper is linear, it also can be applicable to nonlinear fractional integro-differential equations.

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