A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY DIFFERENTIAL SĂLĂGEAN OPERATOR

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Abstract. By means of the Sălăgean differential operator we define a new class $\mathcal{BS}(p,m,\mu,\alpha)$ involving functions $f\in A(p,n)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

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1. Introduction and definitions

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:\ |z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$A(p,n)=\{f\in\mathcal{H}(U):\ f(z)=z^p+\sum_{j=p+n}^{\infty}a_jz^j,\quad z\in U\},\tag{1}$$

with $A(1,n)=A_n$ and

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}(U):\ f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,\ z\in U\},$$

where $p,n\in\mathbb{N}$, $a\in\mathbb{C}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}_n^*(p,\alpha)$ we denote a subclass of $A(p,n)$ consisting of p-valently starlike functions of order α, $0\le\alpha<p$ which satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad z\in U.\tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be p-valently convex of order α in U, if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha,\quad z\in U\tag{3}$$

for some α, $(0\le\alpha<p)$. We denote by $\mathcal{K}_n(p,\alpha)$ the class of functions in $\mathcal{S}$ which are p-valently convex of order α in U and denote by $\mathcal{R}_n(p,\alpha)$ the class of functions in $A(p,n)$ which satisfy

$$\operatorname{Re}f'(z)>\alpha,\quad z\in U.\tag{4}$$

It is well known that $\mathcal{K}_n(p,\alpha) \subset \mathcal{S}_n^*(p,\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let D^m be the Sălăgean differential operator [7], $D^m : A(p,n) \to A(p,n)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, defined as

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \\ D^m f(z) &= D(D^{m-1}f(z)) = z\left(D^{m-1}f(z)\right)', \quad z \in U. \end{aligned}$$

We note that if $f \in A(p,n)$, then

$$D^m f(z) = z^p + \sum_{j=n+p}^{\infty} j^m a_j z^j, \quad z \in U.$$

To prove our main theorem we shall need the following lemma.

Lemma 1 [6] *Let u be analytic in U with $u(0) = 1$ and suppose that*

$$\mathrm{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $\mathrm{Re}\, u(z) > \alpha$ *for* $z \in U$ *and* $1/2 \le \alpha < 1$.

2. Main results

Definition 1 *We say that a function $f \in A(p,n)$ is in the class $\mathcal{BS}(p,m,\mu,\alpha)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \ge 0$, $\alpha \in [0,1)$ if*

$$\left| \frac{D^{m+1}f(z)}{z^p} \left(\frac{z^p}{D^m f(z)} \right)^{\mu} - p \right| < p - \alpha, \qquad z \in U. \tag{6}$$

Remark 1 *The family $\mathcal{BS}(p,m,\mu,\alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BS}(1,0,1,\alpha) \equiv \mathcal{S}_n^*(1,\alpha)$, $\mathcal{BS}(1,1,1,\alpha) \equiv \mathcal{K}_n(1,\alpha)$ and $\mathcal{BS}(1,0,0,\alpha) \equiv \mathcal{R}_n(1,\alpha)$. Another interesting subclass is the special case $\mathcal{BS}(1,0,2,\alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus* [5] *and also the class $\mathcal{BS}(1,0,\mu,\alpha) \equiv \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri* [6].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BS}(p,m,\mu,\alpha)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2 *For the function* $f \in A(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ *if*

$$\frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu\frac{D^{m+1}f(z)}{D^{m}f(z)} + p(\mu-1) + 1 \prec 1 + \beta z, \quad z \in U, \tag{7}$$

where

$$\beta = \frac{3\alpha-1}{2\alpha}, \tag{8}$$

then $f \in \mathcal{BS}(p,m,\mu,\alpha)$.

Proof. If we consider

$$u(z) = \frac{D^{m+1}f(z)}{z^p}\left(\frac{z^p}{D^m f(z)}\right)^{\mu}, \tag{9}$$

then $u(z)$ is analytic in U with $u(0)=1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu\frac{D^{m+1}f(z)}{D^{m}f(z)} + p(\mu-1). \tag{10}$$

Using (7) we get

$$\operatorname{Re}\left(1+\frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}. \tag{11}$$

Thus, from Lemma 1 we deduce that

$$\operatorname{Re}\left\{\frac{D^{m+1}f(z)}{z^p}\left(\frac{z^p}{D^m f(z)}\right)^{\mu}\right\} > \alpha. \tag{12}$$

Therefore, $f \in \mathcal{BS}(p,m,\mu,\alpha)$, by Definition 1.

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 3 *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{13}$$

then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{14}$$

That is, f is convex of order $\frac{1}{2}$.

Corollary 4 [1] *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{\frac{2z^2 f''(z)+z^3 f'''(z)}{zf'(z)+z^2 f''(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{15}$$

then

$$\operatorname{Re}\left\{f'(z)+zf''(z)\right\} > \frac{1}{2}, \quad z \in U. \tag{16}$$

Corollary 5 [1] *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{17}$$

then

$$\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{18}$$

In another words, if the function f *is convex of order* $\frac{1}{2}$, *then* $f \in \mathcal{BS}(1,0,0,\frac{1}{2}) \equiv \mathcal{R}_n\left(1,\frac{1}{2}\right)$.

Corollary 6 [1] *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U, \tag{19}$$

then f *is starlike of order* $\frac{1}{2}$.

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